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INCOMPLETE PREFERENCE MATRIX ON ALO-GROUP AND ITS APPLICATION TO RANKING OF ALTERNATIVES

Abstract

Pairwise comparison is a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns a value from the given scale which is an Abelian linearly ordered group (Alo-group) of the real line to any pair of alternatives representing an element of the preference matrix (P-matrix). Both non-fuzzy and fuzzy multiplicative and additive preference matrices are generalized. Then we focus on situations where some elements of the P-matrix are missing. We propose a general method for completing fuzzy matrix with missing elements, called the extension of the P-matrix, and investigate some important particular cases of fuzzy preference matrix with missing elements. Eight illustrative numerical examples are included.

Keywords: multi-criteria optimization, pairwise comparison, preference matrix, incomplete matrix, Alo-group.

1 Introduction

In various selection and prioritization processes the decision maker(s) (DM) try to find the best alternative(s) from a finite set of alternatives. DM problems and procedures have been established to combine opinions about alternatives related to different DM criteria. These procedures are often based on pairwise comparisons, in the sense that the processes are linked to some preference values from a given scale of one alternative over another. According to the nature

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of the information expressed by the DM, for every pair of alternatives different representation formats can be used to express preferences, e.g. multiplicative preference relations, Herrera-Viedma et al. (2001), fuzzy preference relations, see Chiclana et al. (2009), Herrera-Viedma et al. (2004), Ma et al. (2008), interval-valued preference relations, Xu (2008), and also linguistic preference relations, see Alonso et al. (2008).

In this paper we consider pairwise comparison matrices over an *Abelian linearly ordered group* (*Alo-group*) and, in this way, we provide a general framework for all the above mentioned cases. By introducing this more general setting, we provide a consistency measure that has a natural meaning: it corresponds to the consistency indices presented in the literature, see e.g. Ramik (2014); it is easy to calculate it in the additive, multiplicative and fuzzy cases. This setting is based on the papers of Cavallo et al. (2009), Cavallo et al. (2012), and Ramik (2014).

Usually, experts are characterized by their personal background and experience of the problem to be solved. Expert opinions may differ substantially: some of them would not be able to efficiently express a preference degree between two or more of the available options. This may be due to an expert's not possessing a precise or sufficient level of knowledge of part of the problem, or because these experts are unable to estimate the degree to which some options are better than others. In these situations an expert will provide an incomplete preference matrix, see Alonso et al. (2008), Kim et al. (1999), Xu (2008).

Usual procedures for DM problems correct this lack of knowledge of a particular expert using the information provided by the other experts together with aggregation procedures, see Saaty (2008). In the literature, see Xu et al. (2008), the problem is solved by using the least deviation method to obtain a priority vector of the corresponding preference relation. In this paper, we put forward a general procedure that attempts to estimate the missing information in any of the above formats of incomplete preference relations. Our proposal is different to the above mentioned procedures in Alonso et al. (2008), Kim et al. (1999), Xu (2008) because the estimation of missing values in an expert incomplete preference matrix is done using only the preference values provided by these particular experts. By doing this, we assume that the reconstruction of the incomplete preference matrix is compatible with the rest of the information provided by the experts.

The paper is organized as follows. Some basic information on Alo-groups is summarized in Section 2. In Section 3, preference matrices with elements from an Alo-grup are investigated, and reciprocity and consistency conditions are defined as well as the inconsistency index of the P-matrix. The priority vector for ranking the alternatives is also defined. In Section 4, a special notation for the matrix with missing elements is introduced and the concept of the extension of a P-matrix with missing elements is defined. This concept is based on a particular representation of the consistent matrix; the missing elements of the extended matrix are calculated by applying the generalized least squares method. In Section 5, two special cases of the P-matrix with missing elements are investigated. Here, for an $n \times n$ P-matrix the expert evaluates only $n-1$ pairs of alternatives. In this section, two numerical examples illustrating the necessary and sufficient conditions for elements to be evaluated in the P-matrix are presented. In Section 6, some concluding considerations and remarks are presented.

2 Abelian linearly ordered groups

In this section we summarize basic information on Abelian linearly ordered groups (Alo-groups). The content of this section is based mainly on Cavallo et al. (2012), and Bourbaki (1990).

An *Abelian group* is a set $\mathbf{G}$, together with an operation $\odot$ (read: operation *odot*) that combines any two elements $a, b \in \mathbf{G}$ to form another element denoted by $a \odot b$. The symbol $\odot$ is a placeholder for a concrete operation. The set and the operation $(\mathbf{G}, \odot)$, satisfy the following requirements known as the *Abelian group axioms*:

- If $a, b \in \mathbf{G}$, then $a \odot b \in \mathbf{G}$ (*closure*).
- If $a, b, c \in \mathbf{G}$, then $(a \odot b) \odot c = a \odot (b \odot c)$ (*associativity*).
- There exists an element $e \in \mathbf{G}$ called the *identity element*, such that for all $a \in \mathbf{G}$, $e \odot a = a \odot e = a$ (*identity element*).
- If $a \in \mathbf{G}$, then there exists an element $a^{(-1)} \in \mathbf{G}$ called the *inverse element to* a such that $a \odot a^{(-1)} = a^{(-1)} \odot a = e$ (*inverse element*).
- If $a, b \in \mathbf{G}$, then $a \odot b = b \odot a$ (*commutativity*).

The *inverse operation* $\div$ to $\odot$ is defined for all $a, b \in \mathbf{G}$ as follows:

$$a \div b = a \odot b^{(-1)}.$$

A nonempty set $\mathbf{G}$ is *linearly (totally) ordered* under the order relation $\leq$, if the following statements hold for all $a, b, c \in \mathbf{G}$:

- If $a \leq b$ and $b \leq a$, then $a = b$ (*antisymmetry*).
- If $a \leq b$ and $b \leq c$, then $a \leq c$ (*transitivity*).

- $a \leq b$ or $b \leq a$ (*totality*).

The *strict order* relation $<$ is defined for $a, b \in \mathbf{G}$: $a < b$ if $a \ \leq \ b$ and $a \ \neq \ b$.

Let $(\mathbf{G}, \odot)$ be an Abelian group, $\mathbf{G}$ be linearly ordered under $\leq$. $(\mathbf{G}, \odot, \leq)$ is said to be an *Abelian linearly ordered group*, *Alo-group* for short, if for all $c \in \mathbf{G}$: $a \leq b$ implies $a \odot c \leq b \odot c$.

If $\mathcal{G} = (\mathbf{G}, \odot, \leq)$ is an Alo-group, then $\mathbf{G}$ is naturally equipped with the order topology induced by $\leq$ and $\mathbf{G} \times \mathbf{G}$ is equipped with the related product topology. We say that $\mathcal{G}$ is a *continuous Alo-group* if $\odot$ is continuous on $\mathbf{G} \times \mathbf{G}$.

Because of the associative property, the operation $\odot$ can be extended by induction to n-ary operations, $n > 2$. Then, for a positive integer n, the (n)-*power* $a^{(n)}$ *of* $a \in \mathbf{G}$ is defined. We can extend the meaning of power $a^{(s)}$ to the case when s is a negative integer.

$\mathcal{G} = (\mathbf{G}, \odot, \leq)$ is *divisible* if for each positive integer n and each $a \in \mathbf{G}$ there exists the (n)-*th root of* a denoted by $a^{(1/n)}$, i.e. $\left(a^{(1/n)}\right)^{(n)} = a$. Moreover, the function $\|.\| : \mathbf{G} \to \mathbf{G}$ defined for each $a \in \mathbf{G}$ by:

$$\|a\| = \max\{a, a^{(-1)}\}$$

is called a $\mathcal{G}$-*norm.* The operation $d : \mathbf{G} \times \mathbf{G} \to \mathbf{G}$ defined by $d(a, b) = = \|a \div b\|$ for all $a, b \in \mathbf{G}$ is called a $\mathcal{G}$-*distance.* It is easy to show that d satisfies the usual distance properties.

Example 1 Additive Alo-group
$\mathcal{R} = (]-\infty, +\infty[, +, \leq)$ *is a continuous Alo-group with:* $e = 0, \ a^{(-1)} = -a$, $a^{(n)} = n.a$.

Example 2 Multiplicative Alo-group
$\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ *is a continuous Alo-group with:* $e = 1$, $a^{(-1)} = a^{-1} = 1/a, \ a^{(n)} = a^n$. *Here, the symbol* $\bullet$ *denotes the usual multiplication.*

Example 3 Fuzzy additive Alo-group
$\mathcal{R}_a$=$(]-\infty, +\infty[, +_f, \leq)$, *see Ramik et al. (2014), is a continuous Alo-group with:* $a +_f b = a + b - 0.5, \ e = 0.5, \ a^{(-1)} = 1 - a, \ a^{(n)} = n.a - \frac{n-1}{2}$.

Example 4 Fuzzy multiplicative Alo-group
$]0, 1[_m$=$(]0, 1[, \bullet_f, \leq)$, *is a continuous Alo-group with:* $a \bullet_f b = \frac{ab}{ab+(1-a)(1-b)}$, $e = 0.5, \ a^{(-1)} = 1 - a, \ a^{(n)} = \frac{a^n}{a^n+(1-a)^n}$.

3 P-matrix on Alo-groups over a real interval

Let $\mathbf{G}$ be an open interval of the real line $\mathbf{R}$ and $\leq$ be the total order on $\mathbf{G}$ inherited from the usual order on $\mathbf{R}$, $\mathcal{G} = (\mathbf{G}, \odot, \leq)$ be a real Alo-group. We also assume that $\mathcal{G}$ is a divisible and continuous Alo-group. Then $\mathbf{G}$ is an open interval, see Cavallo et al. (2012).

The DM problem can be formulated as follows. Let $X = \{x_1, x_2, ..., x_n\}$ be a finite set of alternatives. These alternatives have to be classified from best to worst, using the information given by a DM in the form of pairwise comparison matrix.

The preferences over the set of alternatives X, can be represented in the following way. Let us assume that the preferences on X are described by a preference relation on X given by an $n \times n$ matrix $A = \{a_{ij}\}$, where $a_{ij} \in \mathbf{G}$ for all $i, j = 1, 2, ..., n$ indicates a preference intensity for the alternative x_i over x_j, i.e. it is interpreted as "x_i is a_{ij} times better than x_j". The elements of $A = \{a_{ij}\}$ satisfy the following reciprocity condition, see Cavallo et al. (2012).

An $n \times n$ matrix $A = \{a_{ij}\}$ is $\odot$-*reciprocal*, if:

$$a_{ij} \odot a_{ji} = e \text{ for all } i, j = 1, 2, ..., n, \tag{1}$$

or, equivalently,

$$a_{ji} = a_{ij}^{(-1)} \text{ for all } i, j = 1, 2, ..., n. \tag{2}$$

An $n \times n$ matrix $A = \{a_{ij}\}$ is $\odot$-*consistent* Cavallo et al. (2012), if:

$$a_{ik} = a_{ij} \odot a_{jk} \text{ for all } i, j, k = 1, 2, ..., n. \tag{3}$$

Here, $a_{ii} = e$ for all $i = 1, 2, ..., n$, and also (3) implies (1), i.e. an $\odot$-consistent matrix is $\odot$-reciprocal (but not vice-versa).

The following result gives a characterization of a $\odot$-consistent matrix by the vectors of weights, see Cavallo et al. (2012).

Proposition 1 *A P-matrix $A = \{a_{ij}\}$ is $\odot$-consistent if and only if there exists a vector $w = (w_1, w_2, ..., w_n)$, $w_i \in \mathbf{G}$ such that:*

$$w_i \div w_j = a_{ij} \textit{ for all } i, j = 1, 2, ..., n. \tag{4}$$

If for some $i, j, k = 1, 2, ..., n$ (3) is not satisfied we say that the P-matrix $A = \{a_{ij}\}$ is *inconsistent*.

The inconsistency of A will be measured by the $\odot$-mean distance of the *ratio matrix* $W = \{w_i \div w_j\}$ to the matrix $A = \{a_{ij}\}$.

Let $A = \{a_{ij}\}, w = (w_1, ..., w_n), w_i \in \mathbf{G}$ for all $i = 1, 2, ..., n$, denote:

$$I_{\odot}(A, w) = \left(\bigodot_{1 \le i < j \le n} \|a_{ij} \div (w_i \div w_j)\| \right)^{(2/n(n-1))} . \quad (5)$$

Now, we define the concept of a priority vector. Consider the following optimization problem:

(P1) $\quad I_{\odot}(A, w) \longrightarrow min_e;$
subject to:

$\bigodot_{k=1}^{n} w_k = e,$
$w_i \in \mathbf{G}, i = 1, 2, ..., n.$

If an optimal solution of (P1) exists, then the *$\odot$-consistency index of A*, $I_{\odot}(A)$, is defined as:

$$I_{\odot}(A) = I_{\odot}(A, w^*), \quad (6)$$

where $w^* = (w_1^*, ..., w_n^*)$ is the optimal solution of (P1). Notice that "minimization" in (P1) is carried out with respect to the identity element e.

The optimal solution w^* of (P1) is called the *$\odot$-priority vector of A*. In (P1), $\bigodot_{k=1}^{n} w_k = e$, is a normalization condition reducing the number of the priority vectors (uniqueness), on condition that the optimal solution exists. The proof of the following theorem is evident and it is left to the reader.

Proposition 2 *A P-matrix $A = \{a_{ij}\}$ is $\odot$-consistent if and only if:*

$$I_{\odot}(A) = e.$$

4 P-matrix with missing elements

Usually, in many decision-making procedures, experts are capable of providing preference degrees for any pair of given alternatives. However, this may not be always true. A missing value can be the result of the inability of an expert to quantify the degree of preference of one alternative over another. In this case he/she may decide not to guess the preference degree between some pairs of alternatives. When an expert is not able to express a particular value a_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than alternative x_j , this does not mean that he/she prefers both options with the same intensity. In order to model these situations, in the following we introduce the incomplete preference matrix. Here, we use a different approach and notation as compared to e.g. Alonso et al. (2008); on the other hand, our approach is similar to that of Ramik (2014).

We are going to define the P-matrix with missing elements. For the sake of simplicity of presentation we identify the alternatives $x_1, x_2, ..., x_n$ with integers $1, 2, ..., n$, i.e. by $X = \{1, 2, ..., n\}$ we denote the set of alternatives, $n > 1$. Moreover, let $X \times X = X^2$ be the Cartesian product of X, i.e. $X^2 = \{(i,j)|i,j \in X\}$. Let $K \subset X^2$, $K \neq X^2$ and let $\mathcal{A}$ be the preference relation on K given by the (membership) function $\mu_{\mathcal{A}} : K \to \mathbf{G}$, $\mathbf{G}$ is an Alo-group. The preference relation $\mathcal{A}$ is represented by the $n \times n$ *preference matrix* $A(K) = \{a_{ij}\}_K$ *with missing elements depending on* K as follows:

$$a_{ij} = \begin{cases} \mu_{\mathcal{A}}(i,j) & \text{if} \quad (i,j) \in K, \\ \times & \text{if} \quad (i,j) \notin K. \end{cases}$$

In what follows we shall assume that each P-matrix $A(K) = \{a_{ij}\}_K$ with missing elements is $\odot$-reciprocal, i.e.:

$$a_{ij} \odot a_{ji} = e \text{ for all } (i,j) \in K.$$

If $L \subset K$, and $L = \{(i_1, j_1), (i_2, j_2), ..., (i_q, j_q)\}$ is a set of pairs (i,j) of alternatives such that there exist a_{ij}, with $a_{ij} \in \mathbf{G}$ for all $(i,j) \in L$, then the subset L' symmetric to L, i.e. $L' = \{(j_1, i_1), (j_2, i_2), ..., (j_q, i_q)\}$ is also a subset of K, i.e. $L' \subset K$. By reciprocity, each subset K of X^2 can be represented as follows: $K = L \cup L' \cup D$, where L is the set of pairs of alternatives (i,j) of given preference degrees a_{ij} of the P-matrix $A(K)$ and D is the diagonal of this matrix, i.e. $D = \{(1,1), (2,2), ..., (n,n)\}$, where $a_{ii} = e$ for all $i \in X$. The reciprocity property means that the expert is able to quantify both a_{ij} and a_{ji} as well as a_{ii}. The elements a_{ij} with $(i,j) \in X^2 - K$ are called *the missing elements of the matrix* $A(K)$. Note that the missing elements of $A(K)$ are denoted by the symbol $\times$ ("ex"). On the other hand, those elements which express the preference degrees given by the experts are denoted by a_{ij}, where $(i,j) \in K$. By $\odot$-reciprocity it is sufficient that the expert quantifies only those elements a_{ij}, where $(i,j) \in L$, such that $K = L \cup L' \cup D$. In what follows we shall investigate two important particular cases: $L = \{(1,2), (2,3), ..., (n-1, n)\}$, and $L = \{(1,2), (1,3), ..., (1,n)\}$.

Now we shall deal with the problem of finding the values of missing elements of a given P-matrix so that the extended matrix is as much $\odot$-consistent as possible. In the ideal case the extended matrix will become $\odot$-consistent.

Let $K \subset X^2$, let $A(K) = \{a_{ij}\}_K$ be a P-matrix with missing elements. The matrix $A^e(K) = \{a^e_{ij}\}_K$, called the $\odot$-*extension of* $A(K)$, is defined as follows:

$$a^e_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i,j) \in K, \\ v^*_i \div v^*_j & \text{if} \quad (i,j) \notin K. \end{cases}$$

Here, $v^* = (v_1^*, v_2^*, ..., v_n^*)$ is called the *$\odot$-priority vector with respect to* K, if it is an optimal solution of the following problem:

(P2) $\quad d(v, K) \longrightarrow min_e$;

subject to:

$$\bigodot_{j=1}^{n} v_j = e,$$

$v_i \in \mathbf{G}$ for all i=1,2,...,n.

Here, $d(v, K) = (\bigodot_{i,j \in K} \|a_{ij} \div (v_i \div v_j)\|)^{(1/|K|)}$, $|K|$ denotes the cardinality of K. Note, that the $\odot$-consistency index of the matrix $A^e(K) = \{a_{ij}^e\}_K$ is defined by (6) as $I_{\odot}(A^e(K))$. Minimization in (P2) is carried out with respect to the identity element e.

The proof of the following proposition follows directly from Proposition 2.

Proposition 3 $A^e(K) = \{a_{ij}^e\}_K$ *is $\odot$-consistent, (i.e. $I_{\odot}(A^e(K)) = e$) if and only if:*

$$d(v^*, K) = e.$$

5 Special cases of preference matrices with missing elements

For a complete $n \times n$ reciprocal preference matrix we need $N = \frac{n(n-1)}{2}$ pairs of elements to be evaluated by an expert. For example, if $n = 12$, then $N = 66$, which is a considerable number of pairwise comparisons. We ask that the expert evaluates only "around n"pairwise comparisons of alternatives which seems to be a reasonable number. In this section we shall investigate two important particular cases of a fuzzy preference matrix with missing elements where the expert will evaluate only $n-1$ pairwise comparisons of alternatives. Here we generalize the approach presented in Ramik (2014). Let $K \subset X^2$ be a set of indices given by an expert, $A(K) = \{a_{ij}\}_K$ be a P-matrix with missing elements. Moreover, let $K = L \cup L' \cup D$. In fact, it is sufficient to assume that the expert will evaluate only a chain of matrix elements of L, i.e. $a_{12}, a_{23}, a_{34}, ..., a_{n-1,n}$.

5.1 Case $L = \{(1, 2), (2, 3), ..., (n - 1, n)\}$

Here, we assume that the expert evaluates $n-1$ chain elements of the P-matrix $A(K)$, i.e. $a_{12}, a_{23}, a_{34}, ..., a_{n-1,n}$. First, we investigate the $\odot$-extension of

$A(K)$. We derive the following result.

Proposition 4 *Let $L = \{(1,2),(2,3),...,(n-1,n)\}$, $a_{ij} \in \mathbf{G}$ with $a_{ij} \odot a_{ji} = e$ for all $(i,j) \in K$, $K = L \cup L' \cup D$, and $L' = \{(2,1),(3,2),...,(n,n-1)\}$, $D = \{(1,1),...,(n,n)\}$. Then the $\odot$-priority vector $v^* = (v_1^*, v_2^*, ..., v_n^*)$ with respect to K is given as:*

$$v_1^* = \left(\bigodot_{i=2}^{n} (a_{12} \odot ... \odot a_{i-1,i}) \right)^{(1/n)}, \tag{7}$$

$$v_i^* = a_{i-1,i}^{(-1)} \odot v_{i-1}^* \text{ for } i = 2,3,...,n. \tag{8}$$

Proof
If (7) and (8) are satisfied, then:

$$v_i^* = a_{i-1,i} \odot a_{i-2,i-1} \odot ... \odot a_{1,2} \odot v_1^* \text{ for } i = 2,...,n,$$

hence for all $i = 1,2,...,n, v_i^* \in \mathbf{G}$ and:

$$\bigodot_{i=1}^{n} v_i^* = e.$$

Also,

$$a_{i-1,i} = v_{i-1}^* \div v_i^* \text{ for } i = 2,...,n.$$

Then $v = (v_1^*, ..., v_1^*)$ is an optimal solution of (P2).

As a simple consequence, we obtain the following corollary.

Corollary 5 *Let $\mathcal{R} = (]-\infty,+\infty[,+,\leq)$ be an additive Alo-group, see Example 1, i.e. $\odot = +$. Then we obtain (7), (8) in the following form:*

$$v_1^* = \frac{1}{n} \sum_{i=2}^{n} (n-i+1) a_{i-1,i}, \tag{9}$$

$$v_i^* = v_{i-1}^* - a_{i-1,i} \text{ for } i = 2,3,...,n. \tag{10}$$

Example 5 *Let $\odot = +$, $L = \{(1,2),(2,3),(3,4)\}$, see Example 1. Let the chain evaluations be $a_{12} = 9, a_{23} = 8, a_{34} = 5$, with $a_{ij} + a_{ji} = 0$ for all $(i,j) \in L$, $K = L \cup L' \cup D$. Hence $A(K) = \{a_{ij}\}_K$ is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0 & 9 & \times & \times \\ -9 & 0 & 8 & \times \\ \times & -8 & 0 & 5 \\ \times & \times & -5 & 0 \end{pmatrix}.$$

By (9), (10) we obtain $+$*-priority vector* v^* *with respect to* K*, particularly,* $v^* = (12, 3, -5, -10)$*. By (4) we obtain the following* $+$*-extension of* $A(K)$*:*

$$A^e(K) = \begin{pmatrix} 0 & 9 & 17 & 22 \\ -9 & 0 & 8 & 13 \\ -17 & -8 & 0 & 5 \\ -22 & -13 & -5 & 0 \end{pmatrix},$$

where $A^e(K)$ *is* $+$*-consistent, and* $d(v, B(K)) = 0$*, hence* $I_+(A^e(K)) = 0$*. The corresponding ranking of the alternatives is* $x_1 > x_2 > x_3 > x_4$*.*

Also, as a simple consequence, we obtain the following corollary.

Corollary 6 *Let* $\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ *be a multiplicative Alo-group, see Example 2, i.e.* $\odot = \bullet$*. Then we obtain (7), (8) in the following form:*

$$P_1 = 1, \;\; P_i = P_{i-1}a_{i-1,i}, \;\; \text{for } i = 2, 3, ..., n, \tag{11}$$

$$v_1^* = \left(\prod_{i=1}^{n} P_i \right)^{\frac{1}{n}}, \tag{12}$$

$$v_i^* = \frac{v_{i-1}^*}{a_{i-1,i}} \;\; \text{for } i = 2, 3, ..., n. \tag{13}$$

Example 6 *Let* $\odot = \bullet$*,* $L = \{(1, 2), (2, 3), (3, 4)\}$*, see Example 2. Let the chain evaluations be* $a_{12} = 4, a_{23} = 3, a_{34} = 2$*, with* $a_{ij} \bullet a_{ji} = 1$ *for all* $(i, j) \in L$*,* $K = L \cup L' \cup D$*. Hence* $A(K) = \{a_{ij}\}_K$ *is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 1 & 4 & \times & \times \\ \frac{1}{4} & 1 & 3 & \times \\ \times & \frac{1}{3} & 1 & 2 \\ \times & \times & \frac{1}{2} & 1 \end{pmatrix}.$$

By (11), (12), (13) we obtain the $\bullet$*-priority vector* v^* *with respect to* K*, in case,* $v^* = (5.826, 1.456, 0.485, 0.243)$*. By (4) we obtain the following* $\bullet$*-extension of* $A(K)$*:*

$$A^e(K) = \begin{pmatrix} 1 & 4 & 12 & 24 \\ \frac{1}{4} & 1 & 3 & 6 \\ \frac{1}{12} & \frac{1}{3} & 1 & 2 \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{2} & 1 \end{pmatrix},$$

where $A^e(K)$ *is* $\bullet$*-consistent, and* $d(v, B(K)) = 1$*, hence* $I_\bullet(A^e(K)) = 1$*. The corresponding ranking of the alternatives is* $x_1 > x_2 > x_3 > x_4$*.*

Corollary 7 *Let $\mathcal{R}_a = (]-\infty, +\infty[, +_f, \leq)$ be a fuzzy additive Alo-group, see Example 3, i.e. $\odot = +_f$. Then we obtain (7), (8) in the following form:*

$$S_1 = 0,\; S_i = S_{i-1} + a_{i-1,i},\; \textit{for } i = 2, 3, ..., n, \tag{14}$$

$$v_1^* = \frac{3-n}{4} + \frac{1}{n}\sum_{i=1}^{n} S_i, \tag{15}$$

$$v_i^* = v_{i-1}^* - a_{i-1,i} + 0.5 \textit{ for } i = 2, 3, ..., n. \tag{16}$$

Example 7 *Let $\odot = +_f$, $L = \{(1,2),(2,3),(3,4)\}$, see Example 3. Let the chain evaluations be $a_{12} = 0.9, a_{23} = 0.5, a_{34} = 0.3$, with $a_{ij} +_f a_{ji} = 0.5$ for all $(i,j) \in L$, $K = L \cup L' \cup D$. Hence $A(K) = \{a_{ij}\}_K$ is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0.5 & 0.9 & \times & \times \\ 0.1 & 0.5 & 0.5 & \times \\ \times & 0.5 & 0.5 & 0.3 \\ \times & \times & 0.7 & 0.5 \end{pmatrix}.$$

By (14), (15), (16) we obtain the $+_f$-priority vector v^* with respect to K, in case, $v^* = (0.75, 0.35, 0.35, 0.55)$. By (4) we obtain the following $+_f$-extension of $A(K)$:

$$A^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.9 & 0.7 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.3 & 0.7 & 0.7 & 0.5 \end{pmatrix},$$

where $A^e(K)$ is $+_f$-consistent, and $d(v, B(K)) = 0.5$, hence $I_{+_f}(A^e(K)) = = 0.5$. The corresponding ranking of the alternatives is $x_1 > x_4 > x_2 \sim x_3$.

We obtain also the following corollary.

Corollary 8 *Let $]0,1[_m = (]0,1[, \bullet_f, \leq)$ be a fuzzy multiplicative Alo-group, see Example 4, i.e. $\odot = \bullet_f$. Then for $i = 2, 3, ..., n$ we obtain (7), (8) in the following form:*

$$P_i = \frac{(1-a_{12}) \cdot ... \cdot (1-a_{i-1,i})}{(1-a_{12}) \cdot ... \cdot (1-a_{i-1,i}) + a_{12} \cdot ... \cdot a_{i-1,i}}, \tag{17}$$

$$P = \frac{P_1 \cdot ... \cdot P_n}{(1-P_1) \cdot ... \cdot (1-P_n) + P_1 \cdot ... \cdot P_n}, \tag{18}$$

$$v_1^* = \frac{(1-P)^{1/n}}{(1-P)^{1/n} + P^{1/n}}, \tag{19}$$

$$v_i^* = \frac{(1-a_{i-1,i})v_{i-1}^*}{(1-a_{i-1,i})v_{i-1}^* + a_{i-1,i}(1-v_{i-1}^*)}. \tag{20}$$

Formulas (17), (18), (19) and (20) can be easily calculated e.g. using Excel.

Example 8 *Let* $\odot = \bullet_f$, $L = \{(1,2),(2,3),(3,4)\}$, *see Example 4. Let the chain evaluations be* $a_{12} = 0.9, a_{23} = 0.5, a_{34} = 0.3$, *with* $a_{ij} \bullet_f a_{ji} = 0.5$ *for all* $(i,j) \in L$, $K = L \cup L' \cup D$. *Hence* $A(K) = \{a_{ij}\}_K$ *is the following P-matrix with missing elements:*

$$A(K) = \begin{pmatrix} 0.5 & 0.9 & \times & \times \\ 0.1 & 0.5 & 0.5 & \times \\ \times & 0.5 & 0.5 & 0.3 \\ \times & \times & 0.7 & 0.5 \end{pmatrix}.$$

By (18), (19) we obtain the $\bullet$*-priority vector* v^* *with respect to* K, *in this case,* $v^* = (0.808, 0.318, 0.318, 0.522)$. *By (4) we obtain the following* $\bullet_f$*-extension of* $A(K)$*:*

$$A^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.9 & 0.794 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.1 & 0.5 & 0.5 & 0.3 \\ 0.206 & 0.7 & 0.7 & 0.5 \end{pmatrix},$$

where $A^e(K)$ *is* $\bullet$*-consistent, and* $d(v, B(K)) = 0.5$, *hence* $I_{\bullet_f}(A^e(K)) = 0.5$. *The corresponding ranking of the alternatives is* $x_1 > x_4 > x_2 \sim x_3$.

5.2 Case $L = \{(1,2),(1,3),...,(1,n)\}$

Now we assume that the expert evaluates the pairs consisting of a given fixed element and the remaining $n-1$ elements, i.e. the P-matrix $A(K)$ is given by $a_{12}, a_{13}, ..., a_{1n}$. We investigate the extension of $A(K)$ and obtain the following result.

Proposition 9 *Let* $L = \{(1,2),(1,3),...,(1,n)\}$, $a_{ij} \in \mathbf{G}$ *with* $a_{ij} \odot a_{ji} = e$ *for all* $(i,j) \in K$, $K = L \cup L' \cup D$, *and* $L' = \{(2,1),(3,1),...,(n,1)\}$, $D = \{(1,1),...,(n,n)\}$. *Then the* $\odot$*-priority vector* $v^* = (v_1^*, v_2^*, ..., v_n^*)$ *with respect to* K *is given as:*

$$v_1^* = \left(\bigodot_{i=2}^{n} a_{1i}\right)^{(1/n)}, \tag{21}$$

$$v_i^* = a_{1,i}^{(-1)} \odot v_1^* \text{ for } i = 2, 3, ..., n. \tag{22}$$

Proof
If (21) and (22) are satisfied, then:

$$v_i^* = a_{1,i-1} \odot a_{1,i-2} \odot ... \odot a_{1,2} \odot v_1^* \text{ for } i = 2, ..., n,$$

hence for all $i = 1, 2, ..., n, v_i^* \in \mathbf{G}$, moreover,

$$\bigodot_{i=1}^{n} v_i^* = e,$$

and also:

$$a_{1,i-1} = v_1^* \div v_i^* \text{ for } i = 2, ..., n.$$

Then $v = (v_1^*, ..., v_1^*)$ is an optimal solution of (P2).

As a simple consequence, we obtain the following corollary.

Corollary 10 *Let $\mathcal{R} = (]-\infty, +\infty[, +, \leq)$ be an additive Alo-group, see Example 1, i.e. $\odot = +$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \frac{1}{n} \sum_{i=2}^{n} a_{1,i}, \tag{23}$$

$$v_i^* = v_1^* - a_{1,i} \text{ for } i = 2, 3, ..., n. \tag{24}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\odot$-consistent.

Example 9 *$\odot = +$, $L = \{(1,2), (1,3), (1,4)\}$, let the expert evaluations be $b_{12} = 9, b_{13} = 8, b_{14} = 5$, with $b_{ij} + b_{ji} = 0$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 0 & 9 & 8 & 5 \\ -9 & 0 & \times & \times \\ -8 & \times & 0 & \times \\ -5 & \times & \times & 0 \end{pmatrix}.$$

By (23), (24) we obtain the $+$-priority vector w^* with respect to K, in this case, $w^* = (5.5, -3.5, -2.5, 0.5)$. By (4) we obtain the following $+$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 0 & 9 & 8 & 5 \\ -9 & 0 & -1 & -4 \\ -8 & 1 & 0 & -3 \\ -5 & 4 & 3 & 0 \end{pmatrix},$$

where $B^e(K)$ is $+$-consistent, and $d(v, B(K)) = 0$, hence $I_+(B^e(K)) = 0$. The corresponding ranking of the alternatives is $x_1 > x_4 > x_3 > x_2$.

Corollary 11 *Let $\mathcal{R}^+ = (]0, +\infty[, \bullet, \leq)$ be a multiplicative Alo-group, see Example 2, i.e. $\odot = \bullet$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \left(\prod_{i=2}^{n} a_{1,i}\right)^{1/n}, \tag{25}$$

$$v_i^* = \frac{v_1^*}{a_{1,i}} \text{ for } i = 2, 3, ..., n. \tag{26}$$

Moreover, the extension of $A(K)$, i.e. the matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\bullet$-consistent.

Example 10 *$\odot = \bullet$, $L = \{(1,2), (1,3), (1,4)\}$, see Example 2. Let the expert evaluations be $b_{12} = 4, b_{13} = 3, b_{14} = 2$, with $b_{ij} \bullet b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ \frac{1}{4} & 1 & \times & \times \\ \frac{1}{3} & \times & 1 & \times \\ \frac{1}{2} & \times & \times & 1 \end{pmatrix}.$$

By (25), (26) we obtain the $\bullet$-priority vector w^* with respect to K, in this case, $w^* = (2.213, 0.553, 0.738, 1.107)$. By (4) we obtain the following $\bullet$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 1 & 4 & 3 & 2 \\ \frac{1}{4} & 1 & \frac{3}{4} & \frac{1}{2} \\ \frac{1}{3} & \frac{4}{3} & 1 & \frac{2}{3} \\ \frac{1}{2} & 2 & \frac{3}{2} & 1 \end{pmatrix},$$

where $B^e(K)$ is $\bullet$-consistent, and $d(v, B(K)) = 1$, hence $I_\bullet(B^e(K)) = 1$. The corresponding ranking of the alternatives is $x_1 > x_2 \sim x_3 > x_4$.

Corollary 12 *Let $\mathcal{R}_a = (]-\infty, +\infty[, +_f, \leq)$ be a fuzzy additive Alo-group, see Example 3, i.e. $\odot = +_f$. Then we obtain (21), (22) in the following form:*

$$v_1^* = \frac{1}{2n} + \frac{1}{n}\sum_{i=2}^{n} a_{1,i}, \tag{27}$$

$$v_i^* = v_1^* - a_{1,i} + 0.5. \text{ for } i = 2, 3, ..., n. \tag{28}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $+_f$-consistent.

Example 11 *$\odot = +_f$, $L = \{(1,2),(1,3),(1,4)\}$, let the expert evaluations be $b_{12} = 0.9, b_{13} = 0.5, b_{14} = 0.3$, with $b_{ij} +_f b_{ji} = 0.5$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements:*

$$B(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & \times & \times \\ 0.4 & \times & 0.5 & \times \\ 0.6 & \times & \times & 0.5 \end{pmatrix}.$$

By (27), (28) we obtain the $+_f$-priority vector w^* with respect to K, in this case, $w^* = (0.6, 0, 2, 0.5, 0.7)$. By (4) we obtain the following $+_f$-extension of $B(K)$:

$$B^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & 0.2 & 0.0 \\ 0.4 & 0.8 & 0.5 & 0.3 \\ 0.6 & 1.0 & 0.7 & 0.5 \end{pmatrix},$$

where $B^e(K)$ is $+_f$-consistent, and $d(v, B(K)) = 0.5$, hence $I_{+_f}(B^e(K)) = = 0.5$. The corresponding ranking of the alternatives is $x_4 > x_1 > x_3 > x_2$.

Corollary 13 *Let $]0,1[_m = (]0,1[, \bullet_f, \leq)$ be a fuzzy multiplicative Alo-group, see Example 3, i.e. $\odot = \bullet_f$. Then for $i = 2, 3, ..., n$ we obtain (21), (22) in the following form:*

$$P_i = \frac{a_{1,i}^{1/n}}{a_{1,i}^{1/n} + (1 - a_{1,i})^{1/n}}, \tag{29}$$

$$v_1^* = \frac{P_1 \cdot ... \cdot P_n}{P_1 \cdot ... \cdot P_n + (1 - P_1) \cdot ... \cdot (1 - P_n)}, \tag{30}$$

$$v_i^* = \frac{(1 - a_{1,i})v_1^*}{(1 - a_{1,i})v_1^* + a_{1,i}(1 - v_1^*)}. \tag{31}$$

Moreover, the extension of $A(K)$, i.e. matrix $A^e(K) = \{a_{ij}^{ac}\}_K$ is $\bullet_f$-consistent.

Example 12 *Let $\odot = \bullet_f$, $L = \{(1,2),(1,3),(1,4)\}$, $b_{12} = 0.9$, $b_{13} = 0.6, b_{14} = 0.4$, with $b_{ij} \bullet_f b_{ji} = 0.5$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Let $B(K) = \{b_{ij}\}_K$ be the following P-matrix with missing elements (see Example 4 and 10):*

$$B(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & \times & \times \\ 0.4 & \times & 0.5 & \times \\ 0.6 & \times & \times & 0.5 \end{pmatrix}.$$

By (29), (30), (31) we obtain the $\bullet_f$*-priority vector* w^* *with respect to* K*, in this case,* $w^* = (0.634, 0.161, 0.536, 0.722)$*. By (4) we obtain the following* $+$*-extension of* $B(K)$*:*

$$B^e(K) = \begin{pmatrix} 0.5 & 0.9 & 0.6 & 0.4 \\ 0.1 & 0.5 & 0.143 & 0.069 \\ 0.4 & 0.857 & 0.5 & 0.308 \\ 0.6 & 0.931 & 0.692 & 0.5 \end{pmatrix},$$

where $B^e(K)$ *is* $\bullet_f$*-consistent, and* $d(v, B(K)) = 0.5$*, hence* $I_{\bullet_f}(B^e(K)) = = 0.5$*. The corresponding ranking of the alternatives is* $x_4 > x_1 > x_3 > x_2$*.*

6 Conclusions

In this paper we have dealt with some properties of P-matrices, namely reciprocity and consistency, with the entries from an Alo-group. We have shown how to measure the degree of consistency and also how to evaluate pairs of elements using values taken from an Alo-group if some elements are missing. Moreover, we have dealt with two particular cases of the incomplete P-matrix, and we have proposed some special methods for dealing with such cases. Finally, eight numerical examples have been presented to illustrate our approach.

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