Uniwersytet
Ekonomiczny
w Katowicach

Studia Ekonomiczne. Zeszyty Naukowe
Uniwersytetu Ekonomicznego w Katowicach
ISSN 2083-8611 Nr 247 · 2015

Informatyka i Ekonometria 4

Agnieszka Orwat-Acedańska

University of Economics in Katowice
Faculty of Informatics and Communication
Department of Demography and Economic Statistics
agnieszka.orwat@ue.katowice.pl

THE ANALYSIS OF ROBUST PORTFOLIOS RISK IN THE STOCHASTIC PROGRAMMING METHOD

Summary: The paper discusses an application of stochastic programming to the portfolio selection problem involving estimation risk. The paper focuses on problems where a portfolio risk should not exceed some prespecified level with high probability. Based on the real data on daily returns from American sector stock indices it is analyzed whether the stochastic programming methods truly guarantee to reach the goal regarding portfolios risk. The results show that the discussed methods indeed lower probability of exceeding the risk level compared to the classical approach. However in most cases the excess fractions were still higher from the level expected by an investor.

Keywords: robust portfolios, stochastic programming, sampling, Monte Carlo method.

Introduction

In practice of the portfolio analysis, the classical assumptions behind the quantitative methods and models often turn out to be too strong. In many cases, that results from the nature of financial markets. Asset returns are characterized by fat tails, leptokurtosis and strong asymmetries. Therefore, the classical theories based on normality or independence assumptions, to name but a few, should no longer be applied. Similarly, discrepancies between theoretical and empirical distributions of returns may lead to severe errors in estimation of their characteristics. In such case, the term estimation risk is frequently used, which is associated with the loss possibility that stems from estimation errors. As a consequence, taking into account the estimation errors, portfolios based on the procedures involving classical estimation and Markowitz [1952] optimization are only sub-optimal. Non-classical estimation and optimization methods offer a variety of tools tailored to reduce the estimation risk or its consequences. Among them, sampling methods and stochastic programming play important roles, as they take into account the stochastic nature of the parameter estimates obtained from finite samples. Therefore, these methods can be treated as robust against the estimation risk.

In the paper, we apply the stochastic programming tools for the portfolio selection problem, where the portfolio risk should not exceed some predetermined level taking into account the estimation risk. The aim of the paper is to verify to what extent the stochastic programming allows for controlling the portfolios risk by running a pseudo-real-time experiment with the long time series of returns. In other words, we examine if the method is truly able to deliver the portfolios which risk does not exceed some predefined upper bound. For solving the stochastic programming problem, the sample approximation method is employed as in Orwat-Acedańska, Acedański [2013]. However, the current paper differs from the cited one, because new we use real data to test the robust portfolios characteristics. In particular, we utilize daily data on the sector indices from the US stock exchanges spanning the years 1964-2014.

The paper is structured as follows. The first chapter contains a description of the stochastic programming portfolio problem. Then, we present solution of the stochastic programming problem. In the third chapter the verification procedure is discussed. Finally, we present our pseudo-real-time investment experiment and show the results.

1. Stochastic programming portfolio problem

In the paper, we maximize the expected returns of the portfolios subject to the variance constraints taking into account the estimation risk.

The portfolio shares are defined as a solution to the classic Markowitz portfolio problem:

$$\max_{\mathbf{x}\in C}\{\mathbf{x}'\boldsymbol{\mu}\} \text{ s.t. } \sqrt{\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x}} \le v . \tag{1}$$

The shares are denoted with $\mathbf{x}^{(kl)}$ and simply called *classic portfolios*. In problem (1), v denotes the upper bound for the portfolio standard deviation, $C = \{\mathbf{x}:\mathbf{x} \ge 0, \mathbf{x1}= 1\}$ represents the set of admissible solutions, $\boldsymbol{\mu}$ stands for vector of the expected asset returns and $\boldsymbol{\Sigma}$ is their covariance matrix.

Stochastic programming counterpart of the Markowitz problem can be defined as follows:

$$\max_{\mathbf{x}\in C}\{E(\mathbf{x}'\tilde{\boldsymbol{\mu}})\} \text{ p.w. } P\left(\sqrt{\mathbf{x}'\tilde{\boldsymbol{\Sigma}}\mathbf{x}} \le v\right) \ge 1-\alpha\,, \tag{2}$$

where:

$\tilde{\boldsymbol{\mu}} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_k)'$ – random vector of the assets expected returns,

$\tilde{\boldsymbol{\Sigma}}$ – random covariance matrix of the assets returns,

α – probability that the portfolio's standard deviation exceeds the upper bound v,

E – expectations operator,

P – probability operator.

Problem (2) is the classic stochastic programming problem with probability constraints [Shapiro, Dentcheva, Ruszczyński, 2009; Luedtke, Ahmed, 2008; Pagoncelli, Ahmed, Shapiro, 2009; Yu, Ji, Wang, 2003].

2. Solving the stochastic programming problem

Generally, analytical solutions to problem (2) do not exist. The sample approximation is one of the possible solution methods for the discussed problem. In this approach, we replace the random matrices $\tilde{\boldsymbol{\mu}}$ and $\tilde{\boldsymbol{\Sigma}}$ with their empirical counterparts. Similarly, the probability α that the portfolio risk exceeds the prespecified level is replaced by the fraction of samples q, where the risk constraint is not satisfied. As a result, the solution to the stochastic optimization problem (2) is approximated by its empirical, deterministic counterpart:

$$\max_{\mathbf{x}\in C}\left\{\frac{1}{n}\sum_{j=1}^{n}\mathbf{x}'\boldsymbol{\mu}_j\right\} \text{ p.w. } \frac{1}{n}\sum_{j=1}^{n} I\left(\sqrt{\mathbf{x}'\boldsymbol{\Sigma}_j\mathbf{x}} \le v\right) \ge 1-q\,, \tag{3}$$

where $\boldsymbol{\mu}_j$ and Σ_j, $j = 1, 2, \ldots, n$ denote the characteristics of j-th subsample, and $I(A)$ is the indicator function that takes the value 1 if A is true and 0, otherwise. The subsamples are drawn either from some theoretical distribution (Monte Carlo simulation) or by bootstrap resampling technique. In the former case, the normal distribution is used with the parameters equal to the moments obtained from the data, whereas in the latter approach, we simply draw the returns from the whole sample.

As far as the problem of setting the number n of subsamples and the fraction q of violated constrains is concerned, we choose $q = 0$ and examine a few different values of n. Our conservative choice of q can be motivated by the fact, that we primarily focus on controlling the portfolio risk. Thus we opt for the safest parametrization. The more thorough discussion of the problem of setting n and q can be found in Orwat-Acedańska, Acedański [2013] or particularly in Campi, Garatti [2011].

The portfolios obtained as solutions to (3) are called stochastic portfolios. Their shares are denoted by $\mathbf{x}^{(st)}$, whereas $\boldsymbol{\mu}^{(st)}$ and $\boldsymbol{\Sigma}^{(st)}$ stand for expected returns and covariance matrix, respectively.

3. The verification procedure

The stochastic problem approximation together with the whole verification procedure consists of the following steps:

Step 0. $(T \times k)$ matrix of asset returns is considered.

Step 1. The whole dataset is divided into m rolling training samples of the equal length $d_u < T$. The first subsample covers the periods from 1 to d_u, the second one from $1 + \Delta$ to $d_u + \Delta$, and so on, where Δ represents the sample shift length.

Step 2. For the comparison purpose, for each i-th training sample with the characteristics $\boldsymbol{\mu}_j^{(kl)}$, $\boldsymbol{\Sigma}^{(kl)}{}_j$ and the upper bound for the portfolio standard deviation v, classic portfolio $\mathbf{x}_i^{(kl)}$ is constructed as a solution to problem (1).

Step 3

a) For each i-th training sample, n subsamples of equal length d_u is drawn, either from normal distribution or via bootstrap procedure.

b) For each j-th subsample of the training sample i, given the upper bound v for portfolio standard deviation, the stochastic portfolio $\mathbf{x}^{(st)}$ is constructed as a solution to problem (3).

Step 4. *Ex post* characteristics (mean returns and standard deviation of returns) of classic and stochastic portfolios are calculated on the verification periods of length d_{test}. The verification period for the first sample contains the observations from $d_u + 1$ to $d_u + d_{test}$. Consequently, for the second sample the verification period covers $d_u + \Delta + 1$ do $d_u + \Delta + d_{test}$ observations, and so on. The characteristics are calculated using the standard formulas:

$$\mathbf{x'}^{(p)} \boldsymbol{\mu}_i^{test} \; ; \; \sqrt{\mathbf{x'}^{(p)} \boldsymbol{\Sigma}_i^{test} \mathbf{x}^{(p)}} \,, \tag{4}$$

where $\mathbf{x'}^{(p)}$ represents the portfolios (classic or stochastic) and $\boldsymbol{\mu}_i^{test}$, $\boldsymbol{\Sigma}_i^{test}$ are the asset characteristics calculated on the verification period for i-th sample.

The above procedure is described assuming that the upper bounds for the portfolio standard deviation are given. However, we were shy on how these are set, so far. Setting reasonable values for v is not a trivial task. Because of time-variation in the market risk level, that is evident in our long time series, the bounds on risk cannot be fixed. Instead we examine a series of sample-dependent bounds. For each sample, the lowest bound on the portfolio risk is equal to the standard deviation of the minimum-risk portfolio (without any constraint on the portfolio return). On the other side, the highest bound corresponds to the standard deviation of the maximum-return portfolio. The intermediate bounds are equally distributed between the two extremes.

4. Results of the empirical analysis

Our database consists of seventeen sector indices from the US stock exchanges NYSE, AMEX and NASDAQ obtained from Kenneth French's website. The data covers the period 01.07.1964-31.12.2014. As a result, we have seventeen time series with T = 12 911 observations each. The long time span of the sample allows examining the portfolios' performances under the very different market conditions and assessing the discussed method from a real investor's point of view.

We consider four different stochastic portfolios: three based on the bootstrap resampling technique with n = 100 (*stoch_100* and *stoch_100w*) or n = 1000 (*stoch_1000*) subsamples, where in the portfolio *stoch_100w* we employ a weighted drawing scheme with newer observations having higher weights to account for the volatility clustering effect, and one based on Monte Carlo simulations with n = 100 samples (*stoch_100MC*).

The window span for the training samples is set to d_u = 240 periods, which approximately equals one year, whereas the verification samples contain d_{test} = 20, 60 or 120 observations. The sample jump is set to Δ = 20 observations. As a result, we end up with almost 600 training samples. Later, we also analyse the shorter training sample consisted of 120 observations. Finally, we consider 15 different bounds v on the portfolio standard deviations set as described in the previous section.

In the first step, we analyse the average fraction of samples where the portfolio standard deviation exceeded the bounds v. The results for the three different verification period lengths are depicted on Figures 1a-1c.

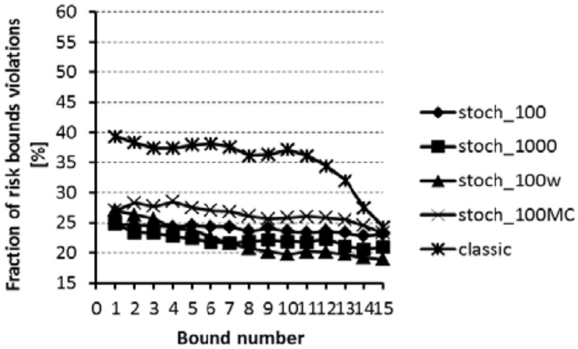

Fig. 1a. Fraction of risk bounds *v* violations for classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 20$

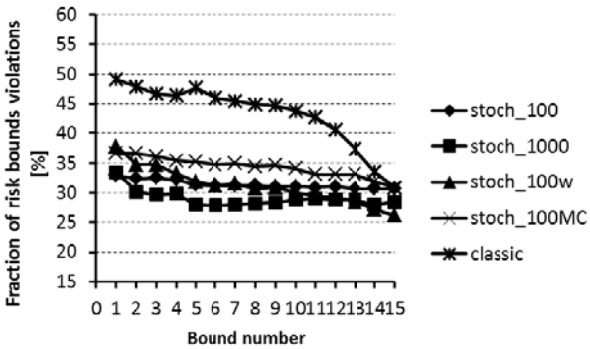

Fig. 1b. Fraction of risk bounds *v* violations for classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 60$

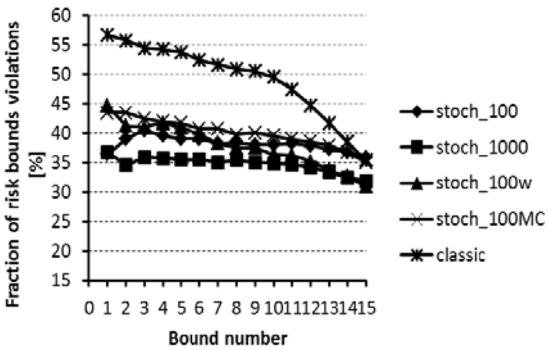

Fig. 1c. Fraction of risk bounds *v* violations for classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 120$

The results depicted on Figures 1a-1c clearly document that, regardless of the verification period length, the stochastic portfolios risk exceeds the bounds considerably less often compared to the classic portfolios for all bounds but the highest. Moreover, the fraction of periods when the risk bound is met is more stable throughout the different bounds v for the stochastic portfolios. However, the differences in performance between the stochastic portfolios are less pronounced. Nonetheless, one can notice that in most cases, the stochastic portfolios based on 1000 samples generate the portfolios with the lowest fraction of the risk bound violations. On the other hand, the Monte-Carlo-based portfolios exhibit the poorest performance compared to the other stochastic portfolios. Finally, it should be acknowledged that the fraction of the verification samples where the risk exceeds the bounds is rather high, regardless of the method, with 20% being the lowest rate attained for the shortest verification period.

In the second step, we repeat the previous exercise, but with shorter training period $d_u = 120$ observations, that approximately corresponds to half year. The results for the three different verification period lengths are illustrated on Figures 2a-2c.

This exercise basically confirms the findings from the previous one. The main difference is that for the shorter training periods the fraction of risk bound violations for the stochastic portfolios depends stronger on the value of the bound. For the higher risk bounds the fraction of the bound violations drops more than before. As a consequence, the fraction of the risk bound violations for the least demanding bounds is lower than for the training samples with 240 observations, although the difference is not particularly impressive. Interestingly, the stochastic portfolio with weighted bootstrap performs rather poorly for the low risk bounds, but outperforms the other portfolios as far as the loose bounds are concerned.

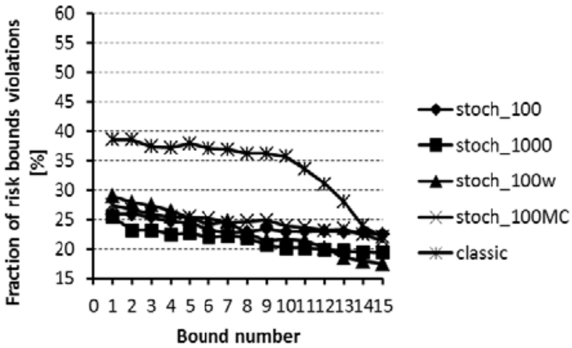

Fig. 2a. Fraction of risk bounds v violations for classic and stochastic portfolios with $d_u = 120$ and $d_{test} = 20$

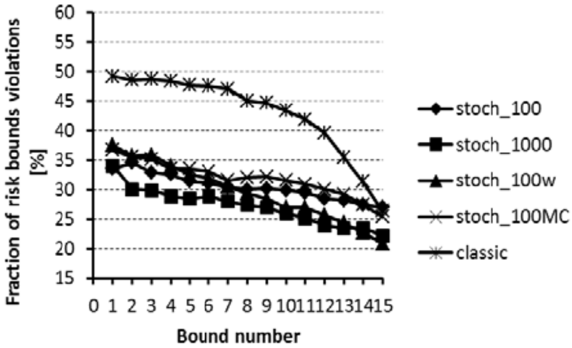

Fig. 2b. Fraction of risk bounds *v* violations for classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 60$

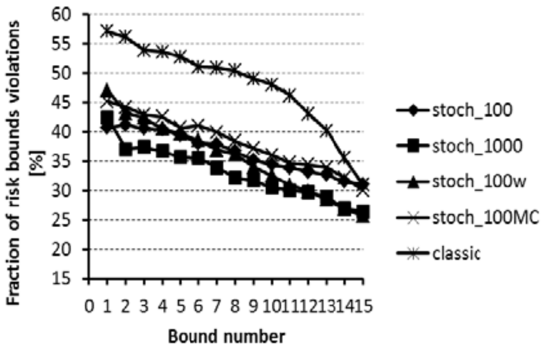

Fig. 2c. Fraction of risk bounds *v* violations for classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 120$

In the final step of our empirical investigation, we compare the average returns and standard deviation of the daily returns of the portfolios. On Figures 3a-3c, we present the results for the training samples with 240 observations.

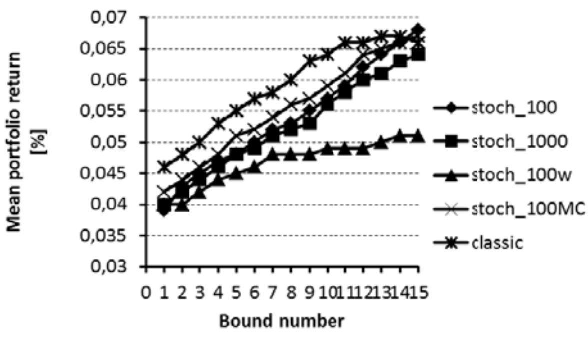

Fig. 3a. Mean daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 20$

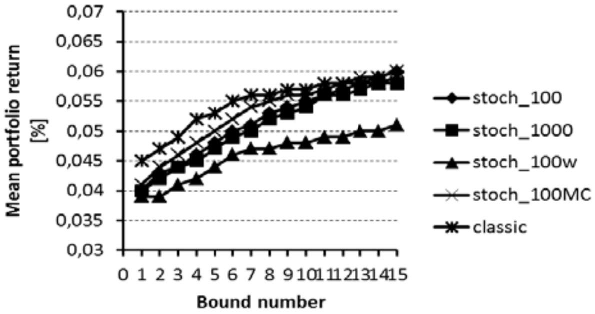

Fig. 3b. Mean daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 60$

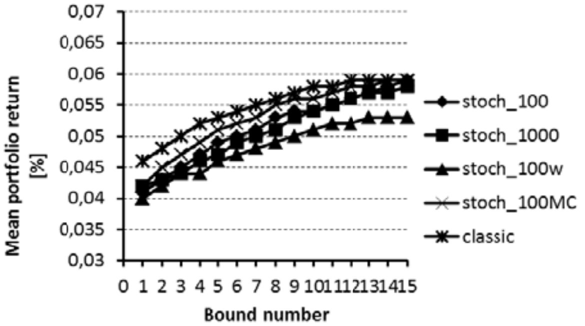

Fig. 3c. Mean daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 120$

As one can expect, the classic portfolios are characterized by the highest returns regardless of the risk bound and the verification period length. This, of course, is the compensation for the excess risk of these portfolios. As far as the stochastic portfolios are concerned, the weighted-bootstrap portfolios exhibit the lowest returns, particularly for the higher risk bounds. On the other hand, the portfolios constructed by the Monte-Carlo simulations are characterized be relatively high returns.

As far as the portfolios risk, depicted on Figures 4a-4c, is concerned, we observe the opposite results. For example, the classic portfolios are characterized by the highest standard deviations of the daily returns, whereas the lower risk is usually associated with the portfolios based on the bootstrap with 1000 subsamples or the weighted bootstrap.

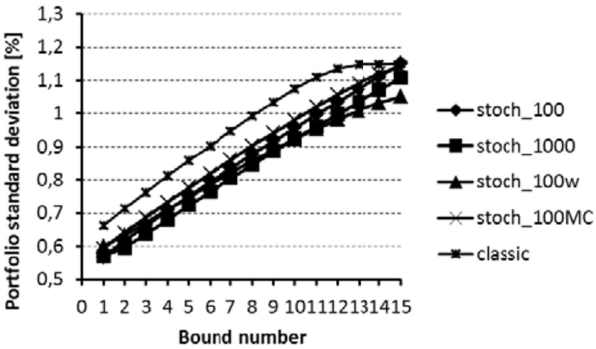

Fig. 4a. Average standard deviations of daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 20$

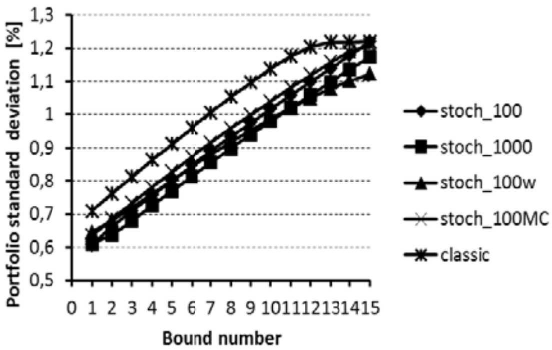

Fig. 4b. Average standard deviations of daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 60$

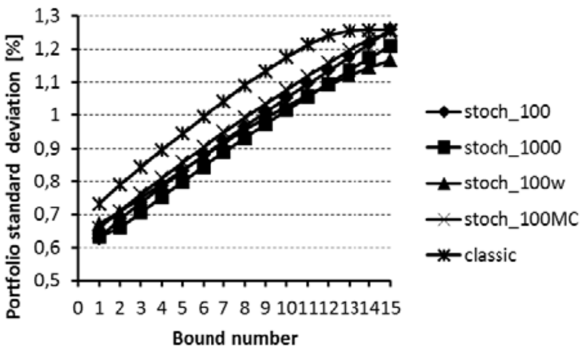

Fig. 4c. Average standard deviations of daily returns of classic and stochastic portfolios with $d_u = 240$ and $d_{test} = 120$

Summary

The problems concerning portfolios that account for the estimation risk and ensure that the portfolio risk does not exceed some predefined level are the central topic of the modern financial statistics, operational research as well as everyday practice of investors. They are crucial for the asset allocation decisions taken by both individual and, in particular, institutional investors, like pension or investment funds.

In the paper, we studied the application of the stochastic programming tools for the portfolio selection problem that accounts for the estimation risk. The problems were solved using the sample approximation method. We focused on problems of maximizing expected returns provided that the portfolio risk does not exceed the predefined level. Three particular sampling methods were investigated: ordinary bootstrap, weighted bootstrap and Monte Carlo. For each method, we considered different lengths of the training as well as verification rolling samples.

Our simulation experiments showed that the portfolio risk can exceed the predefined bounds quite often. The stochastic programming tools were able to mitigate the problem, but only partially. The fraction of samples where the risk constraint is violated was lower compared to the classic portfolios, but still higher than expected. The fraction dropped as the number of subsamples was increased, although the differences were not very pronounced. Of course in all cases, the stochastic portfolios were characterized by lower average returns as it is a normal price for the better control over the portfolio riskiness.

In some cases the weighted bootstrap sampling generated the portfolios with low level of risk. This probably results from the fact that the method can partially account for the time-variation of returns risk. Nonetheless, the approach cannot provide a completely satisfactory solution of the risk nonstationarity problem. Instead, employing the multivariate GARCH models can improve the performance of the stochastic programming methods, which is left for further investigation.

Although the stochastic programming methods are not able to completely mitigate the negative impact of the estimation risk in the portfolio selection process, we believe that they are useful tools that allow for better control over the portfolio riskiness. Therefore, in our opinion, they should especially suite the needs of investors characterized by high degree of risk aversion.

References

Campi M., Garatti S. (2011), *A Sampling-and-Discarding Approach to Chance-Constrained Optimization: Feasibility and Optimality*, "Journal of Optimization Theory and Applications", Vol. 148(2), p. 257-280.

Luedtke J., Ahmed S. (2008), *A Sample Approximation Approach for Optimization with Probabilistic Constraints*, "SIAM Journal of Optimization", Vol. 19, p. 674-699.

Markowitz H. (1952), *Portfolio Selection*, "Journal of Finance", Vol. 7.

Pagoncelli B., Ahmed S., Shapiro A. (2009), *The Sample Average Approximation Method for Chance Constrained Programming: Theory and Applications*, "Journal of Optimization Theory and Applications", Vol. 142, p. 399-416.

Shapiro A., Dentcheva D., Ruszczyński A. (2009), *Lectures on Stochastic Programming: Modelling and Theory*, SIAM, Philadelphia.

Yu L., Ji X., Wang S. (2003), *Stochastic Programming Models in Financial Optimization: A Survey*, "AMO – Advanced Modeling and Optimization", Vol. 5(1).

ANALIZA RYZYKA PORTFELI ODPORNYCH W METODZIE PROGRAMOWANIA STOCHASTYCZNEGO

Streszczenie: W artykule rozważano zastosowanie metod programowania stochastycznego w problemach wyboru portfela uwzględniających ryzyko estymacji. Koncentrowano się na zadaniach, które miały na celu zapewnienie, że ryzyko portfela z dużym prawdopodobieństwem nie przekroczy zadanego poziomu. Bazując na rzeczywistych danych dotyczących dziennych stóp zwrotu amerykańskich indeksów sektorowych, analizowano, czy rozważane metody programowania stochastycznego pozwalają osiągnąć zakładany cel odnośnie do ryzyka portfela. Wyniki wskazują, że w porównaniu do klasycznego podejścia analizowane metody pozwalają zmniejszyć prawdopodobieństwo przekroczenia zadanego poziomu ryzyka. Niemniej jednak w większości przypadków odsetek przekroczeń w dalszym ciągu był wyższy od zakładanego.

Słowa kluczowe: portfele odporne, programowanie stochastyczne, próbkowanie, metoda Monte Carlo.