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BORDERED MATRICES AND SOME THEIR APPLICATIONS

Abstract

In this paper bordered matrices and some their applications are presented. In particular we give information how can be found matrix $F = AB^{-1}C$ without calculation the inverse of matrix B (when B = I this way we obtain Cauchy's product of matrices A and C). We present how to find generalized inverse of the Moore'a-Penrose'a type too and finally calculate value of determinant of given matrix A of order $n \times n$.

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Bordered matrices make it possible to homogenize matrix calculations. Thanks to bordered matrices matrix algebra is more approachable, easier and preferred for example by students.

This article focuses on main applications of bordered matrices, inter alia in econometrics.

We consider a matrix $A = \lfloor a_{ij} \rfloor$ of order $m \times n$. We divide it by p – 1 horizontal and q - 1 vertical lines into blocks order $m_i \times n_j$ where

$$\sum_{i=1}^{p} m_i = m \quad \sum_{j=1}^{q} n_j = n$$

Then we obtain

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \cdots & \cdots & \cdots & \cdots \\ A_{p1} & A_{p2} & \cdots & A_{pq} \end{bmatrix} \tag{1}$$

The matrix given by formula (1) is called a block one. In particular case when the blocks matrix satisfies following conditions

1. block A_{11} is nonsingular matrix of order $s \times s$
2. blocks $A_{12} \ldots A_{1q}$ are s – dimensional vertical vectors
3. blocks $A_{21} \ldots A_{p1}$ are s – dimensional horizontal vectors
4. remaining blocks A_{ij} $i = 2, 3 \ldots p$ $j = 2, 3 \ldots q$ are numbers

it is called multiple bordered matrix. Henceforth the matrix A of order $m \times n$ written as bordered shape may be presented as following:

$$A = \begin{bmatrix} A_1 & f_1 & f_2 & \cdots & f_k \\ g_1 & & & & \\ \vdots & & Q_{rk} & & \\ g_r & & & & \end{bmatrix} \tag{2}$$

In formula (2) the matrix A_1 is nonsingular of order $s \times s$ one; f_j j = 1, 2 … k are s – dimensional vertical vectors, likewise g_i i = 1, 2 … r are s – dimensional horizontal vectors and the matrix $Q = [q_{tu}]$ of order $r \times k$.

On the rows of the bordered matrix given by formula (2) following elementary transformations are done:

α - the matrix A_1 is transformed to upper triangular one. Its diagonal elements are equal one,

β - the vectors g_i i = 1, 2 … r are transformed to zero ones.

Then the matrix $Q_{rk} = [q_{tu}]$ is transformed to the matrix $D_{rk} = [d_{tu}]$ where

$$d_{tu} = q_{tu} - g_t A_1^{-1} f_u \quad t = 1,2\ldots r \quad r = 1,2\ldots k \tag{3}$$

For example the matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & -1 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix} \tag{4}$$

Will be put down as the bordered matrix :

$$A = \left[\begin{array}{cc|c|c} 3 & 2 & 1 & 4 \\ 2 & 0 & 1 & 2 \\ \hline 1 & 0 & -1 & 3 \\ \hline 2 & 1 & 3 & 2 \end{array}\right] \tag{5}$$

where

$$A_1 = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix} \quad f_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad f_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \tag{6}$$

$$g_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 2 & 1 \end{bmatrix} \tag{7}$$

$$Q_{22} = \begin{bmatrix} -1 & 3 \\ 3 & 2 \end{bmatrix} \tag{8}$$

On the rows of matrix A given by formula (5) transformations α and β will be done. Hence

$$A \sim \begin{bmatrix} 3 & 2 & 1 & 4 \\ 2 & 0 & 1 & 2 \\ 1 & 0 & -1 & 3 \\ 2 & 1 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 4/3 \\ 0 & -4/3 & 1/3 & -2/3 \\ 0 & -2/3 & -4/3 & 5/3 \\ 0 & -1/3 & 7/3 & -2/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2/3 & 1/3 & 4/3 \\ 0 & 1 & -1/4 & 1/2 \\ 0 & 0 & -3/2 & 2 \\ 0 & 0 & 9/4 & -1/2 \end{bmatrix}$$

$$D = \begin{bmatrix} -3/2 & 2 \\ 9/4 & -1/2 \end{bmatrix}$$

Currently, we verify the correctness of the results obtained. For:

$$A_1^{-1} = -\frac{1}{4}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}$$

we obtain:

$$d_{11}^{22} = q_{11} - g_1 A_1^{-1} f_1 = -1 + \frac{1}{4}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{-3}{2}$$

$$d_{12}^{22} = q_{12} - g_1 A_1^{-1} f_2 = 3 + \frac{1}{4}\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 2$$

$$d_{21}^{22} = q_{21} - g_2 A_1^{-1} f_1 = 3 + \frac{1}{4}\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{9}{4}$$

$$d_{22}^{22} = q_{22} - g_2 A_1^{-1} f_2 = 2 + \frac{1}{4}\begin{bmatrix} 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = -\frac{1}{2}$$

The formula (3) is basic in the theory of the bordered matrix. It has many various applications. Some of them will be presented in this paper.

First will be shown how we can find the matrix $F = AB^{-1}C$ without a calculation of the inverse of matrix B, where A, B and C are given and are respectively of order $m \times n, \quad n \times n, \quad n \times p$.

For this aim we construct the bordered matrix M.

$$M = \begin{bmatrix} B & c^1 & \dots & c^p \\ -a_1 & & & \\ \vdots & & O_{m\times p} & \\ -a_m & & & \end{bmatrix} \tag{9}$$

where c^j $j = 1, 2 \ldots p$ stand for the j – th column of matrix C,
a_i $i = 1, 2 \ldots m$ stand for the i – th row of matrix A.
On the rows of the matrix given by formula (9) elementary transformations will be done according α and β.

Then in place of the $O_{m \times p}$ block we obtain the matrix F.

If $B = I$ on the foregoing way we can obtain Cauchy's product of matrices A and C.

Likewise when $m = n, \quad p = 1$ and $A = I \quad C = c$ we can find vector X satisfying system equations $BX = c$ without calculation inverse of matrix B.
Next on the same way if $A = a$ and $C = c$; a and c are n – dimensional vertical and horizontal vectors respectively, we can find values x and y from:

$$x = aB^{-1}c$$

$$y = aB^{-1}a^T$$

(matrix B is non symmetric one).
The bordered matrix can be used to compute generalized inverse of the Moore'a-Penrose'a type.
For example if we are going to find the generalized inverse of matrix A of order $m \times n$ and of rank r $(r < \min(m, n))$. (A^+) and we know that $A = BC$ (both matrices B and C have rank equal r) we can use following formula:

$$A^+ = A^T B \left(B^T A A^T B \right)^{-1} B^T \tag{10}$$

In this case we utilized the bordered matrix:

$$N = \begin{bmatrix} B^T A A^T B & q_1 & \cdots & q_m \\ -p_1 & & & \\ \vdots & & Q_{n \times m} & \\ -p_n & & & \end{bmatrix} \tag{11}$$

where q_j $j = 1, 2 \ldots m$ stands for the j – th column of B^T and p_i $i = 1, 2 \ldots n$ stands for i – th row of $A^T B$. We avoid computing the inverse of matrix $B^T A A^T B$.
On the rows of the matrix N given by formula (11) we do elementary transformations according to α and β.
The bordered matrix can be used to calculate value of determinant of given matrix A of order $n \times n$. For this aim we put down this matrix $A = A^{(n)}$ as the bordered matrix:

$$A = A^{(n)} = \begin{bmatrix} D^{(n)} & f^{(n)} \\ g^{(n)} & d_{nn}^{(n)} \end{bmatrix} \tag{12}$$

where:

$$D^{(n)}=\left[a_{ij}^{(n)}\right]\quad i=1,2\ldots n-1\quad j=1,2\ldots n-1$$

$$\left(f^{(n)}\right)^T=\left[a_{1n}^{(n)}\quad\ldots\quad a_{n-1n}^{(n)}\right]\tag{13}$$

$$\left(g^{(n)}\right)=\left[a_{n1}^{(n)}\quad\ldots\quad a_{nn-1}^{(n)}\right]$$

we suppose $a_{nn}\neq 0$. If this condition would not be satisfied we have to change rows and columns for reaching it. If this is impossible value of determinant of this matrix is equal zero.

If given the above condition is satisfied then we obtain:

$$\det A=\det A^{(n)}=a_{nn}^{(n)}\det\left(D^{(n)}-\frac{1}{a_{nn}^{(n)}}f^{(n)}g^{(n)}\right)\tag{14}$$

Using over and over again formula (14) we obtain:

$$\det A^{(n)}=a_{nn}^{(n)}a_{n-1n-1}^{(n-1)}\ldots a_{11}^{(1)}\tag{15}$$

where

$$A^{(k)}=\begin{bmatrix}D^{(k)} & f^{(k)}\\ g^{(k)} & a_{kk}^{(k)}\end{bmatrix}\tag{16}$$

$$a_{ij}^{(k-1)}=a_{ij}^{(k)}-\frac{1}{a_{kn}^{(k)}}a_{ik}^{(k)}a_{kj}^{(k)}\tag{17}$$

k = 2, 3, …,n, i, j = 1, 2, …, k-1.

For example we will calculate value of determinant of matrix A given by formula (4). According to formula (12) we have:

$$A=A^{(4)}=\left[\begin{array}{ccc|c}3 & 2 & 1 & 4\\ 2 & 0 & 1 & 2\\ 1 & 0 & -1 & 3\\ \hline 2 & 1 & 3 & 2\end{array}\right]\tag{18}$$

but according to formula (14) we obtain:

$$\det A=\det A^{(4)}=2\det\left(\begin{bmatrix}3 & 2 & 1\\ 2 & 0 & 1\\ 1 & 0 & -1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}4\\ 2\\ 3\end{bmatrix}\begin{bmatrix}1 & 0 & -1\end{bmatrix}\right)=2\det\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & 2\\ -1/2 & 0 & 1/2\end{bmatrix}\tag{19}$$

Next the matrix

$$\begin{bmatrix}1 & 2 & 3\\ 1 & 0 & 2\\ -1/2 & 0 & 1/2\end{bmatrix}\tag{20}$$

will be presented as following bordered matrix:

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 0 & 2 \\ \hline -1/2 & 0 & 1/2 \end{array}\right] \tag{21}$$

The formula (14) will be utilized once more to the matrix given by formula (21). We obtain:

$$\det A = \det A^{(4)} = 2\det\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 1 & 0 & 2 \\ \hline -1/2 & 0 & 1/2 \end{array}\right] = \tag{22}$$

$$= 2\cdot\frac{1}{2}\cdot\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 3 \\ 2 \end{bmatrix}\begin{bmatrix} -1/2 & 0 \end{bmatrix}\right) = \det\begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix}$$

At present the matrix

$$\begin{bmatrix} 4 & 2 \\ 3 & 0 \end{bmatrix} \tag{23}$$

is presented as bordered matrix

$$\left[\begin{array}{c|c} 4 & 2 \\ \hline 3 & 0 \end{array}\right] \tag{24}$$

Formula (15) can’t be directly utilized because $a_{22}^{2} = 0$. Changing second row with first one we obtain

$$\left[\begin{array}{c|c} 3 & 0 \\ \hline 4 & 2 \end{array}\right]$$

Hence according to formula (14) we obtain:

$$\det A = \det A^{(4)} = \det\left[\begin{array}{c|c} 4 & 2 \\ \hline 3 & 0 \end{array}\right] = -\det\left[\begin{array}{c|c} 3 & 0 \\ \hline 4 & 2 \end{array}\right] = -2\det\left([3] - \frac{1}{2}[0][4]\right) = -6$$

Results, which were presented in this paper can be used in econometrics. In particular, bordered matrix can be used to find, without estimation of an econometric model, the value of coefficient r^2, theoretical values of endogenous variable and in many others econometric applications. Many of them are given in the works: (Kolupa, 1982), (Borowiecki & Kaliszyk & Kolupa, 1986), (Kolupa & Łukasik & Michalski, 1988), (Kolupa & Marcinkowska-Lewandowska, 1987), (Kolupa & Nowakowski, 1987), (Kolupa & Szczepańska-Gruźlewska, 1991), (Kolupa & Śleszyński , 2010).

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