Uniwersytet
Ekonomiczny
w Katowicach

Studia Ekonomiczne. Zeszyty Naukowe
Uniwersytetu Ekonomicznego w Katowicach
ISSN 2083-8611 Nr 247 · 2015

Informatyka i Ekonometria 4

Grażyna Trzpiot

University of Economics in Katowice
Faculty of Informatics and Communication
Department of Demography and Economic Statistics
grazyna.trzpiot@ue.katowice.pl

APPLICATION OF FUNCTIONAL BASED ON SPATIAL QUANTILES

Summary: Presented approach describe some functional which is robust measures based on quantiles. We consider the multivariate context and utilize the spatial quantiles. We notice an extension of univariate quantiles and we presented nonparametric measures of multivariate location, spread, skewness and kurtosis. In modeling extreme risk important properties are based on heavy-tailed distribution. We present tailweight and peakedness measures. To aid better understanding of the spatial quantiles as a foundation for nonparametric multivariate inference and analysis, we also provide some basic properties.

Keywords: multivariate distributions; spatial quantiles, descriptive measures.

Introduction

When we make some research using some probability distributions we need robust descriptive measures. Presented approach can be useful when the distributions are unspecified, as in the case of exploratory and nonparametric inference. In the univariate case, many notions of descriptive measures are *quantile-based*, exploiting the natural order in R. For extension to the multivariate case, one must decide which approach we can use, select a particular version of multivariate quantiles. We use the *spatial quantiles*, which were introduced by Chaudhuri [1996] and Koltchinskii [1997] as a certain form of generalization of the univariate case based on the L_1 norm.

The spatial quantiles have induced and the basic descriptive measures is notice in terms of the spatial quantiles. Appling the spatial quantiles as a basis we notice measures of *location*, *spread*, *skewness* and *kurtosis*. Location in this sense already has a representative measure: *spatial median*[1]. Almost as important as location is spread, for which spatial versions are already noted in Chaudhury [1996]. Next most important are skewness and kurtosis, which serve, for example, to characterize the way in which a distribution deviates from normality. Building on our treatment of spatial location and spread, we notice spatial measures of skewness and kurtosis.

The spatial location and volume functional are utilized in formulating measures of asymmetry and a spatial kurtosis functional. We present on the kurtosis as a measure of the degree of shift of probability mass toward the center and/or the tails. Additionally we make some note on tailweight and peakedness as special tail distribution measure.

1. The spatial quantiles

For univariate Z with $E|Z| < 1$, and for $0 < p < 1$, the L_1-based definition of univariate quantiles characterizes the p^{th} quantile as any value θ minimizing [Ferguson, 1967]:

$$E\{|Z-\theta| + (2p-1)(Z-\theta)\}. \tag{1}$$

As an extension to R^d, "spatial" or "geometric" quantiles were introduced by Chaudhuri [1996] as follows. We can rewrite the previous formula as:

$$E\{|Z-\theta| + u(Z-\theta)\}, \tag{2}$$

where $u = 2p - 1$, thus re-indexing the univariate pth quantiles for $p \in (0, 1)$ by u in the open interval $(-1, 1)$. Then d-dimensional quantiles are formulated by extending this index set to the open unit ball $\mathbf{B}^{d-1}(0)$ and minimizing a generalized form of (2),

$$E\{\Phi(u, X-\theta) - \Phi(u, X)\}, \tag{3}$$

where X and θ are R^d-valued and $\Phi(u, t) = \|t\| + \langle u, t\rangle$ with $\|\cdot\|$ the usual Euclidean norm and $\langle \cdot,\cdot \rangle$ the usual Euclidean inner product. (Subtraction of $\Phi(u, X)$ in (3) eliminates the need of a moment assumption.) This yields, corresponding to the underlying distribution function F for X on R^d, and for $u \in \mathbf{B}^{d-1}(0)$, a u^{th} quantile $Q_F(u)$, having both direction and magnitude. In particular, the well-known spatial median is given by $Q_F(0)$, which we shall also denote by M_F.

---

[1] See Small [1990] for an overview of multidimensional medians.

It can be checked that, for each $u \in \mathbf{B}^{d-1}(0)$, the quantile $Q_F(u)$ may be represented as the solution[2] x of:

$$-E\{(X-x)/\|X-x\|\} = u. \tag{4}$$

So to each point x in R^d a spatial quantile have some interpretation: spatial quantile $Q_F\,(u_x)$ indexed by the average unit vector u_x pointing to x from a random point having distribution F. Since u_x is uniquely determined by (4) and satisfies $x = Q_F\,(u_x)$, we interpret u_x as the inverse at x of the spatial quantile function Q_F and denote it by $Q_F^{-1}(x)$. When the solution x of (4) is not unique, multiple points x can have a common value of $Q_F^{-1}(x)$.

Additionally we can say that "central" and "extreme" quantiles $Q_F(u)$ correspond to $\|u\|$ being close to 0 and 1, respectively. Thus we may think of the quantiles $Q_F(u)$ as indexed by a directional "outlyingness" parameter u whose magnitude measures outlyingness quantitatively, and thus we may measure the outlyingness of any point x quantitatively by the corresponding magnitude $\|u_x\| = \|Q_F^{-1}(x)\|$. As consequence of (4) is that $Q_F(u)$ is obtained by inverting the map:

$$t \to -E\{(X-t)/\|X-t\|\}, \tag{5}$$

from which it is seen that spatial quantiles are a special case of the "M-quantiles" introduced by Breckling and Chambers [1988].

The function $Q_F^{-1}(x) = -E\{(X-x)/\|X-x\|\}$ is called the "spatial rank function" [Möttönen, Oja, 1995], as it generalizes the univariate centered rank function, $2F(x) - 1$, and similarly indicates the average direction and distance of an observation from the median. The spatial quantile function and the spatial rank function are simply inverses of each other.

In the setting of the multivariate location model $F(x-\theta)$, the sample analogue rank function evaluated at a point θ_0 provides a "spatial sign test" statistic for the hypothesis H_0: $\theta = \theta_0$.

Further, (4) yields the following useful property of the spatial quantile function. For the case that F is *centrally symmetric* about M_F, that is, $X - M_F$ and $M_F - X$ are identically distributed, the corresponding median-centered spatial quantile function Q_F is *skew-symmetric*:

$$Q_F(-u) - M_F = -(Q_F(u) - M_F),\ u \in \mathbf{B}^{d-1}(0). \tag{6}$$

[2] The solution $Q_F(u)$ to (4) always exists for any u, and it is unique if $d \geq 2$ and F is not supported on a straight line Chaudhuri [1996].

When the distribution F is transformed by $x \to Ax + b$, with A proportional to an orthogonal matrix and b an arbitrary vector, then the same mapping applied to the original quantile function at u yields the quantile function of the transformed distribution, subject to the reindexing $u \to u' = (\|u\|/\|Au\|)Au$ then:

$$Q_{AX+b}((\|u\|/\|Au\|)Au) = AQ_X(u) + b,\ u \in \mathbf{B}^{d-1}(0). \tag{7}$$

For convenience we denote Q_G also by Q_Y for Y having distribution G. In particular, the spatial median of the transformed distribution is given by the same mapping applied to the spatial median of the original distribution: $M_{AX+b} = AM_X + b$. Note that the quantity $\|u\|$ is preserved under the reindexing, that is, $\|u'\| = \|u\|$, having the interpretation that the outlyingness measure associated with a given point x is *invariant* under the given linear transformation, that is, $\|Q^{-1}_{AX+b}(x)\| = \|Q^{-1}_X(x)\|$ for each $x \in \mathrm{R}^d$. So the spatial quantiles are *equivariant* with respect to *shift*, *orthogonal*, and *homogeneous scale transformations*.

In terms of a data set (a cloud) in R^d, the sample spatial quantile function changes as was prescribed by (7) if the cloud of observations becomes translated, or homogeneously rescaled, or rotated about the origin, or reflected about a $(d - 1)$-dimensional hyperplane through the origin. For the singular value decomposition of matrices, equivariance with respect to an *arbitrary affine* transformation $x \to Ax + b$ fails only in the case that the action by A includes *heterogeneous* scale transformations of the coordinate variables.

Computation of the sample spatial quantile function for a data set $X_1, \ldots, X_n$ via:

$$-\frac{1}{n}\sum_{i=1}^{n}\frac{X_i - x}{\|X_i - x\|} = u \tag{8}$$

is straightforward [see Chaudhuri, 1996], whereas, for example, many of the depth-based notions of multivariate quantiles are computationally intensive. We note that the left-hand side of (8) is the *sample* version of the centered rank function discussed above. Likewise,

$$-\frac{1}{2n}\left[\sum_{i=1}^{n}\frac{X_i - x}{\|X_i - x\|} + \sum_{i=1}^{n}\frac{-X_i - x}{\|-X_i - x\|}\right] \tag{9}$$

gives the sample *spatial signed-rank function* [Möttönen and Oja, 1995].

We note a *robustness property* of $Q_n(u)$: its value remains unchanged if the points X_i are moved outward along the rays joining them with $Q_n(u)$. Moreover, it has *favorable breakdown point* (50% for the median) and *bounded influence function* [Koltchinskii, 1997].

The formulation of spatial quantiles as a solution of an L_1 optimization problem is quite different from that of multivariate quantiles defined in terms of *statistical depth functions* as boundary points of depth-based central regions of specified probability, in Serfling [2002] it is seen that the spatial quantiles indeed possess a useful depth-based representation, in terms of a new "spatial depth function" which is quite natural: $D(x, F) = 1 - \left\| Q_F^{-1}(x) \right\|$.

Sample generalized spatial quantiles are consistent and asymptotically Gaussian with an intractable dispersion parameter. The generalized bootstrap can be used for inference and obtaining all statistical properties of these quantiles. Projection quantiles have one to one relationship like univariate quantiles. Projection quantiles based confidence sets have exact coverage.

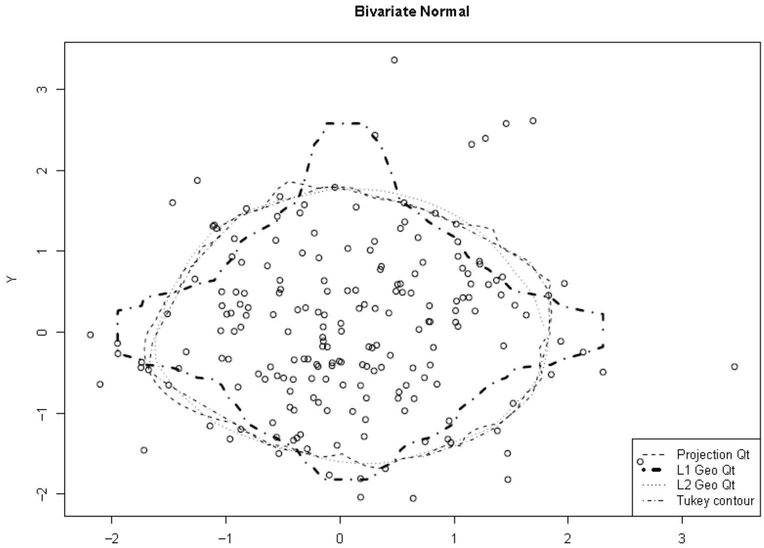

Fig. 1. Example scatter plot

Source: Simultaneous quantiles of several variables. S. Chatterjee, School of Statistics, University of Minnesota.

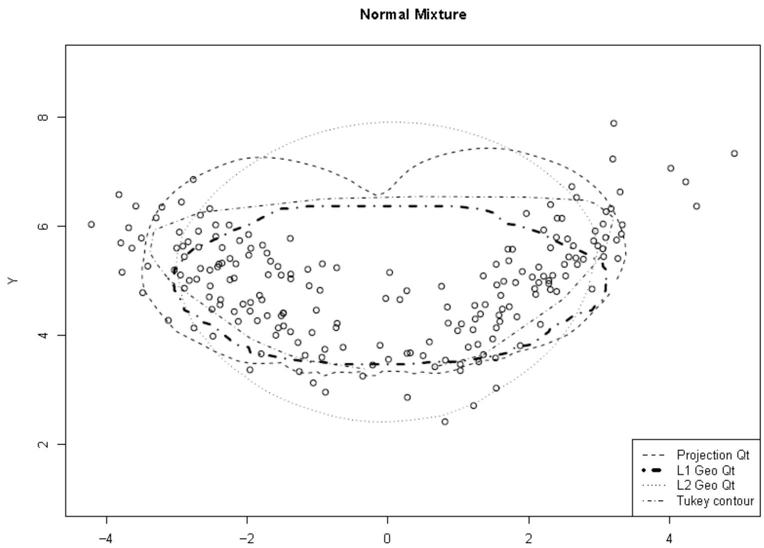

Fig. 2. Example scatter plot

Source: Simultaneous quantiles of several variables. S. Chatterjee, School of Statistics, University of Minnesota.

2. Spatial location measures

The traditional "spatial" location measure is the well-known *spatial median* given by $Q_F(0) = M_F$. Additional forms of location measure are generated by quantile-based "L-functionals" which in the present context are given by (vector-valued) weighted averages of the spatial quantile function, $\int_{\mathbf{B}^{d-1}(0)} Q_F(u)\mu du$ R, with respect to signed measures $\mu(du)$ on the index set $\mathbf{B}^{d-1}(0)$.

Definition 1. Particular class of *location measures*, is defined by:

$$l_F(r) = \int_{\mathbf{S}_r^{d-1}(0)} Q_F(u)m(du),\ 0 \le \mathrm{r} < 1,$$

where $S_r^{d-1}(0)$ is the sphere (the surface of the ball) of radius r centered at the origin 0, and $m(du)$ is the uniform measure on this sphere.

Note that $l_F(0)$ is just M_F. Moreover, in the case of centrally symmetric F, $l_F(\mathrm{r}) \equiv M_F$. Considered as a function of r, we call $l_F(\cdot)$ the *location functional* corresponding to F through its associated spatial quantile function. It is easily seen that $l_F(r)$ is equivariant with respect to shift, orthogonal and homogeneous scale transformations.

Various applications are supported by the spatial location functional. For example, a spatial version of multivariate trimmed mean is given by the integral of $Q_F(u)$ with respect to the uniform measure on a subset of $\mathbf{B}^{d-1}(0)$ of form $\{u : \|u\| \leq \beta\}$. In terms of the location functional, this is just $\int_0^\beta l_F(r)dr$. Further, we use this location functional in defining a spatial skewness measure.

We can find other notions of location functional which can be associated with the spatial median. We notice extending of the univariate interquantile intervals to multivariate "median balls" indexed by their radii, as a family of "central regions" which provide *optimal* summaries in a certain L_1 sense, and a corresponding location functional is defined by the centers of the balls. This location functional also is identically M_F in the case of a centrally symmetric F.

3. Spatial central regions and spread measures

Corresponding to the spatial quantile function Q_F , we call:

$$C_F(r) = \{Q_F(u) : \|u\| \leq r\}$$

the r^{th} *central region*.

When F is centrally symmetric, the skew-symmetry of $Q_F - M_F$, given by (6) yields that the regions $C_F(r)$ additionally have the useful property of being *symmetric* sets, in the sense that for each point x in $C_F(r)$ its reflection about M_F is also in $C_F(r)$. It is clear that the central regions $C_F(r)$ are equivariant under shift, orthogonal and homogeneous scale transformations.

Definition 2. The (real-valued) *volume functional* corresponding to Q_F is defined by:

$$v_F(r) = \text{volume}\ (C_F(r)),\ 0 \leq r < 1.$$

For each r, $v_F(r)$ provides a dispersion measure.

It is invariant under shift and orthogonal transformations, and $v_F(r)^{1/d}$ is equivariant under homogeneous scale transformations. As an increasing function of the variable r, $v_F(r)$ characterizes the dispersion of F in terms of expansion of the central regions $C_F(r)$.

Analogous to the scale curve introduced by Liu, Parelius and Singh [1999] in connection with *depth-based* central regions indexed by their *probability weight*, the spatial volume functional may likewise be plotted as a "scale curve" over $0 \leq r < 1$, thus providing a convenient two-dimensional device for the viewing or comparing of multivariate distributions of any dimension.

Alternatively, two multivariate distributions F and G may be compared via a *spread-spread plot*, the graph of $v_G\, v^{-1}{}_F$. Since the central regions are ordered and increase with respect to the spatial "outlyingness" parameter r that describes their boundaries, i.e., $r < r'$ implies $CF(r) \subset CF(r')$, their probability weights p increase with r. Thus the central regions and associated volume functional and scale curve can equivalently be indexed by the probability weight of the central region. This relationship may be described by a mapping $\psi_F : r \to p_r \in [0, 1)$, with inverse $\psi_F^{-1} : p \to r_p$ (thus $p_r = \psi_F(r)$ and $r_p = \psi_F^{-1}(p)$), but characterization of this mapping is complicated.

Different notion of spatial dispersion function based on the median balls was developed by Avérous and Meste [1997]. Under regularity conditions on F, the probability weight of a median ball is a nondecreasing function of its radius, even in cases when the balls are not ordered by inclusion. This yields an analogue of the scale curve described above.

Definition 3. *Matrix-valued dispersion measures*. As an analogue of the usual covariance matrix, one can also consider *matrix-valued* dispersion measures based on the spatial quantiles:

$$S(F) = \int_{\mathrm{B}^{d-1}(0)} \left(Q_F(u) - M_F\right)\left(Q_F(u) - M_F\right)' \lambda du$$

for measures $\lambda(du)$ on $\mathbf{B}^{d-1}(0)$.

For example, a suitable choice of $\lambda(\cdot)$ yields a *trimmed dispersion measure*. Such scatter matrices contain information on the shape and orientation of the probability distribution as well as on the variations and mutual dependence of the coordinate variables.

Real-valued "generalized variance" measures are provided by the corresponding determinants. We note that the spatial version of $S(F)$ satisfies the "*covariance equivariance*":

$$S(F_{AX+b}) = AS(F_X)A'$$

for all $d \times d$ (proportionally) *orthogonal* A and all $b \in \mathrm{R}^d$.

4. Spatial skewness and kurtosis functional measures

In general, a skewness measure should be location and scale free and in the case of a "symmetric" distribution, equal zero. Classical *univariate* quantitative skewness measures thus have the form of a difference of two location measures divided by a scale measure, whereby skewness then is characterized by a sign indicating *direction* and a magnitude measuring *asymmetry*. Along with such measures, associated notions of the *ordering* of distributions according to their skewness have been developed. Extension of the above notion of a skewness measure to the *multivariate* case should in principle yield a *vector*, thus again characterizing skewness by both a direction and an asymmetry measure.

We must specify a notion of multivariate symmetry relative to which skewness represents a deviation. So, we require a quantitative skewness measure to reduce to the null vector in the case of *central symmetry.*

Avérous and Meste [1997] open up a broader treatment by introducing two *vector-valued skewness functionals* oriented to the spatial median, along with corresponding definitions of quantitative skewness, directional qualitative skewness, and directional ordering of multivariate distributions. In particular, one of skewness functionals is given by the difference of the "median balls" location functional and the spatial median *MF*, divided by a fixed real-valued scale parameter, the inverse of the density of *F* evaluated at the spatial median.

Definition 4. Utilizing instead the *spatial* location and volume functionals, a *spatial skewness functional* is defined as:

$$s_F(r) = 2\frac{l_F(r) - M_F}{v_F(r)^{1/d}},\ 0 < r < 1, \tag{10}$$

which in the case of centrally symmetric *F* reduces appropriately to the null vector, each *r*.

Note that the scale factor in the denominator is allowed to depend on *r*. The power $1/d$ for $v_F(r)$ makes $s_F(r)$ invariant under any homogeneous scale transformation. For each $r = r_0$, $s_F(r_0)$ represents a *quantitative* vector-valued skewness measure, indicating an overall direction of skewness. More generally, such a measure is given by any weighted average, $\beta_\mu(F) = \int_0^1 s_F(r)\mu(dr)$, taken with respect to a probability measure $\mu(dr)$ on [0, 1) not depending on *F*. Further, we obtain quantitative *real-valued* measures of the skewness of *F* in any particular direction *h*, taken from the median M_F, by taking scalar products with the vector measures:

$$\langle s_F(r), h\rangle,\ 0 < r < 1, \text{ and } \langle \beta_\mu(F), h\rangle.$$

We can take $\langle s_F(r), h\rangle$, $0 < r < 1$, as a *functional* real-valued measure of skewness in the direction *h*. This provides a basis for straightforward *qualitative* notions of skewness.

Definition 5. *Asymmetry measures*. Vector-valued spatial skewness functional yields a corresponding real-valued *asymmetry functional*, which we can notice in the formula:

$$\|s_F(r)\| = 2\frac{\left\| \int_{\|S_r^{d-1}(0)} Q_F(u) + Q_F(-u) - M_F \right\|}{v_F(r)^{1d}}, \ 0 < r < 1,$$

from which is obtained a real-valued *index of asymmetry* $A_F = \sup_{0<r<1} \|s_F(r)\|$.

The latter index extends a measure of asymmetry for the univariate case suggested by MacGillivray [1986] and likewise may be used to order distributions: "F is less asymmetric than G", written $F \prec_A G$, if $A_F \leq A_G$.

Like $\|s_F(r)\|$, by (6) it equals zero in the case of centrally symmetric F. As seen below, it coincides with $\|s_F(r)\|$ in the univariate case. Also, its supremum over r yields an alternative asymmetry measure.

Definition 6. *Asymmetry curves*. Analogous to the scale curve, a plot of the asymmetry functional $\|s_F(r)\|$, $0 < r < 1$, as a "spatial skewness curve" provides a convenient two-dimensional summary of the skewness of a multivariate distribution.

Likewise we may plot a *directional* version $<s_F(r), h>$, $0 < r < 1$, for any selected direction h. An alternative summary, related to (11), is given by a plot of:

$$\sup_{\|u\|=r} \frac{\|Q_F(u) - M_F\|}{\|Q_F(-u) - M_F\|}, \ 0 < r < 1.$$

By (6), we see that in the case of centrally symmetric F, this curve follows the constant level 1. In the univariate case it is equivalent to a plot of $F^{-1}(1-p)-F^{-1}(1/2)$ versus $F^{-1}(1/2)-F^{-1}(p)$.

Another type of asymmetry curve is obtained by adapting one given by Liu, Parelius and Singh [1999] in the context of depth-based central regions. For each r, let $I_F(r)$ denote the intersection of the central region $C_F(r)$ and its reflection about M_F, and let $w(r)$ denote the ratio of the volume of $I_F(r)$ to that of $C_F(r)$, over $0 < r < 1$. For centrally symmetric F the intersection $I_F(r)$ coincides with $C_F(r)$ and thus $w(r) \equiv 1$, a departure of F from central symmetry about M_F is indicated by the degree to which the curve $w(r)$ lies below the constant level 1.

Turning now to the multivariate case, and drawing upon the above discussion, we consider kurtosis to characterize the relative degree, in a location- and scale-free sense, to which probability mass of a distribution is diminished in the "shoulders" and heavier in the either the center or tails or both. We thus distinguish *peakedness*, *kurtosis* and *tailweight* as distinct, although very much interrelated, features of a distribution.

Definition 7. Based on the spatial volume functional, a *spatial kurtosis functional* can be notice as:

$$k_F(r) = \frac{v_F\left(\frac{1}{2}-\frac{r}{2}\right)+v_F\left(\frac{1}{2}+\frac{r}{2}\right)-2v_F\left(\frac{1}{2}\right)}{v_F\left(\frac{1}{2}+\frac{r}{2}\right)-v_F\left(\frac{1}{2}-\frac{r}{2}\right)}, \; 0 < r < 1.$$

We interpret kurtosis as measuring a feature which is interrelated with peakedness and tailweight but not to be equated with either of these. Here we comment on peakedness and tailweight as separate from kurtosis.

Definition 8. A family of *tailweight measures* based on the spatial quantiles is given by:

$$t_F\,(r, s) = v_F(r)/v_F(s), \; 0 < r < s < 1,$$

which reduces in the univariate case to ratios of the spread functional (9) evaluated at different points.

Using the term "kurtosis" for tailweight, a similar multivariate extension using *depth-based* central regions is given by Liu, Parelius and Singh [1999], who introduce a "fan plot" exhibiting the curves $t_F\,(r, s)$ for a fixed choice of r and selected choices of s. They also introduce other forms of tailweight measures, i.e, a Lorenz curve and a "shrinkage plot", which likewise may be formulated analogously in terms of the spatial quantile function. Several multivariate distributions or data sets may be compared with respect to tailweight on the basis of their respective (either spatial or depth-based) fan plots, Lorenz curves, or shrinkage plots. Asymptotics for sample versions of the kurtosis functional $k_F\,(\cdot)$ and these other transforms of the volume functional may be derived from the asymptotics for the scale curve.

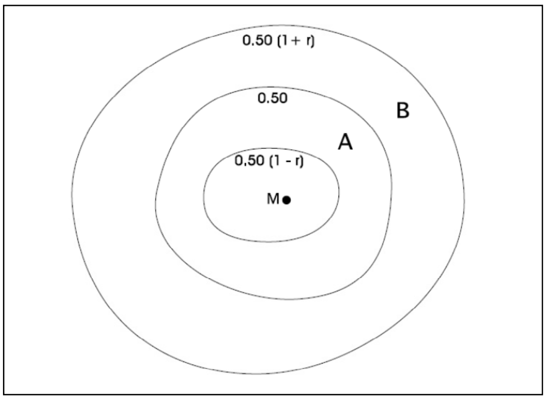

Fig. 3. Median M and central regions C_F (1/2 − r/2), C_F (1/2), C_F (1/2+ r/2), with $A = C_F$ (1/2) − C_F (1/2 − r/2), and $B = C_F$ (1/2 + r/2) − C_F (1/2)

It can be suggest that a measure of "kurtosis" (meaning tailweight) is given by any suitable ratio of two scale measures. Typical tailweight measures indeed are of this form, but such a restriction is too restrictive for the more refined notion of kurtosis as distinct from tailweight. The term "peakedness" is traditionally used synonymously with "concentration" or inversely with "dispersion" or "scatter".

In particular, the latter authors provide a depth-based notion for ordering distributions by "more scattered": relative to a depth function $D(x, \cdot)$, the distribution F on R^d is more scattered than the distribution G if the D-based volume functional for F lies above that of G. As an appropriate analogue in terms of the spatial quantile functional, we thus define:

Definition 9. F is more scattered (**less peaked**) than G if $v_F(r) \geq v_G(r)$, $0 < r < 1$.

Concluding remarks

We tried to present useful functional measures based on spatial quantiles. It can be used as a basis for a variety of useful upgrades and methodological techniques. We can extend the *regression quantiles* of Koenker and Bassett [1978] to many different case [Trzpiot, 2011a, 2011b, 2013a) and also to spatial *quantiles regression* [Trzpiot, 2012, 2013b]. As an analogue of procedures widely used in univariate data analysis, the use of *bivariate QQ-plots* based on spatial quantiles, can be illustrates along with some related devices. Chakraborty [2001] develops similar methods based on a modified type of sample spatial quantile. Also, as noted above, notions of *multivariate ranks* may be based on spatial quantiles. This suggests the possibility of *spatial rank-rank plots*. Finally, in the present paper we see that the spatial quantile function may serve effectively as a *descriptive measure*. Some of them can be used in actuarial application as a risk measures [Wolny-Dominiak, Trzpiot, 2013c].

References

Avérous J. and Meste M. (1997), *Median Balls: An Extension of the Interquantile Intervals to Multivariate Distributions*, "Journal of Multivariate Analysis", 63, p. 222-241.

Breckling J. and Chambers R. (1988), *M-quantiles*, "Biometrika", 75, p. 761-771.

Breckling J., Kokic P. and L¨ubke O. (2001), *A Note on Multivariate M-quantiles*, "Statistics & Probability Letters", 55, p. 39-44.

Chakraborty B., Chaudhuri P., and Oja H. (1998), *Operating Transformation and Retransformation on Spatial Median and Angle Test*, "Statistica Sinica", 8, p. 767-784.

Chakraborty B. (2001), *On Affine Equivariant Multivariate Quantiles*, "Annals of the Institute of Statistical Mathematics", 53, p. 380-403.

Chaudhuri P. (1996), *On a Geometric Notion of Quantiles for Multivariate Data*, "Journal of the American Statistical Association", 91, p. 862-872.

Ferguson T.S. (1967), *Mathematical Statistics: A Decision Theoretic Approach*, Academic Press, New York.

Koenker R. and Bassett G. (1978), *Regression Quantiles*, "Journal of the American Statistical Association", 82, p. 851-857.

Koltchinskii V. (1997), *M-estimation, Convexity and Quantiles*, "Annals of Statistics", 25, p. 435-477.

Liu R.Y., Parelius J.M. and Singh K. (1999), *Multivariate Analysis by Data Depth: Descriptive Statistics, Graphics and Inference (with Discussion)*, "Annals of Statistics", 27, p. 783-858.

MacGillivray H.L. (1986), *Skewness and Asymmetry: Measures and Orderings*, "Ann. Statist.", Vol. 14, No 3, p. 994-1011.

Möttönen J. and Oja H. (1995), *Multivariate Spatial Sign and Rank Methods*, "Journal of Nonparametric Statistics", 5, p. 201-213.

Serfling R. (2002), *A Depth Function and a Scale Curve Based on Spatial Quantiles* [in:] *Statistical Data Analysis Based On the L*1*-Norm and Related Methods*, ed. Y. Dodge, Birkhaüser, Basel, p. 25-38.

Small C.G. (1990), *A Survey of Multidimensional Medians*, International Statistical Institute.

Trzpiot G. (2011a), *Some Tests for Quantile Regression Models*, "Acta Universitatis Lodziensis", Łódź, Folia Economica, 255, p. 125-135.

Trzpiot G. (2011b), *Bayesian Quantile Regression*, Studia Ekonomiczne. Zeszyty Naukowe nr 65, Uniwersytet Ekonomiczny w Katowicach, p. 33-44.

Trzpiot G. (2012), *Spatial Quantile Regression*, "Comparative Economic Research. Central and Eastern Europe", Vol. 15, No 4, p. 265-279, Wydawnictwo Uniwersytetu Łódzkiego.

Trzpiot G. (2013a), *Properties of Transformation Quantile Regression Model*, "Acta Universitatis Lodziensis", Łódź, Folia Economica, 285, p. 125-137, Wydawnictwo Uniwersytetu Łódzkiego.

Trzpiot G. (2013b), *Bayesian Spatial Quantile Regression*, "Acta Universitatis Lodziensis", Łódź, Folia Economica, 286, p. 109-117.

Trzpiot G. (2014), *Some Properties of Spatial Quantiles*, "Acta Universitatis Lodziensis", Łódź, Folia Oeconomica, 5 (307), "Spatial Econometrics", Wydawnictwo Uniwersytetu Łódzkiego, p. 141-152.

Wolny-Dominiak A., Trzpiot G. (2013c), *GLM and Quantile Regression Models in a Priori Ratemaking*, „Wiadomości Ubezpieczeniowe", No 4, p. 49-58.

WYBRANE FUNKCJONAŁY − PRZESTRZENNE MIARY KWANTYLOWE

Streszczenie: W artykule przedstawiono wybrane funkcjonały, które są odpornymi miarami opisowymi. Miary te są definiowane z wykorzystaniem wielowymiarowych przestrzennych kwantyli. Zatem zapiszemy zadanie wielowymiarowe i dodatkowo wykorzystamy jeden argument dla opisu przestrzennego. Wykorzystując opis jednowymiarowy, rozszerzymy je do zadania wielowymiarowego, zapisując odpowiednie miary położenia, rozproszenia, skośności i kurtozy. W modelowaniu ekstremalnego ryzyka ważne są własności ogonów rozkładów. Przedstawiono również miary skupione na ogonach rozkładów.

Słowa kluczowe: wielowymiarowe dystrybuanty, przestrzenne kwantyle, miary opisowe.