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Measuring Dynamics of Respondents' Opinions by Means of Nonhomogeneous Markov Chain

Abstract

This paper is focused on a new approach to analyze changes of qualitative characteristic of economic process by means of a special kind of nonhomogeneous Markov chain lated to the concept of switching models.

Qualitative feature of economic process (such as evaluation of economic situation) may be in natural way modeled by multinomial distribution. While observing time series of such variables it is appealing to let that the parameters of multinomial distribution be time-varying. In the paper I propose to model the qualitative variable by means of fitting the probability distribution which is a mixture (or Markov mixture) of multinomial probability distributions with parameters depending on transition probabilities of a Markov chain. Such approach lets treat the observed data as an outcome of a nonhomogeneous Markov chain with transition matrix in each period belonging to a finite set of possible matrices. The choice of the matrix describing the process in each moment of observation is governed by an unobserved regime variable and so the nonhomogeneity of the chain results from switches between different regime transition matrices.

Theis paper presents the maximum likelihood estimators of the model parameters developed for micro data (i.e. when the whole history of responses of each individual respondent is available) and macro data (only the structures of responses are available). It also includes the analysis of the results of business tendency survey in Polish industry. The proposed model has been applied to analyze the dynamics of respondents' opinions concerning volume of production and financial situation under two possible regimes.

Key Words: Markov chain, Switching model, Mixture distribution
JEL Classification: C33, C51, D29

1. Introduction

Switching models appearing in econometric literature for over 40 years now belong to a wide class of models with a common feature – discrete parameters change according to one of possible regimes valid at a given instant. As the notion of "switching" embodies many different concepts on the very mechanism of regime changes (e.g., deterministic – stochastic, observed – unobserved – partially observed) the class of switching models includes:

- structural break models,
- threshold models,
- mixture distribution and Markov mixture distribution models,
- hidden Markov models,
- AR-MSM (Markov switching models),
- latent structure models.

Switching models have found a lot of applications, many of them in measuring business activity or modeling business cycles based on data from business tendency surveys. The majority of such applications (Amstad, 1999, Koskinsen and Oller, 2002) use MSM or AR-MSM with the underlying normal probability distribution under each regime. Other proposals (Dędys and Tarnowska, 2004) consist in applying binary HMM in analysing dynamics of balances. In both cases however it is necessary to quantify survey data transforming the structures of responses into one-dimensional time series.

The most popular measure of qualitative information from business survey data – balance – is the easiest to obtain but it inherently leads to some loss of information as it is possible to get two identical values of balance for two completely different structures of responses. Besides it does not seem obvious how in fact the values of balance should be interpreted. More sophisticated quantification methods, such as probabilistic or regression method are free of these disadvantages but they may generate problems of statistical nature (restrictive assumptions, estimation of additional parameters, possible collinearity or serial correlation).

Above-mentioned restrictions motivate to look for an alternative approach to analyse business tendency surveys data. Instead of transforming data into one-dimensional time series I recommend using more detailed information in a form of multi-dimensional time series (called micro- and macrodata) and I propose a type of switching model which is suitable in this case.

2. Nonhomogeneous switching Markov chain

Markov chains are useful in modeling dynamics of population in which every individual is characterized by attribution to a particular category. In longer time horizon it is usually impossible however to assume homogeneity of the Markov chain involved and so it is necessary to allow time-varying transition rules. An interesting proposal of a model including time-varying transition matrix has been introduced by Tabeau (1987). In his Interval Markov Chain (IMC) model transition rules depend on the state of the process observed in particularly interesting moments of time. The basic idea consists in superposing a Markov chain with a renewal process - after each renewal the transition rules are changed and the time left to the next renewal is represented by a stochastic variable.

The approach proposed in this paper refers to a concept of IMC as the choice of the transition matrix depends on the value of a stochastic variable. In contrast to the IMC model the regime variable is unobserved and so the process "switches" from one homogeneous Markov chain to another according to regime changes.

2.1. Model for microdata

It is assumed that microdata are available, i.e. for each time period <t-1, t) the number of individuals who moved between each pair of possible states

$$n_{ij}(t) \ , t=1,2,\dots T, \quad i,j=1,2,\dots,r \tag{1}$$

is known.

Let

$$\mathbf{P}^{(k)} = \left\lfloor p_{ij}^{(k)} \right\rfloor, k=1,2,\dots,N, \quad i,j=1,2,\dots,r \tag{2}$$

denote a transition matrix for regime k. It is further assumed that regimes changes are controlled by a series of iid stochastic variables $\{S_t\}$,

$$P(S_t = k) = \pi_k, \ k=1,2,\dots,N, \tag{3}$$

$$\sum_{k=1}^{N} \pi_k = 1 . \tag{4}$$

If it is reasonable to admit that individuals move independently, conditional probability distribution of microdata

$$\mathbf{n}(t) = \left[n_{11}(t), n_{12}(t) \dots n_{1r}(t), \dots, n_{r1}(t), n_{r2}(t) \dots n_{rr}(t)\right] \tag{5}$$

for regime k is given by a formula

$$f(\mathbf{n}(t)\,|\,\mathbf{n}(t-1)),S_t=k)=\prod_{i=1}^{r}\frac{n_i(t-1)!}{n_{i1}(t)!\dots n_{ir}(t)!}\cdot\left(p_{i1}{}^{(k)}\right)^{n_{i1}(t)}\cdot\ldots\cdot\left(p_{ir}{}^{(k)}\right)^{n_{ir}(t)} \quad (6)$$

where

$$n_i(t)=\sum_{i=1}^{r}n_{ij}(t-1)$$

denotes the number of individuals in state i.

Unconditional probability distribution is then

$$f(\mathbf{n}(t)\,|\,\mathbf{n}(t-1))=\sum_{m=1}^{N}\pi_m\left(\prod_{i=1}^{r}\frac{n_i(t-1)!}{n_{i1}(t)!\dots n_{ir}(t)!}\cdot\left(p_{i1}{}^{(m)}\right)^{n_{i1}(t)}\cdot\ldots\cdot\left(p_{ir}{}^{(m)}\right)^{n_{ir}(t)}\right). \quad (7)$$

It has been proved (Decewicz, 2005) that maximization of likelihood function for this model by means of EM algorithm leads to calculating improved estimators of the model parameters in each iteration of algorithm according to formulae

$$\pi_k=\frac{1}{T}\sum_{t=1}^{T}P(S_t=k\,|\,\mathbf{n}(t),\mathbf{n}(t-1)),\ k=1,2,\ldots,N \quad (8)$$

$$p_{ij}{}^{(k)}=\frac{\sum_{t=1}^{T}P(S_t=k\,|\,n(t),\mathbf{n}(t-1))\cdot n_{ij}(t)}{\sum_{t=1}^{T}P(S_t=k\,|\,n(t),\mathbf{n}(t-1))\cdot n_i(t-1)},\ k=1,2,\ldots,N,\quad i,j=1,2,\ldots,r. \quad (9)$$

2.2. Model for macrodata

In this case it is assumed that only macrodata

$$y_j(t)=\frac{n_j(t)}{N(t)},\ \mathrm{j}=1,2,\ldots,\mathrm{r},\quad \mathrm{t}=1,2,\ldots,\mathrm{T} \quad (10)$$

are available. $N(t)$ denotes the number of individuals in the population.

The probability $q_j{}^{(k)}(t)$ that a single individual is in state j under regime k might be then expressed by the formula

$$q_j{}^{(k)}(t)=\sum_{i=1}^{r}y_i(t-1)p_{ij}{}^{(k)} \quad (11)$$

and hence, the conditional probability distribution of macrodata

$$\mathbf{n}(t)=\left[n_1(t),n_2(t),\ldots,n_r(t)\right] \quad (12)$$

under regime k is given by

$$f(\mathbf{n}(t)\,|\,\mathbf{n}(t-1),S_t=k)=\frac{N(t)!}{n_1(t)!...n_r(t)!}q_1^{(k)}(t)^{n_1(t)}...q_r^{(k)}(t)^{n_r(t)}, \tag{13}$$

and the unconditional

$$f(\mathbf{n}(t)\,|\,\mathbf{n}(t-1))=\sum_{k=1}^{N}\pi_k\frac{N(t)!}{n_1(t)!...n_r(t)!}q_1^{(k)}(t)^{n_1(t)}...q_r^{(k)}(t)^{n_r(t)}. \tag{14}$$

In this case maximization of likelihood function does not guarantee nonnegativity of transition probability estimators however it is possible to prove (Decewicz, 2005) that conditional maximization of likelihood function by means of EM algorithm leads to solving in each iteration a series of quadratic programming problems of a form

$$(2\mathbf{X}_*^T\mathbf{V}_*^{(k)^{-1}}\mathbf{y}_*-\mathbf{X}_*^T\mathbf{V}_*^{(k)^{-1}}\mathbf{X}_*\mathbf{p}_*^{(k)})^T\mathbf{p}_*^{(k)}\to\max \tag{15.a}$$

$$\mathbf{p}_*^{(k)}\geq\mathbf{0}, \tag{15.b}$$

$$\sum_{j=1}^{r-1}p_{ij}^{(k)}\leq 1,\ i=1,2,..,r, \tag{15.c}$$

$$\pi_k=\tfrac{1}{T}\sum_{t=1}^{T}P(S_t=k\,|\,\mathbf{n}(t),\mathbf{n}(t-1)), \tag{15.d}$$

where

$$\mathbf{X}_*=\begin{bmatrix}\mathbf{X}_1 & \ldots & \mathbf{0}\\ \vdots & \cdots & \vdots\\ \mathbf{0} & \ldots & \mathbf{X}_{r-1}\end{bmatrix},\quad \mathbf{X}_j=\begin{bmatrix}y_1(0) & \cdots & y_r(0)\\ \vdots & \ddots & \vdots\\ y_1(T-1) & \cdots & y_r(T-1)\end{bmatrix}, \tag{16.a}$$

$$\mathbf{y}_*=\begin{bmatrix}\mathbf{y}_1\\ \vdots\\ \mathbf{y}_{r-1}\end{bmatrix},\quad \mathbf{y}_j=\begin{bmatrix}y_j(1)\\ \vdots\\ y_j(T)\end{bmatrix}, \tag{16.b}$$

$$\mathbf{p}_*^{(k)}=\begin{bmatrix}\mathbf{p}^{(k)}{}_1\\ \vdots\\ \mathbf{p}^{(k)}{}_{r-1}\end{bmatrix},\quad \mathbf{p}_j^{(k)}=\begin{bmatrix}p_{rj}^{(k)}\\ \vdots\\ p_{rj}^{(k)}\end{bmatrix}, \tag{16.c}$$

$$\mathbf{V}_*^{(k)^{-1}}=\mathbf{S}_*^{(k)}\boldsymbol{\Sigma}_*^{(k)^{-1}},\quad \mathbf{S}_*^{(k)}=\begin{bmatrix}\mathbf{S}^{(k)} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \mathbf{S}^{(k)}\end{bmatrix}_{(r-1)T\times(r-1)T}, \tag{16.d}$$

$$\mathbf{S}^{(k)}=diag\{P(S_t=k\,|\,\mathbf{n}(t),\mathbf{n}(t-1)),t=1,2,...,T\}_{T\times T}, \tag{16.e}$$

$$\Sigma_*^{(k)^{-1}} = \begin{bmatrix} \Sigma_{11}^{(k)} & \cdots & \Sigma_{1,r-1}^{(k)} \\ \vdots & \ddots & \vdots \\ \Sigma_{r-1,1}^{(k)} & \cdots & \Sigma_{r-1,r-1}^{(k)} \end{bmatrix}, \tag{16.f}$$

$$\Sigma_{ij}^{(k)} = \begin{cases} diag\left\{\dfrac{N(t)}{q_r^{(k)}(t)}, t = 1,2,\dots,T\right\}_{T\times T} & \text{for i} \neq \text{j} \\ diag\left\{\dfrac{N(t)}{q_r^{(k)}(t)} + \dfrac{N(t)}{q_j^{(k)}(t)}, t = 1,2,\dots,T\right\}_{T\times T} & \text{for i} = \text{j}, \quad \text{i, j} = 1,2,\dots,\text{r - 1}. \end{cases} \tag{16.g}$$

Let us point out that there is no need to use model for macrodata if microdata are available.

3. Application of nonhomogeneous switching Markov chain to analysis of business tendency survey data

To apply the model introduced in section 2 to modeling the dynamics of respondents' opinions it is necessary to accept two general assumptions:

- tendency of choosing a particular response differs subject to unobserved directly regime reflecting the general business climate which is independent on a single respondent's choice;
- tendency of choosing a particular response differs among respondents subject to their indication in a previous period. Such one-period memory makes it possible to distinguish respondents according to their attitude to recent changes of their situation.

The model has been applied to measure the dynamics of the opinions of the Polish industrial enterprises taking part in business tendency survey conducted by the Research Institute for Economic Development (RIED) at the Warsaw School of Economics. For empirical analysis monthly responses from each enterprise since September 1997 to May 2006, concerning two issues

- change in level of production (PROD, EXPROD)
- financial situation of our enterprise (FIN, EXFIN).

For each of them both evaluation of present situation and expectation for the next months have been taken into account. Each time series from a single enterprise has a form of a series of numbers 1, 2, 3 denoting:

1 – increase/improvement

2 – no change

3- decrease/worsening.

Data coming from all respondents taking part in the survey have been then collected and transformed into microdata, i.e. vectors

$$\mathbf{n}(t) = [n_{11}(t) \quad n_{12}(t) \quad n_{13}(t) \quad n_{21}(t) \quad n_{22}(t) \quad n_{23}(t) \quad n_{31}(t) \quad n_{32}(t) \quad n_{33}(t)],$$

where $n_{ij}(t)$ denotes the number of respondents who chose response i at time t-1 and response j at time t.

Assuming that there are two possible regimes (which may be interpreted as relatively better and worse general business climate) affecting respondents' choices, a nonhomogeneous switching Markov chain with two three-states transition matrices has been estimated. The parameters of each transition matrix $\mathbf{P}^{(1)}$ and $\mathbf{P}^{(2)}$ denote probabilities of moving from state i to state j, i.e. probabilities that respondent changes opinion (or expectation) from increase to no change, increase to decrease etc. within one time period. For each variables considered (PROD, EXPROD, FIN, EXFIN) a homogeneous Markov chain has been estimated as well.

The following results have been obtained:

1. Changes in level of production – evaluation of present situation

Nonhomogeneous model:

$$\hat{\pi}_{\mathrm{PROD}} = 0.514$$

$$\hat{\mathbf{P}}^{(1)}_{\mathrm{PROD}} = \begin{bmatrix} 0.521 & 0.330 & 0.149 \\ 0.213 & 0.630 & 0.157 \\ 0.243 & 0.330 & 0.427 \end{bmatrix}$$

$$\hat{\mathbf{P}}^{(2)}_{PROD} = \begin{bmatrix} 0.400 & 0.351 & 0.249 \\ 0.152 & 0.608 & 0.240 \\ 0.182 & 0.285 & 0.533 \end{bmatrix}$$

Homogeneous model:

$$\hat{\mathbf{P}}_{PROD} = \begin{bmatrix} 0.464 & 0.340 & 0.196 \\ 0.183 & 0.619 & 0.198 \\ 0.210 & 0.305 & 0.485 \end{bmatrix}$$

The interpretation of the results obtained for nonhomogeneous model is following: regime 1 is valid for about 51% of time. The exact separation of the analyzed period between two regimes has been illustrated in Figures 1 and 2. The parameters of the first rows of transition matrix $\hat{\mathbf{P}}^1_{PROD}$ and $\hat{\mathbf{P}}^2_{PROD}$ lead to a conclusion that under regime 1 about 52% of respondents who chose the response "increase" previously do not change their opinion in the following month, 33% of them change their choice to "no change" and only 15% indicate "decrease" while under regime 2 the corresponding percentages are 40, 35 and 25. Similarly, from the second rows of transition matrices results that among the respondents who chose "no change" previously 21% change opinion to "increase" under regime 1 while only 15% under regime 2. The percentage of respondents who maintain the response "no change" is 63 under regime 1, and 61 under regime 2. Finally the last rows of transition matrices give the estimated percentage of the opinions "increase", "no change", "decrease" among the respondents who indicated "decrease" in the previous month. The interpretation of the parameters of matrix $\hat{\mathbf{P}}_{PROD}$ for homogeneous model is similar, however it does not take into account the impact of regime switches.

The interpretation of the results for other variables is analogous and thus will not be repeated.

2. Changes in level of production – expectations for next months

Nonhomogeneous model:

$$\hat{\pi}_{EXPROD} = 0.637$$

$$\hat{\mathbf{P}}^{(1)}_{EXPROD} = \begin{bmatrix} 0.506 & 0.334 & 0.159 \\ 0.208 & 0.624 & 0.168 \\ 0.266 & 0.374 & 0.360 \end{bmatrix}$$

$$\hat{\mathbf{P}}^{(2)}_{EXPROD} = \begin{bmatrix} 0.389 & 0.350 & 0.261 \\ 0.140 & 0.609 & 0.251 \\ 0.203 & 0.312 & 0.485 \end{bmatrix}$$

Homogeneous model:

$$\hat{\mathbf{P}}_{EXPROD} = \begin{bmatrix} 0.464 & 0.340 & 0.196 \\ 0.183 & 0.619 & 0.198 \\ 0.240 & 0.349 & 0.410 \end{bmatrix}$$

3. Financial situation of our enterprise – evaluation

Nonhomogeneous model:

$$\hat{\pi}_{FIN} = 0.488$$

$$\hat{\mathbf{P}}_{FIN}^{(1)} = \begin{bmatrix} 0.527 & 0.411 & 0.062 \\ 0.118 & 0.762 & 0.120 \\ 0.075 & 0.374 & 0.551 \end{bmatrix}$$

$$\hat{\mathbf{P}}_{FIN}^{(2)} = \begin{bmatrix} 0.421 & 0.465 & 0.114 \\ 0.084 & 0.729 & 0.187 \\ 0.043 & 0.287 & 0.670 \end{bmatrix}$$

Homogeneous model:

$$\hat{\mathbf{P}}_{FIN} = \begin{bmatrix} 0.477 & 0.437 & 0.087 \\ 0.100 & 0.745 & 0.155 \\ 0.055 & 0.321 & 0.624 \end{bmatrix}$$

4. Financial situation of our enterprise – expectations

Nonhomogeneous model:

$$\hat{\pi}_{EXFIN} = 0.492$$

$$\hat{\mathbf{P}}_{EXFIN}^{(1)} = \begin{bmatrix} 0.514 & 0.336 & 0.149 \\ 0.129 & 0.777 & 0.094 \\ 0.097 & 0.366 & 0.537 \end{bmatrix}$$

$$\hat{\mathbf{P}}_{EXFIN}^{(2)} = \begin{bmatrix} 0.488 & 0.388 & 0.124 \\ 0.113 & 0.742 & 0.145 \\ 0.061 & 0.301 & 0.638 \end{bmatrix}$$

Homogeneous model:

$$\hat{\mathbf{P}}_{EXFIN} = \begin{bmatrix} 0.501 & 0.362 & 0.137 \\ 0.120 & 0.759 & 0.121 \\ 0.075 & 0.327 & 0.598 \end{bmatrix}$$

It is worth to note that conclusions resulting from estimation of both nonhomogeneous and homogeneous models are similar for all the variables taken into account. First of all, the matrices estimated have different rows which proves that using a Markov chain (both homo- and nonhomogeneous) is justified as respondents seem to condition their choice on the response indicated in the previous month. It is controversial however if the reasons for such behavior are of psychological nature or the differences in conditional probabilities reflect discrepancy between the situation of enterprises rather.

In all the cases the largest elements of each row lie on diagonal – this proves strong tendency of maintaining the same choice. Particularly large values are always connected with probability p_{22} of staying in state "no change".

Comparing the matrices obtained for regime 1 and 2 it may be noticed that estimators of the elements of upper triangle of matrix $\mathbf{P}^{(1)}$ - denoting the probabilities of moving from "better" states to "worse" ones - are smaller than in matrix $\mathbf{P}^{(2)}$ while lower triangle with probabilities of moving from "worse" to "better" has the opposite feature. Such observation lets identify regime 1 as the periods when the general economic situation is better (or expectations are more optimistic) than in case of regime 2.

Table 1 Periods of domination of regime 1

PROD	FIN	EXPROD	EXFIN
X 97	X 99 – I 98	X 97, I 98	X 97 – I 98
I 98, IV 98, VI 98	IV 98 – V 98	III 98 – VI 98	IV 98 – V 98
VIII 98 –X 98	VII 98 – X 98	VIII 98 – X 98	VII 98 – IX 98
IV 99, VI 99	IV 99	III 99 – IV 99	III 99 – V 99
IX 99 – X 99	IX 99 – XII 99	IV 99 – X 99	VII 99
III 00 – IV 00	VI 00 – X 00	II 00 – VI 00	I 00 – II 00
VI 00	IX 02	VIII 00 – X 00	V 00
VI 01, IX 01	V 03 – XII 03	II 01, IV 01, VI 01	VII 00 – IX 00
III 02 – III 02	II 04 – X 04	VIII 01 – IX 01	III 02
VIII 02 – X 02	XII 04	III 02 – IV 02	I 03, III 03
III 03 – X 03	IV 05	VIII 02 – X 02	V 03 – X 03
IV 04 – X 04	VI 05 – VIII 05	II 03 – X 03	I 04 – X 04
IV 05	X 05 – XII 05	I 04 – X 04	I 05, III 05, V 05, VII 05
VI 05 – X 05	II 06	II 05 – X 05	IX 05 – X 05
II 06 – V 06	IV 06 – V 06	II 06 – V 06	XII 05 – V 06

Figure 1 shows comparison of the probabilities of generating the observation by the first regime for variables PROD-FIN and Figures 2, 3 – for variables PROD-EXPROD and FIN-EXFIN respectively resulting from nonhomogeneous models. Regime switches are generally irregular and do not lead to a clear explanation. Perhaps the most regular switches might be observed in case of variable EXPROD and their nature is strongly seasonal. Table 1 shows the periods of domination of regime 1 identified by means of a classifying rule which attributes an observation to a specified regime if the evaluated probability of generating it by this regime exceeds 0.5.

Figure 1 Probabilities of regime 1 – variables PROD and FIN

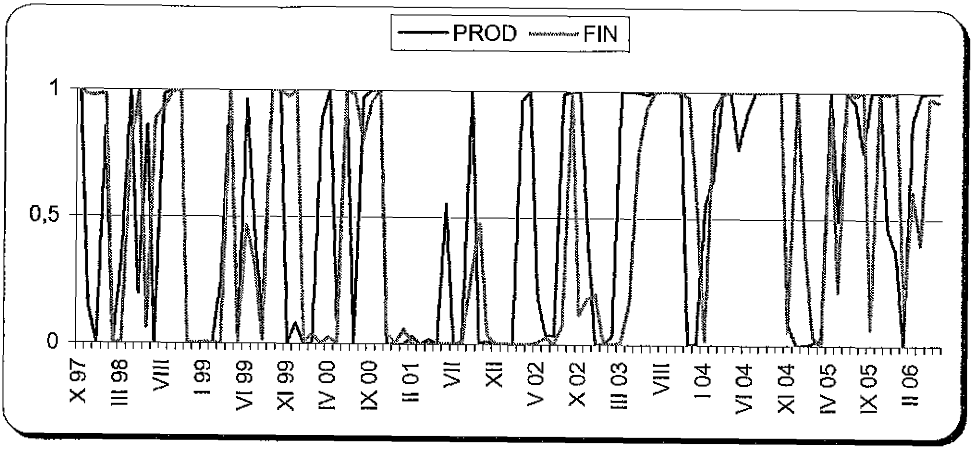


Figure 2 Probabilities of regime 1 – variables PROD and EXPROD

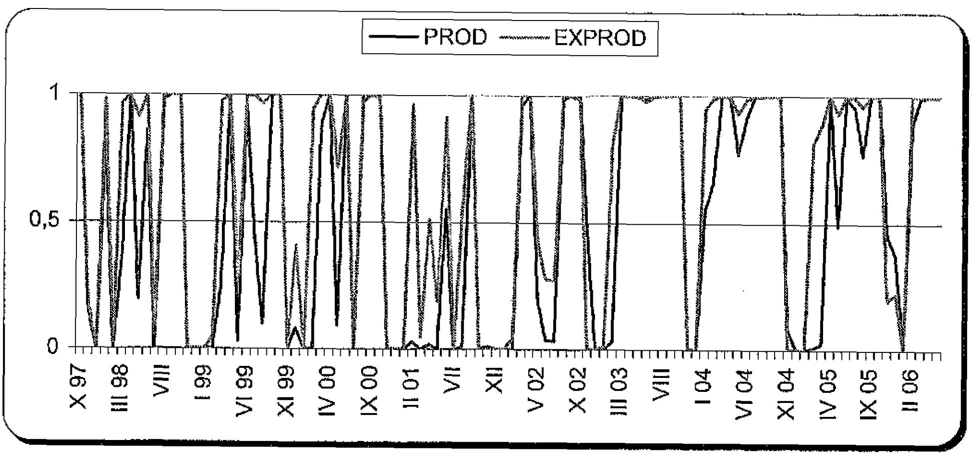
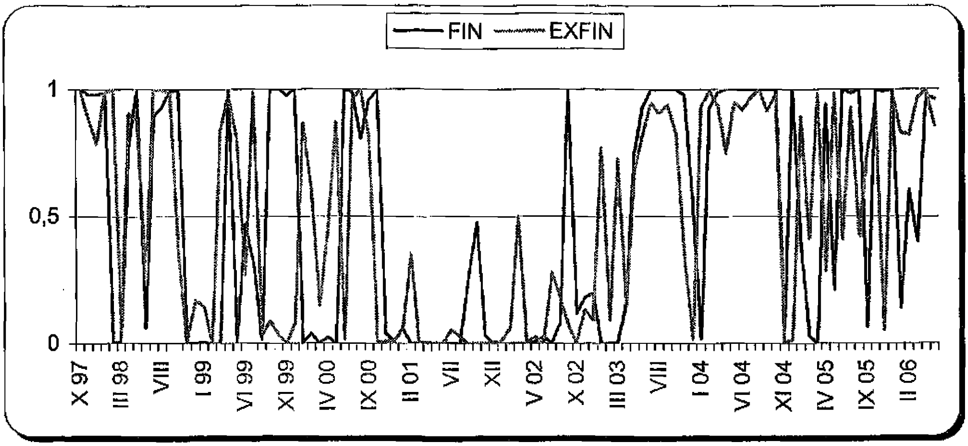


Figure 3 Probabilities of regime 1 – variables FIN and EXFIN



Finally, to show the advantage of nonhomogeneous model the RMSE has been calculated in the following manner. For each time period vector of errors

$$\mathbf{e}(t+1) = \mathbf{n}(t+1) - \mathbf{P}_t\mathbf{n}(t)$$

has been constructed, with $\mathbf{P}_t$ denoting a transition matrix valid for period t. In case of homogeneous model transition matrix is stable, for nonhomogeneous model the choice of transition matrix has been indicated by a classifying rule mentioned above. Then, the mean quadratic error for all the elements of vectors $\mathbf{e}(t)$ has been calculated according to the formula

$$RMSE = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T-1}\sum_{i=1}^{r}[e_i(t)]^2}\,.$$

Table 2 shows that in all the cases nonhomogeneous model performs significantly better than homogeneous one.

Table 2 RMSE for homo- and nonhomogeneous model

Variable	Homogeneous model	Nonhomogeneous model
PROD	21.86	16.86
EXPROD	20.11	18.53
FIN	19.20	15.92
EXFIN	17.69	16.01

4. Summary and conclusions

The nonhomogeneous switching Markov chain presented in the paper has proved useful in analyzing dynamics of responses of business survey data in Polish industry. Analysis of microdata referring to respondents' opinions and expectations on changes in level of production and financial situation showed that conditional probabilities of moving from state to state reflecting tendencies of changing choices among respondents differ depending on their previous choices. Allowing time-varying transition matrix has given more flexibility to the proposed approach and proved the advantage of nonhomogeneous model over a homogeneous one.

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