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INTERACTIVE PROCEDURE FOR MULTIOBJECTIVE DYNAMIC PROGRAMMING WITH THE MIXED ORDERED STRUCTURE

Abstract

The paper presents a multiobjective dynamic programming problem with the values of the criteria function in ordered structures. The first problem is a model with deterministic values; the second, one with triangular fuzzy numbers; and the third, one with discrete random variables with the k-th absolute moment finite. The fourth model is a product of the three models listed above. The aim of the paper is to present an interactive procedure which uses trade-offs and which allows to determine the final solution in the mixed ordered structure. The ordered structures and the proposed procedure are illustrated by numerical examples.

Keywords: multiobjective dynamic programming, interactive procedures, partially ordered criteria space, mixed partially ordered structures.

1 Introduction

Multiobjective, multicriteria decision problems are usually investigated as models of multicriteria dynamic programming, using the vector version of Bellman's principle of optimality (1957), non-dominated evaluations (in the criteria space) and efficient solutions (in the decision space). An example of this approach can

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be found in the papers by Trzaskalik (1991) for the deterministic case, and by Trzaskalik and Hoa (1999) for the stochastic case. Another group of problems consists of scalarization problems which allow to transform the given multicriteria problem into the corresponding single-criterion problem. An overview bibliography on this topic can be found in the papers by Li and Haimes (1989) and by Trzaskalik (1998).

Another method of generalization of single-criterion dynamic programming consists in regarding the evaluations as elements of a partially ordered space. First papers on this subject were written by Mitten (1974) and Sobol (1975), Steinberg and Parks (1979), and Henig (1985). Discrete dynamic programming with evaluations in a partially ordered space was also considered in the papers by Trzaskalik and Sitarz (2002; 2007).

A problem that appears in many decision models is that of the simultaneous occurrence of deterministic, stochastic, and fuzzy values in the set of multidimensional evaluations. Such situations are described by Zaraś (2004). A question arises: Can such mixed evaluations be used in optimal control of a multiobjective decision process according to a homogeneous scheme in ordered structures? This issue was discussed in detail in the paper by Trzaskalik and Sitarz (2004). The authors considered first the situation with a homogeneous one- or multidimensional evaluation space, consisting of real numbers, triangular fuzzy numbers, or first-order stochastic dominances. Further in the paper, examples of combinations of such structures are given, allowing to obtain product structures which are also ordered structures.

The fundamental problem in multicriteria decision making is the selection of the final solution. Commonly used for this purpose are interactive methods, which allow to include the decision maker in the process of obtaining the final decision. Worth mentioning here are selected interactive methods described in the literature. Benayoun et al. (1971) suggested the STEM method, consisting in reducing the set of admissible solutions using the Chebyshev metric. Steuer (1977) described an interactive method based on the determination of weight intervals for the criteria, to obtain a reduced criteria cone. The essence of the Korhonen and Laakso method (1986), on the other hand, is computer visualization of the information on the final solution, obtained in the consecutive iterations. Miettinen and Makela (2000) designed the NIMBUS method, which operates together with the decision maker using the Internet. In this method, the decision maker divides the obtained solutions into five classes describing his/her preferences. In turn, Ozpeynirci et al. (2017) constructed an interactive method based on narrowing the criteria cone by pairwise comparisons of selected admissible solutions.

Interactive methods are used mainly to solve deterministic multicriteria problems. It happens often, however, that the data available to the decision maker do not allow to formulate the problem in such categories. Interactive methods for decision making under risk, when evaluations of alternatives are expressed by probability distributions, have been proposed, for instance, in the papers by Nowak (2006; 2007; 2010). In the last paper, trade-offs are used to determine a new candidate solution. A similar approach is used in the present paper. The paper by Nowak and Trzaskalik (2013), on the other hand, presents an interactive approach to the dynamic decision-making problem under risk.

Trzaskalik and Sitarz (2004) focused on finding maximal solutions in various ordered structures. In the present paper, which is a continuation of the previous paper, we discuss finding the final solution in a mixed model with deterministic, stochastic, and fuzzy criteria. The goal of the present paper is to propose an interactive procedure with trade-offs, which allows to determine the final solution.

Further on, in Section 2, we discuss a multiobjective decision process in an ordered structure, as well as the problem of finding the set of maximal evaluations and the corresponding set of efficient realizations. In Section 3 we present three ordered structures: one with the set of reals as the fundamental set, another one with the set of triangular fuzzy numbers, and the third one with the set of discrete random variables with the k-th absolute moment finite. Next, we present a structure which is a product of the three ordered structures listed above. In the illustrative example in that section we find all the maximal values and the corresponding efficient realizations. In Section 4 we describe the proposed interactive procedure which allows to find the final solution. Section 5 is a summary of the paper.

2   Multiobjective decision process in an ordered structure

We consider a finite, discrete dynamic Markov process whose sets of states and decisions at each stage are finite, and whose transfer function is deterministic. A stage realization is defined as a pair consisting of a process state and a feasible decision. A process realization is a sequence of stage realizations such that the state at the beginning of the next stage is a consequence of the stage realizations at the previous stage, described by the transfer function. Each discrete multiobjective decision process can be assigned a graph whose vertices are process states and edges are decisions. Each process realization corresponds to a path joining two vertices.

Example 1

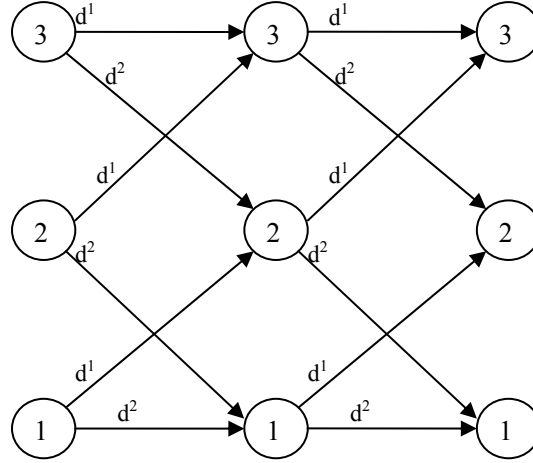An example of a multiobjective process is shown in Figure 1.

Figure 1. An example of a dynamic process

We consider a two-stage dynamic process with three admissible states at the beginning of each stage: 1, 2, 3. In each state, we can make one of the following two decisions: d^1, d^2. Depending on our current state and the decision made, we proceed to the next state according to the transfer function:

$$\Omega_t(s_t, d_t) = s_{t+1} \tag{1}$$

which is given by the following formulas:

$$\Omega_t(1, d^1) = 2, \Omega_t(1, d^2) = 1$$
$$\Omega_t(2, d^1) = 3, \Omega_t(2, d^2) = 1$$
$$\Omega_t(3, d^1) = 3, \Omega_t(3, d^2) = 2$$

To each stage realization of the process we assign a stage evaluation, while the combined evaluation of the process is an aggregate of stage evaluations. Stage and multistage evaluations are elements of a space W. We assume that a binary operator ∘ combining stage evaluations, as well as an ordering relation ≼, are defined in W. Let $a, b \in$ W. If $a \preccurlyeq b$, we say that element b is not worse than element a.

The structure (W, ∘, ≼) is called an ordered structure, if it satisfies the following condition (further referred to as TS):

$$\forall_{a \in W} \; a \preccurlyeq a \tag{2}$$
$$\forall_{a,b \in W} \; a \preccurlyeq b \; \wedge b \preccurlyeq a \implies a = b \tag{3}$$
$$\forall_{a,b,c \in W} \; a \circ (b \circ c) = (a \circ b) \circ c \tag{4}$$

$$\forall_{a,b,c \in W} \; a \preccurlyeq b \implies a \circ c \preccurlyeq b \circ c \land c \circ a \preccurlyeq c \circ b \tag{5}$$

Using relation $\preccurlyeq$ we define the following relation $\prec$:

$$a \prec b \iff a \preccurlyeq b \land a \neq b \tag{6}$$

Let $a, b \in W$. If $a \prec b$, we say that element b is better than element a. Using relation $\prec$ we determine maximal elements of set $A \subset W$:

$$\max(A) = \{a^* \in A: \; \sim \exists_{a \in A} \; a^* \prec a\} \tag{7}$$

In what follows, we will use the following notational convention:

$$\max\{a_1, \dots, a_k\} = \max(\{a_1, \dots, a_k\}) \tag{8}$$

We want to find all maximal evaluations of the process and the corresponding realizations, called efficient realizations. For this purpose, we use dynamic programming and Bellman's principle of optimality. A formal description of the procedure can be found in the papers by Trzaskalik and Sitarz (2002; 2007).

3   Examples of ordered structures

3.1 Structure S_1: $(\mathfrak{R}, +, \leq)$

As the first structure, we will consider a structure with the set of real numbers as set W. This is illustrated by a process with the same sets of admissible states and decisions, and the same transfer function as in Example 1.

Example 2
We consider a two-stage process shown in Figure 2. The values on the edges are real numbers expressing stage evaluations of the process.
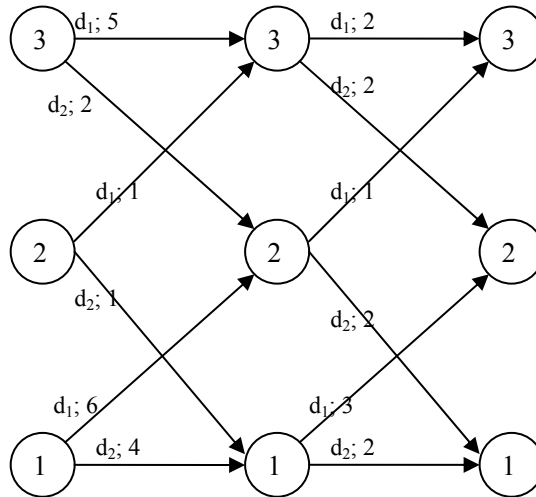

Figure 2. An illustration of the ordered structure S_1

Using dynamic programming (starting with the last stage) we find the maximal values of partial realizations of the process which start at the given state and which proceed until the end of the process, as well as the corresponding decisions. Next, we find the maximal value of the process, which is equal to 8, and the corresponding efficient realization $(1, d_1, 2, d_2)$. Detailed calculations can be found in Trzaskalik and Sitarz (2004).

3.2 Structure S_2: $(W_F, +_F, \leq_F)$

Our next ordered structure is the structure $(W_F, +_F, \leq_F)$, where:
$$W_F = \{(m, \alpha, \beta): m \in \Re, \alpha > 0, \beta > 0\} \tag{9}$$
is a set of triangular fuzzy numbers, where m is the center of the fuzzy number, and α, β are its spreads. The operator $+_F$ combining the values of the criteria function is the sum of triangular fuzzy numbers (m_1, α_1, β_1), (m_2, α_2, β_2) and is defined as follows:
$$(m_1, \alpha_1, \beta_1) +_F (m_2, \alpha_2, \beta_2) = (m_1 + m_2, \alpha_1 + \alpha_2, \beta_2 + \beta_1) \tag{10}$$
The ordering relation $\leq_F$ is defined as follows:
$$(m_1, \alpha_1, \beta_1) \leq_F (m_2, \alpha_2, \beta_2) \Leftrightarrow (m_1 \leq m_2 \wedge m_1 - \alpha_1 \leq m_2 - \alpha_2 \wedge m_1 + \beta_1 \leq m_2 + \beta_2) \tag{11}$$
This is illustrated by a process with the same sets of admissible states and decisions, and the same transfer function, as in Example 1.

Example 3
We consider a two-stage process shown in Figure 3. The values on the edges are triangular fuzzy numbers, expressing stage evaluations of the process.
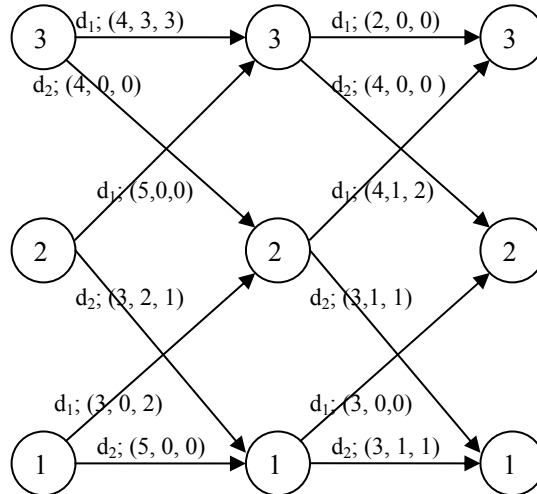

Figure 3. An illustration of the ordered structure S_2.

Using dynamic programming we obtain the following efficient realizations and the corresponding maximal values:

$$(1, d_1, 2, d_1) \rightarrow (7, 1, 4) \qquad (2, d_1, 3, d_2) \rightarrow (9, 0, 0)$$
$$(3, d_1, 3, d_2) \rightarrow (8, 3, 3) \qquad (3, d_2, 2, d_1) \rightarrow (8, 1, 2)$$

Detailed calculations can be found in Trzaskalik and Sitarz (2004).

3.3 Structure S_3: (W_R, $+_R$, $\leq_R$)

In this structure, the set W_R contains discrete random variables with the k-th absolute moment finite. We consider discrete random variables which can admit only a finite number of values from the set $\{0, 1, 2, ...\}$. The set of such random variables can be expressed as a set of probability sequences:

$$W_R = \{(p_0, p_1, p_2, ..., p_n): p_n > 0, p_i \geq 0, \sum_{i=0}^{n} p_i = 1\} \tag{12}$$

where p_i is the probability of the number $i \geq 0$.

As the operator $\circ$ we take addition of random variables.

Let $p = (p_0, p_1, p_2, ..., p_n)$, $q = (q_0, q_1, …, q_m)$; then $p \circ q$ is defined as follows:

$$p +_R q = (r_0, r_1, ..., r_{n+m}) \tag{13}$$

where $r_i = \sum_{k,l: \; k+l=i} p_k q_l$.

The ordering relation is determined by the first-order stochastic dominance which can be characterized as follows. Let $p = (p_0, p_1, ..., p_n)$, $q = (q_0, q_1, …, q_m)$.

Then:

$$p \leq_R q \Leftrightarrow (\forall_{i=1,...,\max\{n,m\}} P_i \geq Q_i) \tag{14}$$

where $P_i = \sum_{k=0}^{i} p_k$, $Q_i = \sum_{k=0}^{i} q_k$.

This is illustrated by a process with the same sets of admissible states and decisions, and the same transfer function, as in Example 1.

Example 4

We consider a two-stage process shown in Figure 4. The values on the edges are discrete random variables which can admit only a finite number of values from the set $\{0, 1, 2, ...\}$.
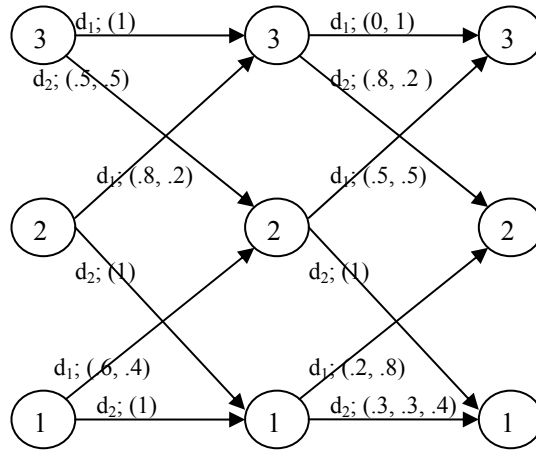
Figure 4. An illustration of the ordered structure S_3

Analogously as in the previous examples, we obtain the following efficient realizations and the corresponding maximal values:

$$(1, d_2, 1, d_2) \rightarrow (.2,.8) \qquad (2, d_2, 1, d_2) \rightarrow (.3, .3, .4)$$
$$(2, d_1, 3, d_2) \rightarrow (0, .8, .2) \qquad (3, d_1, 3, d_2) \rightarrow (.25, .5, .25)$$

Detailed calculations can be found in Trzaskalik and Sitarz (2004).

3.4 Structure S_4: $[(\mathfrak{R}, W_F, W_R), (+, +_F, +_R), (\leq, \leq_F, \leq_R)]$

Structure S_4 is a product of the three structures: with real numbers, with triangular fuzzy numbers, and with random variables with stochastic dominances. Let $[a_1, (m_1, \alpha_1, \beta_1), p_1], [a_2, (m_2, \alpha_2, \beta_2), p_2] \in [(\mathfrak{R}, W_F, W_R)$. Then:

$$[a_1, (m_1, \alpha_1, \beta_1), p_1] (+, +_F, +_R) [a_2, (m_2, \alpha_2, \beta_2), p_2] = \qquad (15)$$
$$= [a_1 + a_2, (m_1, \alpha_1, \beta_1) +_F (m_2, \alpha_2, \beta_2), p_1 +_R p_2]$$

$$[a_1, (m_1, \alpha_1, \beta_1), p_1] (\leq, \leq_F, \leq_R) [a_2, (m_2, \alpha_2, \beta_2), p_2] \Leftrightarrow \qquad (16)$$
$$\Leftrightarrow a_1 \leq a_2 \wedge (m_1, \alpha_1, \beta_1) \leq_F (m_2, \alpha_2, \beta_2) \wedge p_1 \leq_R p_2$$

This case is also illustrated by a process with the same sets of admissible states and decisions, and the same transfer function, as in Example 1.

Example 5

Consider the dynamic process shown in Figure 5. The values on the edges are real numbers, triangular fuzzy numbers, and random variables. Using dynamic programming we obtain efficient realizations and the corresponding maximal values, listed together in Table 1:
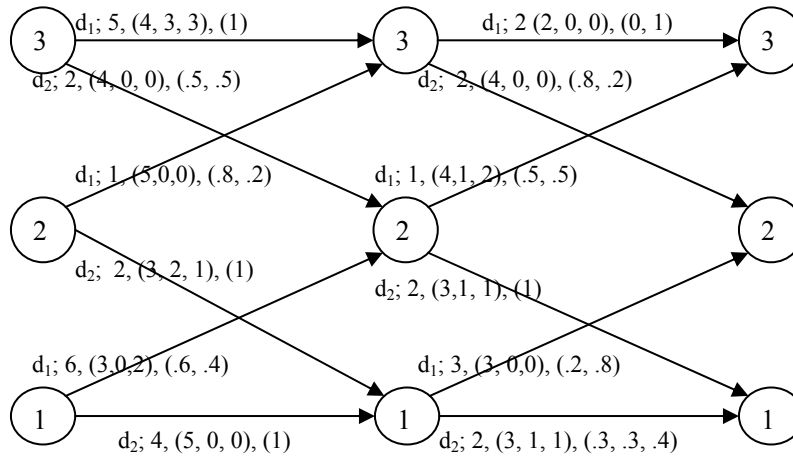

Figure 5. An illustration of the ordered structure S_4

Table 1: Efficient realizations and maximal values for structure S_4

No.	Realization	($\Re$, fuzzy number, random variable)
1	$(1, d_2, 1, d_2)$	6, (8, 1, 1), (.3, .3, .4)
2	$(1, d_2, 1, d_1)$	7, (8, 0, 0), (.2, .8)
3	$(1, d_1, 2, d_2)$	8, (6, 1, 3), (.6, .4)
4	$(1, d_1, 2, d_1)$	7, (7, 1, 4), (.3, .5, .2)
5	$(2, d_2, 1, d_2)$	3, (6, 3, 2), (.3, .3, .4)
6	$(2, d_1, 3, d_2)$	5, (9, 0, 0), (.64, .34, .02)
7	$(2, d_1, 3, d_1)$	5, (7, 0, 0), (0, .8, .2)
8	$(3, d_2, 2, d_1)$	3, (8, 1, 2), (.25, .5, .25)
9	$(3, d_1, 3, d_2)$	8, (8, 3, 3), (.8, .2)
10	$(3, d_1, 3, d_1)$	7, (6, 3, 3), (0, 1)

4   Interactive procedure

The set of efficient realizations is usually so large that choosing one of them as the final solution can be difficult. Therefore, we suggest applying an interactive procedure which enables the decision maker to identify the solution best suited to his/her expectations.

To facilitate the analysis of the solutions presented to the decision maker in the consecutive iterations of the procedure, we will scalarize the evaluations of efficient realizations. That is, we will transform them so that each evaluation will be given as a real number.

The set of real numbers $\Re$ already consists of scalars, therefore we take the identity as the scalarization:

$$a \rightarrow a$$

The set of fuzzy numbers W_F consists of triples of real numbers: center and two spreads (left and right). As the scalarization we will take the map assigning to each fuzzy number its center:

$$(m, \alpha, \beta) \rightarrow m$$

For a random variable from the set W_R, as the scalarization we will take the expected value:

$$p \rightarrow E(p)$$

Let $D = \{d_1, d_2, \ldots, d_m\}$ be the set of efficient realizations of the analyzed process and $F = \{f_1, f_2, \ldots, f_n\}$ the set of criteria used for evaluation. The evaluation of each realization with respect to the criteria can be given as a real number, a discrete random variable with the k-th absolute moment finite, or as a triangular fuzzy number. We denote by $f_j(d_i)$ the scalarized evaluation of realization d_i with respect to criterion f_j.

In the procedure described below, we will also use standardized evaluations of the realizations with respect to criteria $g_j(d_i)$, which will be determined from the following formula:

$$g_j(d_i) = \frac{f_j(d_i) - \min_{i \in \overline{1,m}}\{f_j(d_i)\}}{max_{i \in \overline{1,m}}\{f_j(d_i)\} - min_{i \in \overline{1,m}}\{f_j(d_i)\}} \tag{17}$$

Let $D^{(l)}$ be the set of realizations considered in iteration l. In each iteration of the interactive procedure the DM is shown a certain candidate realization $d^{(l)}$ and a potency matrix $M^{(l)}$ with two rows: the first one groups the largest values of the criteria used for the realizations from set $D^{(l)}$, and the second one, the smallest values:

$$M^{(l)} = \begin{bmatrix} \overline{f}_1^{(l)} & \cdots & \overline{f}_n^{(l)} \\ \underline{f_1}^{(l)} & \cdots & \underline{f_n}^{(l)} \end{bmatrix} \tag{18}$$

where:

$$\overline{f}_j^{(l)} = \max_{d_i \in D^{(l)}} f_j(d_i), j \in \overline{1,n} \tag{19}$$

$$\underline{f_j}^{(l)} = \min_{d_i \in D^{(l)}} f_j(d_i), j \in \overline{1,n} \tag{20}$$

The proposed interactive procedure consists of the following steps:

Preliminary stage:
1. For each efficient realization, calculate the scalar values with respect to all criteria $f_j(d_i)$.
2. Using formula (17), calculate the standardized values of the evaluations of efficient realizations with respect to criteria $g_j(d_i)$.
3. Determine the first candidate realization $d^{(1)}$ using the min-max criterion:
   a) For each realization, determine the minimum of the standardized evaluations with respect to each criterion:
   $$g^{\min}(d_i) = \min_{j\in\overline{1,n}}\{g_j(d_i)\} \tag{21}$$
   b) As the first candidate realization $d^{(1)}$ take that d_i for which the value $g^{\min}(d_i)$ is maximal.
4. Set $l = 1$ and $D^{(1)} = D$ and proceed to the first iteration.

Iteration l
1) Determine the potency matrix $M^{(l)}$.
2) Present the values of the criteria obtained for realization $d^{(l)}$ and potency matrix $M^{(l)}$ to the DM. If the DM is satisfied with the proposed realization, end the procedure.
3) Ask the DM to assign each criterion to one of the following three sets:
   F_1 – the set of criteria whose values should be improved as compared with the value obtained for realization $d^{(l)}$,
   F_2 – the set of criteria whose values should not be made worse as compared with the value obtained for realization $d^{(l)}$,
   F_3 – the set of criteria whose values can be made worse as compared with the value obtained for realization $d^{(l)}$.
4) Determine the set $D^{(l+1)}$ consisting of all the realizations from the set $D^{(l)}$ which satisfy the following conditions:
   $$\forall_{f_j\in F_1} f_j(d_i) > f_j\big(d^{(l)}\big) \tag{22}$$
   $$\forall_{f_j\in F_2} f_j(d_i) \geq f_j\big(d^{(l)}\big) \tag{23}$$
5) If $D^{(l+1)} = \emptyset$, inform the decision maker that no realization exists for which the values of the criteria from F_1 are higher than for realization $d^{(l)}$, and the values of the criteria from F_2 are not lower than those for realization $d^{(l)}$. Return to step (2).
6) If $D^{(l+1)}$ consists of one realization only, take this realization as the next proposed realization $d^{(l+1)}$. Proceed to step (10).

7) For each realization $d_i \in D^{(l+1)}$ and for each criteria pair (f_j, f_k), such that $f_j \in F_1, f_k \in F_3$ and $f_k(d_i) < f_k(d^{(l)})$, calculate the value of the trade-off $t_{jk}(d_i)$ from the formula:

$$t_{jk}(d_i) = \frac{g_j(d_i) - g_j(d^{(l)})}{g_k(d^{(l)}) - g_k(d_i)} \qquad (24)$$

8) For each criteria pair (f_j, f_k) such that $f_j \in F_1, f_k \in F_3$, check if there exists at least one realization $d_i \in D^{(l+1)}$, for which the value of $t_{jk}(d_i)$ has been calculated in step 5. If so, then for each realization $d_m \in D^{(l+1)}$ such that $f_k(d_m) \geq f_k(d^{(l)})$, take as the trade-off $t_{jk}(d_i)$ twice the maximal value of the trade-offs calculated for the pair (f_j, f_k) in step 5. If for every realization we have $d_m \in D^{(l+1)}$, take $t_{jk}(d_m) = 1$.

9) For each realization $d_i \in D^{(l+1)}$, calculate the average of the trade-offs calculated in steps 5 and 6 for each criteria pair (f_j, f_k), such that $f_j \in F_1, f_k \in F_3$. As the next realization $d^{(l+1)}$ to be proposed to the decision maker take the one for which this average is highest.

10) Set $l = l + 1$ and proceed to the next iteration.

The first candidate realization is determined using the min-max criterion. In each iteration, the DM is presented with evaluations of the proposed realization and with the potency matrix, which consists of maximal and minimal criteria values obtained for the currently considered realizations. The DM can either accept the proposed realizations as the solution of the problem, or else determine the direction of improvement, by indicating:

a) which criteria should achieve a value higher than the one obtained for the candidate realization,

b) which criteria should retain the value obtained for the candidate realization,

c) which criteria can have a lower value than the one obtained for the candidate realization.

Of course, since we operate within the set of efficient realizations, the decision maker must indicate at least one criterion whose value can be lowered.

The procedure should continue until the decision maker is satisfied with the proposed realization (step 2). During the dialog it can turn out, however, that the consecutive proposals do not satisfy the decision maker's expectations. He/she can then either end the procedure or else consider once again the realizations proposed earlier and decide to select one of them.

5   An illustration of the interactive procedure

Consider the problem from Example 5. In the preliminary stage, we calculate the values of the scalarized evaluations of efficient realizations (Table 2) and standardized values (Table 3).

Table 2: Values of scalarized evaluations for efficient realizations

d_i	$f_1(d_i)$	$f_3(d_i)$	$f_3(d_i)$
d_1 (1, 2, 1, 2)	6	8	1.1
d_2 (1, 2, 1, 1)	7	8	0.8
d_3 (1, 1, 2, 2)	8	6	0.4
d_4 (1, 1, 2, 1)	7	7	0.9
d_5 (2, 2, 1, 2)	3	6	1.1
d_6 (2, 1, 3, 2)	5	9	0.4
d_7 (2, 1, 3, 1)	5	7	1.2
d_8 (3, 2, 2, 1)	3	8	1.0
d_9 (3, 1, 3, 2)	8	8	0.2
d_{10} (3, 1, 3, 1)	7	6	1.0
Min	3	6	0.2
Max	8	9	1.2

Table 3: Values of standardized evaluations for efficient realizations

d_i	$g_1(d_i)$	$g_3(d_i)$	$g_3(d_i)$	Min
d_1 (1, 2, 1, 2)	0.60	0.67	0.90	0.60
d_2 (1, 2, 1, 1)	0.80	0.67	0.60	0.60
d_3 (1, 1, 2, 2)	1.00	0.00	0.20	0.00
d_4 (1, 1, 2, 1)	0.80	0.33	0.70	0.33
d_5 (2, 2, 1, 2)	0.00	0.00	0.90	0.00
d_6 (2, 1, 3, 2)	0.40	1.00	0.20	0.20
d_7 (2, 1, 3, 1)	0.40	0.33	1.00	0.33
d_8 (3, 2, 2, 1)	0.00	0.67	0.80	0.00
d_9 (3, 1, 3, 2)	1.00	0.67	0.00	0.00
d_{10} (3, 1, 3, 1)	0.80	0.00	0.80	0.00

We include all the efficient realizations in set $D^{(1)}$:
$$D^{(1)} = \{d_1, d_2, d_3, d_4, d_5, d_6, d_7, d_8, d_9, d_{10}\}$$
As the first candidate realization, we take d_1:
$$d^{(1)} = d_1$$
The following calculations are performed in the consecutive iterations:

Iteration 1
1) Determine the potency matrix $M^{(1)}$.
2) Present the criteria values for realization $d^{(1)}$ and the potency matrix $M^{(1)}$ (Table 4) to the DM.

Table 4: Candidate realization and potency matrix from iteration 1

Criterion	f_1	f_2	f_3
$d^{(1)}$	6	8	1.1
Minimal value	3	6	0.2
Maximal value	8	9	1.2

The DM is not satisfied with realization $d^{(1)}$.

3) The DM decides that the value of criterion f_1 should be higher than 6, while the values of the remaining criteria can be lowered as compared with the ones obtained for realization $d^{(1)}$:
$$F_1 = \{ f_1 \}, F_2 = \emptyset, F_3 = \{ f_2, f_3 \}$$

4) Determine the set of variants satisfying the condition formulated by the DM in step 3:
$$D^{(2)} = \{d_2, d_3, d_4, d_9, d_{10}\}$$

5) Since $D^{(2)} \neq \emptyset$, proceed to the next step.

6) Since $D^{(2)}$ contains more than one realization, proceed to the next step.

7) Calculate trade-offs $t_{12}(d_i)$ and $t_{13}(d_i)$ for $d_i \in D^{(2)}$. When calculating the first value, omit the realizations for which $f_2(d_i) \geq f_2(d^{(1)})$, that is, d_2 and d_9. For the remaining realizations from $D^{(2)}$ the trade-offs are:
$$t_{12}(d_3) = 0.60, t_{12}(d_4) = 0.60, t_{12}(d_{10}) = 0.30$$

When calculating the trade-offs, we note that $f_3(d_i) \geq f_3(d^{(1)})$ does not hold for any $d_i \in D^{(2)}$. The trade-offs are:
$$t_{13}(d_2) = 0.67, t_{13}(d_3) = 0.57, t_{13}(d_4) = 1.00, t_{13}(d_9) = $$
$$= 0.44, t_{12}(d_{10}) = 2.00$$

8) For d_2 and d_9, for which trade-offs $t_{12}(d_i)$ have not been calculated in step 7, take:
$$t_{12}(d_2) = t_{12}(d_9) = 2 \cdot \max\{t_{12}(d_3), t_{12}(d_4), t_{12}(d_{10})\} = 1.20$$

9) Calculate the average values of the trade-offs (Table 5).

Table 5: Values of the trade-offs calculated in iteration 1

d_i	$t_{12}(d_i)$	$t_{13}(d_i)$	Average
d_2	1.20	0.67	0.93
d_3	0.60	0.57	0.59
d_4	0.60	1.00	0.80
d_9	1.20	0.44	0.82
d_{10}	0.30	1.00	1.15

The next realization $d^{(2)}$ proposed to the DM is d_{10}.

10) Set $l = 2$ and proceed to the next iteration.

Iteration 2
1) Determine the potency matrix $M^{(2)}$.
2) Present the criteria values obtained for realization $d^{(2)}$ and the potency matrix $M^{(2)}$ (Table 6) to the DM.

Table 6: Candidate realization and potency matrix from iteration 2

Criterion	f_1	f_2	f_3
$d^{(2)}$	7	6	1.0
Minimal value	7	6	0.2
Maximal value	8	8	1.0

The DM decides that realization $d^{(2)}$ does not satisfy his/her expectations.
3) The DM decides that the value of criterion f_2 should be higher than 6, the value of criterion f_1 should not be lowered, but is willing to accept a value lower than 1.0 for criterion f_3:
$$F_1 = \{ f_2 \}, F_2 = \{ f_1 \}, F_3 = \{f_3\}$$
4) Determine the set of variants which satisfy the condition formulated by the DM in step 3:
$$D^{(3)} = \{d_2, d_4, d_9\}$$
5) Since $D^{(3)} \neq \emptyset$, proceed to the next step.
6) Since $D^{(3)}$ contains more than one realization, proceed to the next step.
7) Calculate trade-offs $t_{23}(d_i)$ for $d_i \in D^{(3)}$. For each realization, $f_3(d_i) < f_3(d^{(2)})$ holds. The calculated values are:
$$t_{23}(d_2) = 3.33, t_{23}(d_4) = 3.33, t_{23}(d_9) = 0.83$$
8) Since for each realization, $f_3(d_i) < f_3(d^{(2)})$ holds, there is no need to calculate the next values of the trade-offs.
9) Trade-offs have been calculated for one pair of criteria only. For realizations d_2 and d_4 their values are identical. As the next candidate realization $d^{(3)}$ we take d_2.
10) Set $l = 3$ and proceed to the next iteration.

Iteration 3
1) Determine the potency matrix $M^{(3)}$.
2) Present the criteria values obtained for realization $d^{(3)}$ and the potency matrix $M^{(3)}$ (Table 7) to the DM.

Table 7: Candidate realization and potency matrix from iteration 3

Criterion	f_1	f_2	f_3
$d^{(3)}$	7	8	0.8
Minimal value	7	7	0.2
Maximal value	8	8	0.9

The DM finds realization $d^{(3)}$ satisfactory.

As the solution of the problem, realization d_2 has been finally accepted, according to which the process should begin in state 1, and the decision to be taken is 2. As a result, at the beginning of stage 2, the process will be again in state 1, and the decision to be taken at that state is 1. Eventually, the process will end in state 2. The evaluation of the selected realization with respect to criterion 1 is 6; that with respect to criterion 2 is the fuzzy number (8, 1, 1); and that with respect to criterion 3 is described by the discrete probability distribution (0.3, 0.3, 0.4).

6   Summary

The procedure presented here does not require much effort from the decision maker. When evaluating the solution proposed, all he/she must do is to divide the criteria into three groups: those whose values should be corrected, those whose values should not be made worse, and those whose values can be lowered. To determine the next candidate solution, the values of the trade-offs are analyzed.

One of the fundamental questions which should be answered when constructing a new interactive method is how to present the results to the decision maker and how the decision maker should formulate his/her preferences. This is particularly important when evaluations with respect to criteria are expressed not by real numbers but in another form. In the procedure proposed here, we scalarize the evaluations. In the future, however, we intend to propose other, more advanced tools, using other methods of interacting with the decision maker, which will allow to present him/her with more information as to the consequences of the selection of the solution.

Acknowledgements

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