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REMOVING INCONSISTENCY IN PAIRWISE COMPARISON MATRIX IN THE AHP

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Abstract

The Analytic Hierarchy Process (AHP) allows to create a final ranking for a discrete set of decision variants on the basis of an earlier pairwise comparison of all the criteria and all the decision variants within each criterion. The properties of the obtained ranking depend on the quality of pairwise comparisons; this quality can be evaluated on the basis of consistency measured by means of certain measures. The paper discusses a mathematical model which is the foundation of the AHP and a starting point for a new method which allows to significantly reduce – and even eliminate – the inconsistency of pairwise comparisons measured by the consistency index. The proposed method allows to reduce the consistency index well below the threshold of 0.1.

Keywords: AHP, pairwise comparison, inconsistent pairwise comparison matrices.

1 Introduction

One of the stages of analysis of discrete multicriteria problems can be pairwise comparison. This process requires that the decision maker indicate, on a defined scale and for each pair of objects, the object which is evaluated higher or else that he/she state that they are evaluated identically. However, even for a small number of criteria the number of pairwise comparisons can be fairly large. This, in turn, may cause difficulties with expressing consistent evaluations by the decision maker. This may lead to determining an inconsistent matrix of pairwise comparisons which will therefore lack the assumed properties.

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A well-known method which heavily uses pairwise comparison is the Analytic Hierarchy Process (AHP). An essential obstacle in the application of the AHP is the above mentioned possibility of the occurrence of an inconsistent matrix of pairwise comparisons. Attempts to propose methods reducing the inconsistency of this matrix were made previously. In some papers it was suggested that the AHP itself is incorrectly constructed which leads to difficulties in proper analysis of the decision maker's preferences. One of these papers is Bana e Costa & Vansnick (2008), whose authors state: "we consider that the EM [Eigenvalue Methods] has a serious fundamental weakness that makes the use of AHP as a decision support tool very problematic".

This statement is based on their analysis of the AHP in which an essential role is played by the largest eigenvalue and the corresponding eigenvector of the pairwise comparison matrix. The authors introduced the notion of the Condition of Order Preservation (COP), which was supposed to be used to prove the weakness of the EM, including the AHP. Unfortunately, the authors, in a suggestive example, investigated a pairwise comparison matrix which, on the one hand, does not preserve the COP, and, on the other hand, was regarded in the AHP as consistent, with $c_r = 5\%$ – a value not exceeding the 10% threshold proposed by the author of the method. This example shows very well the problems encountered when analyzing an inconsistent pairwise comparison matrix, even if the degree of inconsistency is small. One can regard the specific values used in Bana e Costa & Vansnick (2008) as revealing the problematic definition of the consistency index and, at the same time, as underscoring the importance of the pairwise comparison matrix. In the present paper, an alternative method of reducing the inconsistency is proposed, which avoids the problems described above (Bana e Costa & Vansnick, 2008).

An interesting proposal of eliminating inconsistency is in the paper Benitez et al. (2011a) in the chapter "Fast computation of the consistent matrix closest to a reciprocal matrix", which describes, in the Matlab language, a function which allows to reduce the inconsistency of the pairwise comparison matrix. This method, however, is not based directly on the EM. Of extreme interest is the mathematical formula from this chapter, since it is similar to the relationship (2), derived in the present paper from the EM. Vector $\boldsymbol{w}$, given by Benitez et al. (2011a), is not based on the eigenvector of the eigenmatrix, but is determined numerically in the above mentioned function. In the proposal described further in the paper, the pairwise comparison matrix will be modified using values based on the eigenvector of the original matrix.

In the paper Zeshui (2004) a variable introducing small perturbations was added to the pairwise comparison matrix. This matrix is corrected using the values of the arithmetic or geometric weighted mean. To improve the pairwise comparison evaluations, the matrix elements with the largest values of the perturbation variables are corrected.

In the papers Saaty (2008, p. 15-16) and Saaty (2003, p. 88-90) three methods of modification of the pairwise comparison matrix have been proposed, which allow to reduce the inconsistency index. In these methods Saaty suggests to determine those elements in the matrix which influence the excessive value of the inconsistency index most. Next, new values are proposed and presented to the decision maker for his/her approval.

To correct an inconsistent pairwise comparison matrix, the paper Ergu et al. (2011) defines an algorithm based on the values of a new matrix containing a certain measure of inaccuracy of the evaluations contained in the original matrix. The authors propose a new method which allows to correct selected evaluations on the basis of the values of the measure proposed.

Another approach, proposed in the paper Siraj et al. (2012), consists in defining a certain heuristics which allows to improve the decision maker's evaluations. This heuristics is based on the ordinal consistency (transitivity) analysis. In this proposal, the relationships between the elements compared are expressed in form of a directed graph, with edges expressing direction and intensity of the decision maker's preferences. By investigating this graph it is possible to determine the number of violations of priority and, on this basis, to correct the values of the pairwise comparison matrix.

The authors of the paper Benitez et al. (2011b) propose to apply a linearization which is supposed to lead to the determination of a consistent pairwise comparison matrix whose distance from the original matrix is small. For this purpose, they define a certain measure based on Frobenius' norm. The paper contains a function in the MatLab language which allows to determine a corrected pairwise comparison matrix.

Among the existing methods of correcting inconsistent evaluations in the pairwise comparison matrix, none is based to a large extent on the eigenvector corresponding to the largest eigenvalue of the original matrix. The present paper attempts to fill this gap.

The purpose of the present paper is to propose a new method of reducing the inconsistency of the pairwise comparison matrix, which is measured with the consistency index c_r. The proposal is based on selected numerical properties of the AHP, which will be described in the next subsection of the paper. Additionally, a new scale is proposed, for the comparison of those elements which differ from each other only slightly. The proposal is based on Saaty's original scale, which introduces two different values (namely 1 and 1.1) for identical objects.

2 Basic properties of the pairwise comparison matrix in the AHP

An essential role in the AHP is played by the scale used for pairwise comparisons. Saaty (2008, p. 257) proposed two versions of the scale, described in Table 1. The first one is used for objects which are clearly different and uses values from 1 to 9. The other one is used for only slightly different objects, for which most evaluations would concentrate between 1 and 2. In this situation Saaty suggested to use values from the interval 1.1-1.9. Unfortunately, undistinguishable objects obtain different values on the two scales, namely 1 and 1.1. The reciprocals of these two values are also different, namely 1 and $\frac{1}{1.1}$, respectively. To solve this problem, in the present paper we use a different form of the second scale, with values smaller by 0.1 as compared with those in the paper Saaty (2008, p. 257), that is, from the interval 1.0-1.8. Thanks to this, identical objects are evaluated as 1, and the reciprocal of this value is also equal to 1.

Table 1: Saaty's Fundamental Scale of Absolute Numbers

Intensity of Importance	**Definition of Importance**	**Explanation**
1	Equal	Both activities contribute equally to the objective
2	Weak or slight	Intermediate importance between 1 and 3
3	Moderate	Experience and judgment slightly favor activity *i* over *j*
4	Moderate plus	Intermediate importance between 3 and 5
5	Strong	Experience and judgment strongly favor activity *i* over *j*
6	Strong plus	Intermediate importance between 5 and 7
7	Very strong or demonstrated	Activity *i* is favored very strongly over *j*; its dominance demonstrated in practice
8	Very, very strong	Intermediate importance between 7 and 9
9	Extreme	The evidence favoring activity *i* over *j* is of the highest possible order of affirmation
1.1-1.9	When all compared activities are very close: a decimal is added to 1 to show their difference as appropriate*	A better alternative way of assigning small decimals is to compare two close activities with other widely contrasting ones, favoring the larger one a little over the smaller one when using the 1-9 values
Reciprocals of above	If activity *i* has one of the above nonzero numbers assigned to it when compared with activity *j*, then *j* has the reciprocal value when compared with *i*	A logical assumption

* Because of different properties of the first degree and its reciprocal in both scales, it is justified to use the range of degrees from the interval 1.0-1.8.

Source: Saaty (2008, p. 257).

In the consecutive subsections of the paper we propose a method supporting the process of correcting inconsistent evaluations of the decision maker. This proposal is based on numerical properties of the AHP, which will be described in the consecutive sections of the paper.

2.1 Analysis of the pairwise comparison matrix in the AHP

The AHP method uses pairwise comparisons of the individual criteria and decision variants. The results of the comparisons are saved in an n by n square matrix, which has ones on the main diagonal, and in which the symmetrical elements are mutually reciprocal. The number of those comparisons is a quadratic function of the number of the elements. The number of the necessary comparisons is expressed by the following formula:

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2} \tag{1}$$

Comparison of two objects results in a consistent pairwise comparison matrix, since only one of the following three cases occurs:

- both objects are identical,
- the first one is evaluated higher than the second one, or
- the second one is evaluated higher than the first one.

Inconsistency of evaluations can occur already in the case of three objects. If the first object is evaluated higher than the second one, and the second one higher than the third one, then the third object cannot be evaluated higher than the first one. If this condition is not satisfied, we obtain an inconsistent pairwise comparison matrix. When investigating the random index described below, we have to generate random pairwise comparison matrices. In simulation experiments with 3×3 matrices, consistent matrices have been obtained in about 20% of cases. For larger matrices, the probability of drawing a consistent pairwise comparison matrix was extremely low. One can observe, therefore, that as the size of the pairwise comparison matrix increases, the problem with the inconsistency of evaluations can grow, too.

A certain inconsistency level was in a sense assumed in the AHP, since the decision maker's evaluations are expressed on a 9-degree scale. This number results from the natural limit of information processing by humans, described by the "seven plus or minus two" rule in Miller (1956).

In the AHP we aim at ordering the discrete decision variants, taking into account a certain hierarchy of criteria. For this purpose, a certain ranking is created, expressed by means of weight coefficients, contained in vector $\boldsymbol{w}$. This vector is normalized, hence the sum of its components is equal to 1.

To describe the AHP, as it was done in Saaty (2008), let us assume that at the beginning the values of vector $\boldsymbol{w}$ are known. An example is the problem of ordering companies with respect to their trade turnover volume. Knowing the values of $\boldsymbol{w}$, we can analytically create the pairwise comparison matrix by dividing the appropriate components of vector $\boldsymbol{w}$. If the turnover volumes are equal, then the quotient of the turnover volumes of the pair of businesses is 1. If the turnover of the first company is greater than that of the second one, the value of this quotient is larger than 1. Otherwise, it is smaller than 1. The results can be written in the form of a pairwise comparison matrix $\boldsymbol{W}$, as in (2) below:

$$\boldsymbol{W} = \boldsymbol{w} \cdot \frac{1}{\boldsymbol{w^T}} = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & & \frac{w_3}{w_n} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \tag{2}$$

From the process of constructing $\boldsymbol{W}$ it follows that its main diagonal consists of ones only, and the symmetric elements are mutually reciprocal: $w_{ij} = \frac{1}{w_{ji}}$. On the basis of (2) we can state that the order of $\boldsymbol{W}$ is exactly 1. Therefore, this matrix has only one non-zero eigenvalue. Additionally, on the basis of calculations in (3), we can see that $\boldsymbol{w}$ is an eigenvector of $\boldsymbol{W}$:

$$\begin{aligned} \boldsymbol{W} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \frac{1}{\boldsymbol{w^T}} \cdot \boldsymbol{w} &= \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & & \frac{w_3}{w_n} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} \\ &= \begin{bmatrix} w_1 + w_1 + w_1 + \cdots + w_1 \\ w_2 + w_2 + w_2 + \cdots + w_2 \\ w_3 + w_3 + w_3 + \cdots + w_3 \\ \vdots \\ w_n + w_n + w_n + \cdots + w_n \end{bmatrix} = \boldsymbol{w} \cdot n \end{aligned} \tag{3}$$

Usually we do not know the components of vector $\boldsymbol{w}$, only the values of matrix $\boldsymbol{W}$. To determine the values of $\boldsymbol{w}$ we analyze the eigenproblem of the form $\boldsymbol{W} \cdot \boldsymbol{w} = n \cdot \boldsymbol{w}$, where n is the eigenvalue corresponding to eigenvector $\boldsymbol{w}$. This relationship is described by formula (3).

From the relationship (3) one can conclude that the order of matrix $\boldsymbol{W}$ is exactly 1. Moreover, from this it follows that all the eigenvalues of $\boldsymbol{W}$, except one, are equal to 0. Since the main diagonal of matrix $\boldsymbol{W}$ contains only 1s, its trace is: $tr(\boldsymbol{W}) = n$. On the other hand, the trace of $\boldsymbol{W}$ is the sum of its eigenvalues, $tr(\boldsymbol{W}) = \sum_i \lambda_i$, and therefore the largest eigenvalue of $\boldsymbol{W}$ is equal to n, and the remaining ones are equal to 0. The problem of constructing the scale vector $\boldsymbol{w}$ is therefore reduced to determining the eigenvector corresponding to the largest eigenvalue of the pairwise comparison matrix $\boldsymbol{W}$.

To determine the eigenvector $\boldsymbol{w}$ it is convenient to use von Mises's exponential method. For a pairwise comparison matrix this method converges, since the difference between the two largest eigenvalues is significantly greater than 0, since $n - 0 > 0$.

It is convenient to start the calculations with the assumptions that the initial vector consists of 1s only: $\boldsymbol{w}_{(0)}^T = [1 \quad 1 \quad 1 \quad \cdots \quad 1]$. We obtain the consecutive approximations of the sought eigenvector from the formula: $\boldsymbol{w}_{(k+1)} = \boldsymbol{W} \cdot \boldsymbol{w}_{(k)}$.

Saaty proposed to normalize matrix $\boldsymbol{W}$ prior to the application of the exponential method, so that the sum of the elements in each column is equal to 1. In a sense this is consistent with the exponential method, since in the consecutive iterations of this method it is necessary to normalize the obtained approximations of the eigenvector. This operation is supposed to prevent a sudden growth of the components of vector $\boldsymbol{w}$. Calculations in (4) show the method of determining the sum in each column of matrix $\boldsymbol{W}$. At the same time, we assume that the sum of the components of vector $\boldsymbol{w}$ is equal to $\boldsymbol{s_w} = \sum_i w_i$:

$$\boldsymbol{s_k} = \boldsymbol{e}^T \cdot \boldsymbol{W} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}^{\boldsymbol{T}} \cdot \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & & \frac{w_3}{w_n} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix} = \begin{bmatrix} \frac{s_w}{w_1} \\ \frac{s_w}{w_2} \\ \frac{s_w}{w_3} \\ \vdots \\ \frac{s_w}{w_n} \end{bmatrix}^T \tag{4}$$

By dividing the columns of matrix $\boldsymbol{W}$ by the sums $\boldsymbol{s_k}$ we obtain the normalized matrix $\boldsymbol{W}$ whose structure is shown in (5):

$$\boldsymbol{W_N} = \boldsymbol{W} \cdot diag\left(\frac{1}{\boldsymbol{s_k}}\right) = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & & \frac{w_3}{w_n} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \cdot diag\begin{pmatrix} \frac{w_1}{s_w} \\ \frac{w_2}{s_w} \\ \frac{w_3}{s_w} \\ \vdots \\ \frac{w_n}{s_w} \end{pmatrix} = \\ = \begin{bmatrix} \frac{w_1}{s_w} & \frac{w_1}{s_w} & \frac{w_1}{s_w} & & \frac{w_1}{s_w} \\ \frac{w_2}{s_w} & \frac{w_2}{s_w} & \frac{w_2}{s_w} & \cdots & \frac{w_2}{s_w} \\ \frac{w_3}{s_w} & \frac{w_3}{s_w} & \frac{w_3}{s_w} & & \frac{w_3}{s_w} \\ \vdots & & & \ddots & \vdots \\ \frac{w_n}{s_w} & \frac{w_n}{s_w} & \frac{w_n}{s_w} & \cdots & \frac{w_n}{s_w} \end{bmatrix} \tag{5}$$

By performing only one iteration of the exponential method, we obtain the result shown in (6):

$$\boldsymbol{w_{(1)}} = \boldsymbol{W_N} \cdot \boldsymbol{w_{(0)}} = \begin{bmatrix} \frac{w_1}{s_w} & \frac{w_1}{s_w} & \frac{w_1}{s_w} & & \frac{w_1}{s_w} \\ \frac{w_2}{s_w} & \frac{w_2}{s_w} & \frac{w_2}{s_w} & \cdots & \frac{w_2}{s_w} \\ \frac{w_3}{s_w} & \frac{w_3}{s_w} & \frac{w_3}{s_w} & & \frac{w_3}{s_w} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{s_w} & \frac{w_n}{s_w} & \frac{w_n}{s_w} & \cdots & \frac{w_n}{s_w} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{n \cdot w_1}{s_w} \\ \frac{n \cdot w_2}{s_w} \\ \frac{n \cdot w_3}{s_w} \\ \vdots \\ \frac{n \cdot w_n}{s_w} \end{bmatrix} \tag{6}$$

Moreover, it is easy to see that when we divide the resulting vector $\boldsymbol{w_{(1)}}$ by *n*, we obtain the normalized scale vector $\boldsymbol{w_n}$ since $\sum_i w_{n_i} = 1$. The relevant calculations are in formula (7):

$$\boldsymbol{w_n} = \frac{\boldsymbol{w_1}}{n} = \begin{bmatrix} \frac{w_1}{s_w} \\ \frac{w_2}{s_w} \\ \frac{w_3}{s_w} \\ \vdots \\ \frac{w_n}{s_w} \end{bmatrix} = \frac{1}{s_w} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_n \end{bmatrix} = \frac{1}{s_w} \cdot \boldsymbol{w} \tag{7}$$

As mentioned before, in general we do not know the scale vector $\boldsymbol{w}$, only the pairwise comparison matrix $\boldsymbol{W}$. However, by performing the calculations shown in formulas (4) through (7), we can determine the scale vector $\boldsymbol{w}$ on the basis of matrix $\boldsymbol{W}$.

2.2 The occurrence of inconsistency in the AHP

The method of determining the scale vector $\boldsymbol{w}$ described in the previous subsection is correct as long as the order of the pairwise comparison matrix $\boldsymbol{W}$ is equal to 1. This is because the pairwise comparisons led to a consistent matrix $\boldsymbol{W}$. Unfortunately, in general, matrix $\boldsymbol{W}$ is not always consistent and therefore it is necessary to find out by how much the eigenvalue obtained exceeds n. In the case of a consistent matrix, the relationships in (8) and (9) are true. On the basis of their construction it is possible to determine the extent to which the maximal eigenvalue differs from the theoretical quantity n:

$$\boldsymbol{W} \cdot \boldsymbol{w}_n = n \cdot \boldsymbol{w}_n \tag{8}$$

$$\boldsymbol{W} \cdot \boldsymbol{w}_n = \begin{bmatrix} \frac{w_1}{w_1} & \frac{w_1}{w_2} & \frac{w_1}{w_3} & & \frac{w_1}{w_n} \\ \frac{w_2}{w_1} & \frac{w_2}{w_2} & \frac{w_2}{w_3} & \cdots & \frac{w_2}{w_n} \\ \frac{w_3}{w_1} & \frac{w_3}{w_2} & \frac{w_3}{w_3} & & \frac{w_3}{w_n} \\ & \vdots & & \ddots & \vdots \\ \frac{w_n}{w_1} & \frac{w_n}{w_2} & \frac{w_n}{w_3} & \cdots & \frac{w_n}{w_n} \end{bmatrix} \cdot \begin{bmatrix} \frac{w_1}{s_w} \\ \frac{w_2}{s_w} \\ \frac{w_3}{s_w} \\ \vdots \\ \frac{w_n}{s_w} \end{bmatrix} = \begin{bmatrix} \frac{n \cdot w_1}{s_w} \\ \frac{n \cdot w_2}{s_w} \\ \frac{n \cdot w_3}{s_w} \\ \vdots \\ \frac{n \cdot w_n}{s_w} \end{bmatrix} \tag{9}$$

For this purpose, we divide the obtained vector (the right-hand side of (9)) by the consecutive components of $\boldsymbol{w_n}$. In the case of a consistent matrix we obtain vector $[n \quad n \quad n \quad \cdots \quad n]^T$, for which the average of the elements λ_{max} is n. In general, this average can have another value, and therefore we determine the consistency index $c_i = \frac{\lambda_{max}}{n-1}$. This index is the arithmetic mean of the eigenvalues of matrix $\boldsymbol{W}$, calculated omitting the largest eigenvalue. If the pairwise comparison matrix is consistent, then $c_i = 0$. Since this index depends on the size of matrix $\boldsymbol{W}$, Saaty proposed to correct the value of c_i by a certain random index which takes into account the size of the matrix under discussion. The consistency index $c_r = \frac{c_i}{r_i}$, where r_i is a certain random index, allows to check if matrix $\boldsymbol{W}$ is inconsistent. We assume that the pairwise comparison matrix is consistent if $c_r < 10\%$.

3 A proposal to eliminate the inconsistency in pairwise comparisons

An essential obstacle in applying the AHP are frequently occurring problems with inconsistency of pairwise comparisons. In many problems, especially those related to large-size matrices, the value of the consistency index c_r significantly exceeds the acceptable threshold of 10%. To obtain a consistent matrix, we have to correct the results of pairwise comparisons. Since matrix $\boldsymbol{W}$ reflects the decision maker's preferences, it is justified to allow him/her to participate in the correction of its contents. This approach requires additional activity from the decision maker. The proposed method analyzes the pairwise comparison matrix and points out the elements to be corrected to the decision maker. Moreover, the method suggests to him/her the values of the evaluations of the elements being corrected.

The proposed algorithm for eliminating inconsistency consists of the following steps:

1. Determine the scale vector $\boldsymbol{w_n}$ using the AHP method and check the consistency index c_r.
2. If $c_r < 0.1$, end the calculations, otherwise go to the next step.
3. Determine the new pairwise comparison matrix $\boldsymbol{W}$ from formula (11).
4. On the basis of matrix $\boldsymbol{W_s}$ and Saaty's scale determine the new proposals of pairwise comparisons.
5. Ask the decision maker to accept the proposed pairwise comparisons or to present the new evaluations of pairwise comparisons from matrix $\boldsymbol{W}$ (in particular, those values which differ most from the proposal).
6. If the decision maker accepts the new comparisons, end the calculations, otherwise go to Step 3.

4 Examples of applications

In the next two subsections we present examples illustrating applications of the proposed algorithm. The first example describes a problem in which the decision maker supplied exceptionally inconsistent evaluations of the individual variants, revealing in the consecutive iterations that according to his/her preferences, the variants compared differ only slightly from each other. During this process a transition from the classic Saaty scale 1-9 to the scale 1.0-1.8 is effected; this scale is proposed in the present paper. The next example deals with a problem described in Saaty (2003, p. 88).

4.1 The problem of an inconsistent pairwise comparison matrix

Let us consider three decision variants: a, b and c, which were evaluated by the decision maker as follows: $a > b$, $b > c$ and $c > a$ (sic!). The pairwise comparison matrix reflecting these preferences is shown in (10):

$$\boldsymbol{W} = \begin{bmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ \mathbf{9} & \frac{1}{9} & 1 \end{bmatrix} \tag{10}$$

Using the AHP we obtain a scale vector of the form $\boldsymbol{w_n} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}^{\boldsymbol{T}}$ and $c_r = 6.84 \gg 0{,}1$ (for $r_i = 0.52$). In our calculations we used the fact that the sums of the elements in the consecutive rows and columns were identical. Since c_r indicates that matrix $\boldsymbol{W}$ is strongly inconsistent, we propose the corrected matrix $\boldsymbol{W_s}$ to the decision maker. Our proposal consists in reconstructing the pairwise comparison matrix on the basis of $\boldsymbol{w_n}$, according to formula (11), which in turn is based on the relationship described in (2):

$$\boldsymbol{W_s} = \boldsymbol{w_n} \cdot \left(\frac{\mathbf{1}}{\boldsymbol{w_n}}\right)^{\boldsymbol{T}} \tag{11}$$

By performing the calculations we obtain the corrected pairwise comparison matrix, for which $c_r = 0$:

$$\boldsymbol{W_s} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{12}$$

It is easy to see that this proposal consists in assuming that all three variants are equivalent: $a = b = c$.

We assume that the decision maker, knowing the new matrix $\boldsymbol{W_s}$ modifies his/her evaluations and expresses them in a new matrix, shown in (13):

$$\boldsymbol{W} = \begin{bmatrix} 1 & 5 & \frac{1}{2} \\ \frac{1}{5} & 1 & 5 \\ 2 & \frac{1}{5} & 1 \end{bmatrix} \tag{13}$$

For this pairwise comparison matrix, the scale vector is $\boldsymbol{w_n} = [0.4 \quad 0.33 \quad 0.27]^{\boldsymbol{T}}$ and the consistency index is $c_r = 1.72 \gg 0.1$. Unfortunately, we have again obtained an inconsistent matrix $\boldsymbol{W}$. The corrected pairwise comparison matrix, shown in (14), has been determined from formula (11):

$$\boldsymbol{W_s} = \begin{bmatrix} 1 & 1.204 & 1.474 \\ 0.830 & 1 & 1.224 \\ 0.678 & 0.817 & 1 \end{bmatrix} \tag{14}$$

Using the 1.0-1.8 scale, we obtain the matrix shown in (15), which we present to the decision maker for evaluation. The consistency index of this matrix is $c_r = 0$:

$$\boldsymbol{W} = \begin{bmatrix} 1 & \mathbf{1.2} & \mathbf{1.5} \\ 0.833 & 1 & \mathbf{1.2} \\ 0.667 & 0.833 & 1 \end{bmatrix} \tag{15}$$

The decision maker finds that the proposed matrix correctly reflects his/her preferences.

4.2 An example from Saaty's paper

The next example is related to the problem of buying a house, with eight criteria taken into account. The decision maker expressed his/her preferences in the form of a pairwise comparison matrix, shown in Table 2.

Table 2: Pairwise comparison matrix $\boldsymbol{W}$ for the problem of buying a single family home for the given criteria

	Size	**Trans.**	**Nbrhd**	**Age**	**Yard**	**Modern**	**Cond.**	**Finance**
Size		5	3	7	6	6		
Trans.				5	3	3		
Nbrhd.		3		6	3	4	6	
Age								
Yard				3				
Modern				4	2			
Cond.	3	5		7	5	5		
Finance	4	7	5	8	6	6	2	

* $\lambda_{max} = 9.618$, $c_i = 0.231$, $r_i = 1.4$, $c_r = 0.165$.

Source: Saaty (2003, p. 88).

Using the AHP we conclude that the matrix in Table 2 is not consistent. From formula (11) we determine the corrected matrix, shown in Table 3.

Using Saaty's scale for the matrix from Table 3, we obtain a new pairwise comparison matrix, shown in Table 4. We assume that the decision maker accepts the proposed corrections. The consistency index decreased from 23.1% to 1%.

Table 3: Pairwise comparison matrix reconstructed from (11)

	Size	Trans.	Nbrhd	Age	Yard	Modern	Cond.	Finance	w_i
Size	1.000	2.639	1.025	9.227	4.947	3.922	0.977	0.558	1.000
Trans.	0.379	1.000	0.388	3.497	1.875	1.486	0.370	0.212	0.379
Nbrhd.	0.976	2.575	1.000	9.003	4.827	3.827	0.953	0.545	0.976
Age	0.108	0.286	0.111	1.000	0.536	0.425	0.106	0.061	0.108
Yard	0.202	0.533	0.207	1.865	1.000	0.793	0.197	0.113	0.202
Modern	0.255	0.673	0.261	2.352	1.261	1.000	0.249	0.142	0.255
Cond.	1.024	2.701	1.049	9.444	5.063	4.015	1.000	0.572	1.024
Finance	1.791	4.726	1.835	16.524	8.859	7.025	1.750	1.000	1.791

Source: Author's own calculations.

Table 4: A correct pairwise comparison matrix, based on Table 3 and after the application of Saaty's scale

	Size	Trans.	Nbrhd	Age	Yard	Modern	Cond.	Finance	w_i
Size		3	1	9	5	4			0.151
Trans.				3	2	1			0.052
Nbrhd.		3		9	5	4			0.151
Age									0.019
Yard				2					0.032
Modern				2	1				0.038
Cond.	1	3	1	9	5	4			0.151
Finance	2	5	2	9	9	7	2		0.259

* $\lambda_{max} = 8.068$, $c_i = 0.010$, $r_i = 1.4$, $c_r = 0.007$.

Source: Author's own calculations.

Analyzing the data from Table 4 we can see that three categories are regarded by the decision maker as equivalent. Table 5 shows the matrix corrected according to Saaty's proposal. For this matrix the consistency index is equal to 8.1% and is significantly higher than that for the matrix from Table 4.

Table 5: The corrected pairwise comparison matrix $\boldsymbol{W}$ for the problem of buying a family home

	Size	Trans.	Nbrhd	Age	Yard	Modern	Cond.	Finance	w_i
Size		5	3	7	6	6			0.175
Trans.				5	3	3			0.062
Nbrhd.		3		6	3	4			0.103
Age									0.019
Yard				3					0.034
Modern				4	2				0.041
Cond.	3	5	2	7	5	5			0.221
Finance	4	7	5	8	6	6	2		0.345

* $\lambda_{max} = 8.811$, $c_r = 0.083$.

Source: Saaty (2003, p. 90).

5 Summary

In this paper the author presented an iterative method of eliminating inconsistency of pairwise comparison matrices. The proposal allows to determine a consistent matrix in a single iteration. By applying the assumed scale of pairwise comparison evaluations we determine the corrected pairwise comparison matrix and present it to the decision maker for acceptance. If the decision maker does not accept the proposed changes, he/she can add necessary corrections of the pairwise comparison matrix, on the basis of the corrections proposed. This process, in which the decision maker plays an active role, lasts until a consistent matrix $\boldsymbol{W}$ is obtained. The proposed method facilitates finding out consistent preferences of the decision maker, especially in large-size problems.

This proposal removes one of the obstacles encountered by users of the AHP in complex problems. Another obstacle is the determination of random indices r_i for matrices of sizes larger than 30. For smaller matrix sizes, these indices are published, but unfortunately various authors give various lists of values for them. Another research direction will be related to the investigation of random indices used in research on consistency of pairwise comparison matrices and on a new construction of the consistency index.

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