ROCZNIKI FILOZOFICZNE
Volume LXVI, number 4 – 2018
DOI: http://dx.doi.org/10.18290/rf.2018.66.4-3en

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ON THE REPRESENTATION OF STATES OF AFFAIRS IN THE ANTINOMY OF *FUTURA CONTINGENTIA*

The problem of the existence of future contingent events constitutes a paradigmatic example of a philosophical problem in which existentially significant questions concerning the possibility of acting freely interweave with abstract principles of formal logic. It is not surprising, then, that the history of philosophy has seen numerous attempts to analyse it. Marcin Tkaczyk, unifying the majority of such attempts (or possibly all of them?), reconstructs in an original way the antinomy of future contingent events within first-order logic as a trilemma of three theorems:

A. Every past (or present) state of affairs is determined.
B. At least some future states of affairs are contingent.
C. Every state of affairs can be represented at any time.

From the formal viewpoint, the trilemma is founded on the observation that the set of the three theorems is inconsistent, even though each one of its subsets is consistent. A philosopher could add that every theorem taken separately seems more or less obvious (or is a consequence of one or another widely accepted philosophical idea); because of that, inconsistency constitutes a non-trivial philosophical problem. In my paper I would like to focus on the last assumption— the most controversial and arguably most interesting one. I will attempt to prove that the formalization of this assumption in the theory presented by Tkaczyk is not adequate for the problem of *futura contigentia*, or, more precisely, that the former is not sufficiently faithful as the latter's formal equivalent.[1]

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[1] Thus, I *assume* that the set of three sentences enumerated above is an adequate description of what has been called the problem of future contingent events in the history of philosophy. An epistemic warrant of such assumption is provided in comprehensive analyses presented in TKACZYK 2015.

1. THE ANTINOMY OF *FUTURA CONTIGENTIA* IN FIRST-ORDER LOGIC

The formalization in question is a first-order theory with two specific predicates: “$\mathbb{P}$” and “$\mathbb{C}$”. The predicates are understood as follows:

I. $\mathbb{P}(x,y)$ — the state of affairs x is earlier than the state of affairs y;
II. $\mathbb{C}(x,y)$ — the state of affairs x is contingent with respect to the state of affairs y.

In addition to that, I will use two defined predicates:

$$\mathbb{S}(x,y) \triangleq \forall z[(\mathbb{P}(x,z) \equiv \mathbb{P}(y,z)) \wedge (\mathbb{P}(z,x) \equiv \mathbb{P}(z,y))] \tag{1}$$

$$\mathbb{E}(x,y) \triangleq \forall z[(\mathbb{C}(x,z) \equiv \mathbb{C}(y,z)) \wedge (\mathbb{C}(z,x) \equiv \mathbb{C}(z,y))] \tag{2}$$

The assumptions of the antinomy mentioned above are formalized by means of the following formulas:

$$\forall x,y\,(\mathbb{P}(x,y) \rightarrow \neg\mathbb{C}(x,y)). \tag{3}$$

$$\exists x,y(\mathbb{P}(x,y) \wedge \mathbb{C}(y,x)). \tag{4}$$

$$\forall x,y \exists z(\mathbb{S}(z,x) \wedge \mathbb{E}(z,y)). \tag{5}$$

Tkaczyk (2015) shows that the theory which consists of these three formulas taken as axioms (and the definitions 1 i 2) is inconsistent and that each one of its subtheories is consistent, which constitutes the trilemma.

In what follows I will argue that the formula 5 is not an adequate formalization of the statement that every state of affairs can be represented at any time. For the sake of simplicity, I will call the formula in question the *representation axiom*, and the statement of which it is a formal expression — the *representation assumption*.

2. FROM THE REPRESENTATION ASSUMPTION TO THE REPRESENTATION AXIOM

At first sight, the representation assumption is not identical or even equivalent to the representation axiom. After all the assumption contains a reference to temporal objects (“at any time”) and has a modal component (“can be”) — the elements at least *prima facie* absent from the axiom. Nonetheless, in the paper published in the this issue of *Roczniki Filozoficzne*, M. Tkaczyk describes in considerable detail the thought process resulting in the representation assumption becoming the representation axiom. The process can be seen as consisting of three steps.

The first step comes down to the description of the relation of representation:

> To put it generally, a representative of the state of affairs x is the state of affairs in which x is reflected. By reflection we should here understand a copy, in particular respects, of the original state of affairs. Thus, saying that a state of affairs y is a representative of a state of affairs x means claiming that in particular respects
>
> a state of affairs y is similar to a state of affairs x, (6a)
>
> a state of affairs y is an effect of a state of affairs x. (6b)
>
> In the condition (6b), instead of saying that a state of affairs y is an effect of a state of affairs x, one could say that a state of affairs x is the cause of the similarity between states of affairs x and y. Not every state of affairs is fit to be a representative of other states of affairs, but only the one that possesses the ability to reflect described above. This ability is possessed by, among other things, propositions, judgments, and beliefs. What is more, it is propositions, judgments, and beliefs that are typical representatives, although they are not the only possible candidates for that role. The term "representative" usually indicates that both conditions (6) are fulfilled. Often it is enough to focus on the condition (6a). In that case one can use the term "equivalent", which expresses a symmetric relation: the original state of affairs and its representative are (in particular respects) equivalents of each other. (TKACZYK 2018, 8).

In the second step, Tkaczyk supplements the description with a modal component:

> In the context of the problem of future contingents a representative has to be similar to the original in terms of modality. If a state of affairs x is a representative of a state of affairs y, then, for any state of affairs z, x is necessary, impossible or contingent with reference to the event z if and only if y is, respectively, necessary, possible or contingent with reference to z. In other words, a state of affairs x agrees with a state of affairs y in terms of modal properties with reference to any state of affairs z. (TKACZYK 2018, 8).

The third step is not clearly marked in the text. It *seems* to consist of (i) his introducing the definition 2, which fulfills the condition of "modal agreement" formulated in the above quote, and (ii) taking the relation in question as the relation of representation: x is a representation of y if and only if x agrees with y in terms of modality, i.e. if and only if $\mathbb{E}(x,y)$. Of course, reducing the relation of representation to $\mathbb{E}$ in this way omits the condition (6b). Such a step, which I consider somehow controversial, has been justified in the following way (TKACZYK 2018, 9):

However, one can limit oneself to the aspect of similarity described in the condition (6a), passing over the aspect of asymmetry between the original state of affairs and its representation expressed in the condition (6b). Because of that, a broader notion of equivalent rather than a narrower notion of representative is formalized in the axiom (9). This does not influence formal results, but makes the constructed theory much simpler and less exposed to errors, taking into consideration all the factors relevant for the antinomy of future contingents.

With this interpretation of the definition, the representation axiom turns out to be a direct formal equivalent of the representation assumption because it has it that for every two events there is an event "temporally equivalent" to one of them which can be a representation of the other one. If we identify the moments in time with the classes of abstraction of the relation $\mathbb{S}$, we will be able to say that the representation axiom states that for every event and every moment there exists an event which takes place at that moment and can be a representation of that first event (in the sense of $\mathbb{E}$).

As I mentioned earlier, I consider the aforementioned argument to be defective. In this section I will present the philosophical reasons for my position, while the next one will describe the formal reasons why such a translation seems to me inadequate.

In order to be specific I will limit my considerations to one type of representations discussed above — namely, to beliefs. Thus limited, the representation assumption states that for any state of affairs, there can exist at any moment the belief that represents that state of affairs. However, then the third step of the discussion becomes doubtful at best. That is because the representation axiom states that for every state of affairs, at any time, there is a state of affairs similar to it with respect to $\mathbb{E}$, where the sense of the similarity in question is established by the definition 2. I claim that, in general, this kind of similarity does not obtain between beliefs and states of affairs which they represent.

Let us consider some necessary state of affairs — e.g. that it is not the case that helium has and does not have 2 as its atomic number — and my current belief representing it.[2] As opposed to the state, my belief is contingent (in the temporal sense): even though it exists at a given moment, it did not exist a day before and will probably not exist next year. If it exists at

[2] If this (admittedly peculiar) example of a state of affairs does not meet someone's definition of states of affairs, we can consider instead the (physically necessary) state of affairs of helium's atomic number being 2 or (ontologically necessary) state of affairs of there being a reason for helium's atomic number being 2.

some point in time, it does not have to exist at that point, since it is ontically dependent on my equally contingent cognitive abilities, volitional whims, more or less random circumstances etc. If we limit ourselves to human beliefs, none of such beliefs inherits the necessity from the states of affairs it represents, since every such belief is at least as contingent as the person who holds it. Perhaps we could speak of some conditional necessity of some features of that belief — e.g. if the belief represents some necessary state of affairs, then, as long as that belief exists, it is true out of necessity. However, the belief itself is not necessary, i.e. it is not the case that it has to exist at every moment of time, and, because of that, none of its attributes are essential to it. Thus, it is not the case that a state of affairs is necessary if and only if its representation is necessary. Assuming that necessary states of affairs are not contingent with respect to other states of affairs, it is not the case that if a state of affairs x is a representation of a state of affairs y, then, for any state of affairs z, x is necessary, impossible or contingent with respect to z if and only if y is, respectively, necessary, possible or contingent with respect to z.

To put it in more general terms, beliefs or propositions can be contingent with respect to events or states of affairs with respect to which the states of affairs represented by them are not contingent — or, to put it more simply: beliefs can be ontologically dependent on events on which events represented by them do not depend.

Similar remarks can be made about the remaining examples of the representations of states of affairs, i.e. about propositions and statements — though I have to say that it is much less clear to me in what sense one can talk about propositions or statements as states or events.

Thus, there are ontological reasons undermining the modal equivalence between states of affairs and their representations. Moreover, as noted by Tkaczyk, the formal features of the relation $\mathbb{E}$ are different from those of the relation of representation: only the former relation is reflexive and symmetrical. Propositions, beliefs etc., represent certain states of affairs or events, but the represented states of affairs or events do not themselves represent propositions or events. In addition to that, it is rare for a proposition or belief to represent itself; an exception from that rule are self-reflexive propositions, including semantic antinomies. The states of affairs or events being represented usually do not have the ontic nature or structure which would allow them to represent the propositions or beliefs by which they are represented. In other words, generally speaking, such states of affairs or events, because of what they are, cannot represent the beliefs or states of affairs which represent them. If that is the case, the relation of representation, as understood in the representation assumption, is not symmetrical or reflexive.

As opposed to Tkaczyk, I do not think that the difficulties I have just described can be solved by generalizing the representation relation or the notion of "representation". With the traditional understanding of the generalization operation, its result is broader in scope than the relation or term being generalized, so if some of the paradigmatic cases of this relation or notion do not meet the definition 2, some paradigmatic cases of its generalization share that characteristic.

Finally, I would like to point out that the representation axiom does not state that for every state of affairs there *can exist* at any time a state of affairs similar to it with respect to the relation $\mathbb{E}$ but that such a state *does* exist. In this sense, the modal component of the representation assumption is omitted (or at least hidden) in the axiom.

3. FORMAL CONSEQUENCES OF THE REPRESENTATION AXIOM

There are also some formal reasons against interpreting the representation assumption by means of the representation axiom. The representation assumption does not explicitly state any formal properties of time. Considering its role in the antinomy, one can probably say that the assumption does not imply by itself (i.e. should not imply) any non-trivial claims concerning such properties. $\mathbb{P}$ can be symmetrical or not, transitive or not, etc. The reason for that is that fact that the representation assumption concerns the relation of representation between states of affairs and states that such a relation can relate states of affairs that occur at any moment—while, as we will shortly see, the representation axiom imposes on the relation of temporal succession certain non-trivial properties, or, more precisely, some intuitively "admissible" structures of relations $\mathbb{P}$ and $\mathbb{C}$ are not among its models. Because of that, the axiom rules out some formal properties of the relations in question—the properties which seem neutral with respect to the problem of future contingent events.

First, I would like to emphasise a certain problem from the methodology of formal philosophy. Let "$\mathbb{M}$" denote the theory consisting of the axiom 5 and definitions 1 and 2. Like every theory formalizing some philosophical discourse, the theory in question has unintended models in the form of the structures which satisfy those formulas but which are not related to time, states of affairs, their modal properties etc. For example, as shown by Tkaczyk, we can interpret the theory $\mathbb{M}$ in the model whose elements are numbers. Of course, it is hard to expect that a formalization of the antinomy of *futura contingentia* be a theory which has only intended models. Nonetheless, if we consider this particular context, it seems that intended models should include, among others, the models that satisfy the following formulas:

$$\exists x, y\ \mathbb{P}(x, y). \tag{6}$$

$$\exists x, y\ \mathbb{C}(x, y). \tag{7}$$

$$\exists x, y\ \neg\mathbb{P}(x, y). \tag{8}$$

$$\exists x, y\ \neg\mathbb{C}(x, y). \tag{9}$$

If 6 is false, then states of affairs are not related by any temporal connections, and, since there are no future events, the problem of whether some future events are contingent is trivial. Thus, 6 is a kind of a presupposition for the problem we are discussing here. Similar remarks can be formulated with respect to the remaining formulas — maybe with the exception of 8.

Coming back to the formal properties of the relation $\mathbb{P}$ determined by the axiom 5, let us consider first the world or situation, in which there are three (not necessarily different!) states of affairs: a,b and c, such that a is earlier than b, b is earlier than c, and no two other states of affairs stand in this relation to each other (i.e. b is not earlier than a, b is not earlier than b, etc.).[3]

Let us consider the theory $\mathbb{M}$ with the following axioms added to it[4]:

$$\mathbb{P}(a, b) \wedge \mathbb{P}(b, c). \tag{10}$$

$$\forall x, y[\mathbb{P}(x, y) \rightarrow (x = a \wedge y = b) \vee (x = b \wedge y = c)]. \tag{11}$$

The theory has an interesting feature: in its every model the relation $\mathbb{C}$ is interpreted as either an empty or a universal set: if we add to those axioms the formulas 7 and 9, the result is going to be an inconsistent theory.[5] The axiom 5 (or, more precisely, the theory $\mathbb{M}$) does not rule out the situation described by 10, 11, 7 and 9.

[3] Technically, as a part of this description one could also add that is earlier than .

[4] The letters and are here understood as individual constants. Thus, strictly speaking, adding these formulas towould require extending the language of the theory to include individual constants (and the symbol of identity).

[5] The proof of this and other theorems from the text can be found in the appendix.

First, let us notice that the representation assumption does not *prima facie* rule out the situation described by 10 and 11. Even though the world we inhabit does not consist solely of the three states of affairs linearly ordered by the relation of temporal succession, the assumptions of the antinomy do not implicate that such a world is impossible. In other words, the assumptions would be satisfied even if a world they concern had such an impoverished temporal structure. In such a case, however, the representation axiom implicates (in the enthymematic sense) the particular solution of the problem of *futura contigentia*: since, according to an elementary ontological intuition, it is not true that everything is contingent with respect to everything, nothing is contingent, i.e. not only future contingent states of affairs do not exist but also no state of affairs is contingent.

In general, the theory $\mathbb{M}$ rules out not only such a world but also the possibility of any world in which (i) the relations $\mathbb{P}$ and $\mathbb{C}$ are non-empty while $\mathbb{C}$ is non-trivial, and (ii) the relation $\mathbb{P}$ is asymmetrical and linear in the future.[6]

In other words, $\mathbb{M}$ when extended by axioms 6, 7, 9, 13, and 14, is inconsistent.

$$\forall x, y[\mathbb{P}(x, y) \rightarrow \neg\mathbb{P}(y, x)]. \tag{13}$$

$$\forall x, y, z[\mathbb{P}(x, y) \wedge \mathbb{P}(x, z) \wedge y \neq z \rightarrow \mathbb{P}(y, z) \vee \mathbb{P}(z, y)]. \tag{14}$$

Things work in a similar way in the case of the linearity in the past: $\mathbb{M}$ extended by 6, 7, 9, 13, and 15 is inconsistent.

$$\forall x, y, z[\mathbb{P}(y, x) \wedge \mathbb{P}(z, x) \wedge y \neq z \rightarrow \mathbb{P}(y, z) \vee \mathbb{P}(z, y)]. \tag{15}$$

In addition to that, $\mathbb{M}$ rules out also asymmetry and linearity (provided that $\mathbb{C}$ is non-empty and non-trivial) because $\mathbb{M} \cup \{7, 9, 13, 16\}$ is inconsistent.

$$\forall x, y(\mathbb{P}(x, y) \vee \mathbb{P}(y, x) \vee x \neq y). \tag{16}$$

Finally, $\mathbb{M}$ rules out (i) $\mathbb{C}$ being non-empty and non-trivial and (ii) $\mathbb{P}$ is asymmetrical and has the first (or, respectively, the last) element, since the theories $\mathbb{M} \cup \{7, 9, 13, 17\}$ and $\mathbb{M} \cup \{7, 9, 13, 18\}$ are inconsistent.

$$\exists x \forall y[x \neq y \rightarrow \mathbb{P}(x, y)]. \tag{17}$$

$$\exists x \forall y[x \neq y \rightarrow \mathbb{P}(y, x)]. \tag{18}$$

[6] It is worth noting that all the statements concerning the extensions of the theorywhich include the feature of asymmetricality can be proven in stronger versions, in which the feature in question is replaced by antisymmetricality; the proofs of both versions can be found in the appendix.

$$\forall x, y(\mathbb{P}(x, y) \wedge \mathbb{P}(y, x) \rightarrow x = y). \tag{12}$$

I have thus shown that there are the classes of the structures of relations $\mathbb{P}$ and $\mathbb{C}$ which are not models of the theory $\mathbb{M}$ even though the relations in question have in these models "admissible" formal properties or at least the properties that are considered in some standard systems of temporal logic.

Let us also note that one can get the antinomy of *futura contingentia* by weakening the assumption 4 to the following form:

$$\exists x, y\mathbb{P}(x, y) \wedge \exists x, y\mathbb{C}(x, y). \tag{19}$$

In other words, the theory of the axioms 3, 19, i 5 (and the definitions 1 and 2), is also inconsistent, and each one of its subtheories is consistent.

While the formula 4 can be a formal expression of the statement "At least some future states of affairs are contingent", the formula 19 cannot. 4 tells us that some state of affairs are contingent with respect to (some) earlier states of affairs. 19 tells us only that some states of affairs are temporally ordered, i.e. earlier than (some) states of affairs, and that some states of affairs are contingent with reference to (some) states of affairs. The contingency in question, however, is not correlated with the temporal order: 19 allows that a state of affairs can be contingent with respect to a state of affairs which is earlier, contemporary, or later or even not correlated temporally in any way. Therefore one can say that 19 can constitute a formal expression of the statement "Some states of affairs are future and some states of affairs are contingent". Still, such a statement does not say much — only that the relations $\mathbb{P}$ and $\mathbb{C}$ are non-empty.

It is interesting that 19 implies 4 in $\mathbb{M}$, i.e., that that (some states of affairs are future and some states of affairs are contingent) implies that some future states of affairs are contingent.

Similarly, in $\mathbb{M}$ 19 implies

$$\exists x, y(\mathbb{P}(x, y) \wedge \mathbb{C}(x, y)). \tag{20}$$

This means that that (some states of affairs are future and some states of affairs are contingent) implies that some past states of affairs are contingent.

Thus, the axiom 5 eliminates the temporal aspect from the problem in question. Speaking somehow metaphorically, one can say that the formalization we are discussing concerns not as much the antinomy of *futura contingentia* as the antinomy of *contingentia*.

4. MODIFICATIONS OF THE REPRESENTATION AXIOM

From the formal viewpoint, it is interesting that some problems stemming from the representation axiom can be solved be a modication of the definition 2 that “weakens” its *definiens*. Namely, the antinomy of *futura contingentia* can be achieved in the theory in which this definition is replaced by 21:

$$\mathbb{E}(x,y) \triangleq \forall z[(\mathbb{C}(y,z) \to \mathbb{C}(x,z)) \wedge (\mathbb{C}(z,y) \to \mathbb{C}(z,x))] \tag{21}$$

The theory $\{1, 21, 3, 4, 5\}$ is inconsistent, while each one of its subtheories is consistent, which leads to a trilemma similar to the previous one.

Such a modification of the representation axiom removes some of the above-mentioned undesirable properties of the models of the axiom 5. In order to see that, let us consider the following structure $< D, \{P, C\} >$, where:

- $D = \{0,1,2,3\}$;
- $P = \{< 2,3 >\}$;
- $C = \{< 0,0 >, < 0,2 >, < 0,3 >, < 2,0 >, < 2,2 >, < 2,3 >, < 3,0 >, < 3,2 >, < 3,3 >\}$.

One can easily see that the model based on this structure (i.e. the one in which P is an interpretation of $\mathbb{P}$ and C is an interpretation of $\mathbb{C}$) is a model of the theory $\{1, 5, 21\}$ (a competitor of $\mathbb{M}$), in spite of the fact that relations P and C are non-empty and non-trivial, whereas P is asymmetrical, linear in the future and linear in the past, while also having the first and last element.

Of course, such a modification is just a formal trick which does not remove all the problems we have discussed. A contradiction can still result from replacing the axiom 4 with 19—so we still have a kind of atemporal version of the antinomy of *futura contingentia*. What is more, the modified theory, just like $\mathbb{M}$, rules out asymmetry and linearity of $\mathbb{P}$ (on the assumption of non-emptiness and non-triviality of $\mathbb{C}$).

The question arises whether the philosophical doubts expressed in the previous sections have been adequately addressed by the modification I described. The representing and the represented states of affairs are now not modally equivalent and the relation of representation is not symmetrical. In this respect, the modification takes into consideration the philosophical context outlined above. Nonetheless, $\mathbb{E}$ is still reflexive, so every representing state of affairs (also) represents itself. These are not the only interpretative difficulties. Let us assume, as before, that the relation $\mathbb{E}$ is a formal equivalent of the relation of representation which occurs in the representation assumption. On such an interpretation, the definition 21 does not state that the representing states of affairs are modally equivalent to the represented states, but only that:

A. a representing state of affairs is contingent with respect to every state of affairs with respect to which a represented state of affairs is contingent;
B. every state of affairs contingent with respect to a represented state of affairs is contingent with reference to a representing state.

To simplify a bit, A states that statements or beliefs depend on everything on which the states of affairs they represent depend. Since this philosophical claim is implied by (the conjunction of) (i) the statement that there is a modal dependency between the representing and represented state of affairs, and (ii) the statement expressing the transitivity of the relation $\mathbb{C}$, it does not seem to be too extravagant. Nonetheless, even if from the philosophical viewpoint 1 seems acceptable, things work very differently in the case of B, which states that (to simplify a little) if one state of affairs depends on another one, it is also dependent on every statement or belief representing that other state of affairs. Let us consider my belief that the Biblical Isaac granted his blessing to Jacob and some state of affairs dependent on Isaac's granting the blessing to Jacob — e.g. that Esau, Jacob's brother, came to hate Jacob. If what is stated by B was true, Esau's coming to hate Jacob would depend not only on Isaac's granting his blessing to Jacob but also on my belief that that was the case. Such case of the modal dependence of a past state of affairs on a somehow random future event would be peculiar and would require separate argument.[7]

The formalization offered by Tkaczyk can be modified even deeper, also by modifying the definition 2. Now we are going to replace it with the *axiom* 22:

$$\forall x, y\, (\mathbb{E}(x,y) \rightarrow \mathbb{C}(x,y)). \tag{22}$$

Of course, after such modification, the symbol "$\mathbb{E}$" becomes a primitive term.

The theory whose axioms are 1, 22, 3, 4 and 5 is inconsistent (and each of its subtheories is consistent).

What is a rationale for such a serious modification? Let us assume (this time, only for a while) that the relation $\mathbb{E}$ is to be understood in the following way:

[7] Of course, Tkaczyk's original argumentation is also open to this objection.

- $\mathbb{E}(x,y)$ — a state of affairs x represents a state of affairs y.

On this interpretation, the axiom 22 (along with 5) states that every state of affairs *is* represented at any time.

This modification focuses on another aspect of the relation of representation, which was omitted in the original axiomatization and which was marked as "(6b)". It states the causal dependency between the represented and representing states of affairs — although that notion should probably be made somehow more general. Let us consider the following way of reading the predicate "$\mathbb{C}$":

- $\mathbb{C}(x,y)$ — a state of affairs x is ontologically dependent on a state of affairs y.

Such interpretation has an advantage over the original proposal (i.e. the interpretation marked by II on the page 56) insofar as it refers to the relation of ontological dependency-the relation well known to philosophers at least since the time of E. Husserl and considered by many to be an indispensable component of every ontological system.[8]

Replacing definition 2 by axiom 4 results in a theory which, like the previous modification, has the models in which its predicates $\mathbb{P}$ and $\mathbb{C}$ are interpreted, respectively, by the relations P and C which are non-empty and non-trivial, whereas P is asymmetrical, linear in the future and linear in the past, while also having the first and last element — e.g. the model based on the structure $< D,\{P,C\} >$, where:

- $D = \{0,1,2,3\}$;
- $P = \{< 2,3 >\}$;
- $C = D \times D \setminus \{< 0,1 >,< 1,0 >,< 1,2 >,< 1,3 >\}$.

Let us note that informal interpretations of the modified axiomatization do not raise any serious doubts:

- 3 states now that no past state of affairs depends ontologically on future states of affairs.
- 4 states now that some future states of affairs do not depend ontologically on past states of affairs.
- 5 states now that at every moment every state of affairs is represented (by some state of affairs).

[8] For the recent debate concerning the relation of ontological dependency, see e.g. HOELTJE et al. 2013.

What can we say about the philosophical interpretation of axiom 22? The latter has it that the representing states of affairs (i.e. beliefs, statements, etc.) are ontologically dependent on the represented states. This claim seems to me much less controversial than the consequences of the definitions 2 and 21 which have been discussed earlier. If we assume that representing states of affairs are not just those which refer to the represented states of affairs but also do that effectively, i.e. are their adequate or true representations, it can be acknowledged that representing states of affairs such as beliefs are ontologically dependent on the states of affairs which they represent. Let us again consider my belief that the Biblical Isaac granted his blessing to Isaac. If my belief is true, it is also ontologically dependent on the past state of affairs which it describes — because the belief would undergo an essential change (i.e. would become false) if Isaac had not granted his blessing to Jacob. Because of that, the relation $\mathbb{E}$ should be understood in the following way:

- $\mathbb{E}(x, y)$ —a state of affairs x truthfully represents a state of affairs y .

Of course, these remarks concerning the philosophical validity of both modifications constitute a digression rather than proper argumentation, since an attempt at the latter should be grounded in a more extensive ontological system. For example, one would probably have to elaborate on the notion of â€śontological dependencyâ€ť used in the characteristics of the relation of representation (4). Let me just note that one option might be to use the notion of "truth-grounding" (see e.g. TAHKO 2013).

4 can be weakened to 6, i.e. the theory $\{1, 22, 3, 6, 5\}$ is also inconsistent. Thus, the current version of the antinomy of *futura contingentia* is in essence an enthymematic trilemma of three philosophical assumptions:

1. the minimal characteristics of the ontological relation of representation (in the form of the axiom 22).
2. the ontological thesis about the ontological independence of the past from the future (3)
3. the theorem about the universality of the relation of representation (5), where the assumed premise is 6.

An interesting consequence of such an interpretation of the antinomy is the lack of a separate assumption of ontic contingency in the world, which is now implied by the theorem about the "universality" of the relation of representation (and its ontological characteristics). Thus, in this version we are showing the incompatibility of the belief about the "closeness" of the past with the belief about the possibility of any state of affairs being represented at any moment in time. That is because if (true) representations of states of affairs can proceed the states of affairs being represented, then (at least) some past state of affairs can be ontologically dependent on future states of affairs, insofar as the relation of representation contains the relation of ontological dependency (in the sense of the axiom 22). In other words, if predicting the future is possible, the future can influence the past and the freedom of choice or the lack of necessity connected to it lose their significance. The antinomy of *futura contigentia* becomes then a kind of the antinomy of *praeterita representanta*, and a part of its metaphysical charm fades away.

Another consequence of the interpretation in question is the possibility of "blocking" the theological version of the antinomy. In some theories of the absolute being its ontic sovereignty (*aseitas*) rules out its ontological dependency and the ontological dependency of all of its attributes or aspects on the beings which are not within its "ontic range". In particular, the beliefs of the absolute being (or the corresponding type of representing states of affairs which can be ascribed to such being) would be ontologically independent from the states of affairs they represent. Thus, if in some accepted system of ontology we consider such a being (about which it can be safely assumed that it somehow represents any state of affairs at any time), we are allowed to question the assumption 22, thus neutralizing the antinomy. The beliefs of the absolute being cannot depend on the states of affairs which they represent. If there is any ontic dependency between the two, it has the opposite vector: it is states of affairs that depend on His beliefs.[9]

5. CONCLUSIONS

The aim of this paper is not as much criticizing the formalization of the antinomy of *futura contingentia* presented by M. Tkaczyk as pointing out at the possibility of a different interpretation of the antinomy — the interpretation which removes (or at least limits) some controversial aspects of the original formalization. All my efforts have focused on the representation axiom; as I mentioned in the introduction, I consider it the most interesting component of the antinomy. I have argued that the representation axiom is not an appropriate formal equivalent of the representation assumption, which

[9] Such ontological dependency can also lead to a certain theological antinomy — however, it is not going to be an antinomy connected to beliefs or judgments but rather one directly concerning human self-agency "confronted" with the universal action of the absolute being.

constitutes one of the three philosophical ideas forming the foundation of the antinomy. Or that the assumption can at least be formalized in a different way. As usual in such cases, the application of the tools of formal logic has revealed that the philosophical idea may have multiple interpretations.

By the way, I should remark that I share the general perspective sketched in (TKACZYK 2015) on how to solve the antinomy, though I do not consider the modification of the assumption 3 — in the form presented by Tkaczyk or in any other reasonable form — to be as epistemically intriguing as the discussion of the representation assumption. That, however, is a topic for another paper.

Translated by Sylwia Wilczewska

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ON THE REPRESENTATION OF STATES OF AFFAIRS IN THE ANTINOMY OF *FUTURA CONTINGENTIA*

Summary

The paper is a comment on the formalization of the antinomy of *futura contigentia* in the form of a (inconsistent) theory formulated by Marcin Tkaczyk in the language of classical predicate calculus. I argue that some features of the formalization in question are controversial from the viewpoint of formal semantics and ontology, and suggest two ways of removing some of those controversies.

O REPREZENTACJI STANÓW RZECZY W ANTYNOMII *FUTURA CONTINGENTIA*

Streszczenie

Artykuł jest komentarzem do formalizacji antynomii futura contigentia w postaci (sprzecznej) teorii sformułowanej przez Marcina Tkaczyka w języku klasycznej logiki predykatów. Argumentuję w nim, że formalizacja ta posiada pewne kontrowersyjne, z punktu widzenia ontologii i semantyki formalnej, własności oraz sugeruję dwa sposoby melioracji niektórych z tych kontrowersji.

Key words: antinomy; future contigents; formalization; classical predicate calculus; formal semantics; ontology.

Słowa kluczowe: antynomia; futura contingentia; formalizacja; semantyka formalna; ontologia.

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APPENDIX

The proofs presented here have been generated by the system of automated theorem proving Prover9. The notation used by the system is outlined in the documentation accessible on the website https://www.cs.unm.edu/mccune/mace4/ (last accessed: April 20, 2018). "$\vdash$" means the classical operator of consequence, and "$\vdash_{\mathbb{M}}$" means the operator of logical consequence for the theory $\mathbb{M}$. "$\perp$" means any counter-tautology.

Proof of theorem {10, 11, 7, 9} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y C(x,y))    [assumption].
5 (exists x exists y -C(x,y))   [assumption].
6 P(a,b) & P(b,c)   [assumption].
7 (all x all y (P(x,y) -> x = a & y = b | x = b & y = c))  [assumption].
10 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
12 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
16 S(f3(x,y),x).    [clausify(3)].
18 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
19 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
25 E(f3(x,y),y).    [clausify(3)].
26 C(c1,c2).    [clausify(4)].
27 -C(c3,c4).   [clausify(5)].
28 P(a,b).  [clausify(6)].
29 P(b,c).  [clausify(6)].
30 -P(x,y) | a = x | b = x.     [clausify(7)].
31 -P(x,y) | a = x | c = y.     [clausify(7)].
51 P(x,f3(y,z)) | -P(x,y).  [resolve(16,a,10,a)].
53 P(f3(x,y),z) | -P(x,z).  [resolve(16,a,12,a)].
70 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,18,a)].
71 C(x,f3(y,z)) | -C(x,z).  [resolve(25,a,19,a)].
72 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,20,a)].
73 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,21,a)].
107 P(b,f3(c,x)).   [resolve(51,b,29,a)].
110 P(f3(a,x),b).   [resolve(53,b,28,a)].
127 -C(c3,f3(x,c4)).    [ur(70,b,27,a)].
128 C(c1,f3(x,c2)).     [resolve(71,b,26,a)].
178 -C(f3(x,c3),f3(y,c4)).  [ur(72,b,127,a)].
203 a = b | f3(c,x) = c.    [resolve(107,a,31,a),flip(b)].
261 f3(a,x) = a | f3(a,x) = b.  [resolve(110,a,30,a),flip(a),flip(b)].
314 C(f3(x,c1),f3(y,c2)).   [resolve(128,a,73,b)].
625 a = b | -C(c,f3(x,c4)).     [para(203(b,1),178(a,1))].
649 a = b | C(c,f3(x,c2)).  [para(203(b,1),314(a,1))].
1228 a = b | -C(c,c).   [para(203(b,1),625(b,2)),merge(b)].
1315 a = b | C(c,c).    [para(203(b,1),649(b,2)),merge(b)].
1318 a = b.     [resolve(1315,b,1228,b),merge(b)].
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1528 f3(b,x) = b.   [back_rewrite(261),rewrite( [1318(1),1318(3),1318(5)]),merge(b)].
1571 -C(b,f3(x,c4)).    [para(1528(a,1),178(a,1))].
1595 C(b,f3(x,c2)).     [para(1528(a,1),314(a,1))].
1671 -C(b,b).   [para(1528(a,1),1571(a,2))].
1708 $F.    [para(1528(a,1),1595(a,2)),unit_del(a,1671)].
```

Proof of theorem {6, 7, 9, 14, 13} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))    [assumption].
5 (exists x exists y C(x,y))    [assumption].
7 (exists x exists y -C(x,y))   [assumption].
8 (all x all y all z (P(x,y) & P(x,z) & y != z -> P(y,z) | P(z,y)))    [assumption].
9 (all x all y (P(x,y) -> -P(y,x)))     [assumption].
13 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
14 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
18 S(f3(x,y),x).    [clausify(3)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
22 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
23 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
27 E(f3(x,y),y).    [clausify(3)].
28 P(c1,c2).    [clausify(4)].
29 C(c3,c4).    [clausify(5)].
31 -C(c7,c8).   [clausify(7)].
32 -P(x,y) | -P(x,z) | z = y | P(y,z) | P(z,y).    [clausify(8)].
33 -P(x,y) | -P(y,x).   [clausify(9)].
48 -P(x,f3(y,z)) | P(x,y).  [resolve(18,a,13,a)].
49 P(x,f3(y,z)) | -P(x,y).  [resolve(18,a,14,a)].
66 -C(f3(x,y),z) | C(y,z).  [resolve(27,a,20,a)].
67 C(f3(x,y),z) | -C(y,z).  [resolve(27,a,21,a)].
68 -C(x,f3(y,z)) | C(x,z).  [resolve(27,a,22,a)].
69 C(x,f3(y,z)) | -C(x,z).  [resolve(27,a,23,a)].
70 -P(x,x).     [factor(33,a,b)].
71 -P(c1,x) | c2 = x | P(c2,x) | P(x,c2).   [resolve(32,a,28,a),flip(b)].
88 P(c1,f3(c2,x)).  [resolve(49,b,28,a)].
105 -C(f3(x,c7),c8).    [ur(66,b,31,a)].
106 C(f3(x,c3),c4).     [resolve(67,b,29,a)].
109 -P(f3(x,y),x).  [ur(49,a,70,a)].
110 -P(x,f3(x,y)).  [ur(48,b,70,a)].
133 f3(c2,x) = c2.  [resolve(88,a,71,a),flip(a),unit_del(b,110),unit_del(c,109)].
213 -C(c2,c8).  [para(133(a,1),105(a,1))].
214 C(c2,c4).   [para(133(a,1),106(a,1))].
217 C(c2,f3(x,c4)).     [resolve(214,a,69,b)].
271 C(c2,c2).   [para(133(a,1),217(a,2))].
309 -C(c2,f3(x,c8)).    [ur(68,b,213,a)].
315 $F.    [para(133(a,1),309(a,2)),unit_del(a,271)].
```

Proof of theorem {6, 7, 9, 14, 12} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))    [assumption].
5 (exists x exists y C(x,y))    [assumption].
6 (exists x exists y -C(x,y))   [assumption].
7 (all x all y all z (P(x,y) & P(x,z) & y != z -> P(y,z) | P(z,y)))    [assumption].
8 (all x all y (P(x,y) & P(y,x) -> x = y)) [assumption].
10 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
11 S(f3(x,y),x).    [clausify(3)].
13 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
15 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
19 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
20 E(f3(x,y),y).    [clausify(3)].
22 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
24 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
28 P(c1,c2).    [clausify(4)].
29 C(c3,c4).    [clausify(5)].
31 -C(c5,c6).   [clausify(6)].
33 -P(x,y) | -P(y,x) | y = x. [clausify(8)].
34 -P(x,y) | -P(x,z) | z = y | P(y,z) | P(z,y).    [clausify(7)].
35 -P(f3(x,y),z) | P(x,z).  [resolve(10,a,11,a)].
37 P(f3(x,y),z) | -P(x,z).  [resolve(13,a,11,a)].
```

```
39 -P(x,f3(y,z)) | P(x,y).  [resolve(14,a,11,a)].
41 P(x,f3(y,z)) | -P(x,y).  [resolve(15,a,11,a)].
55 -C(f3(x,y),z) | C(y,z).  [resolve(19,a,20,a)].
57 C(f3(x,y),z) | -C(y,z).  [resolve(22,a,20,a)].
59 -C(x,f3(y,z)) | C(x,z).  [resolve(23,a,20,a)].
61 C(x,f3(y,z)) | -C(x,z).  [resolve(24,a,20,a)].
76 -P(c1,x) | c2 = x | P(c2,x) | P(x,c2).  [resolve(34,a,28,a),flip(b)].
81 P(c1,f3(c2,x)).  [resolve(41,b,28,a)].
95 -C(f3(x,c5),c6).    [ur(55,b,31,a)].
97 C(f3(x,c3),c4).  [resolve(57,b,29,a)].
115 -C(f3(x,c5),f3(y,c6)).  [ur(59,b,95,a)].
173 C(f3(x,c3),f3(y,c4)).   [resolve(97,a,61,b)].
195 f3(c2,x) = c2 | P(c2,f3(c2,x)) | P(f3(c2,x),c2).   [resolve(76,a,81,a),flip(a)].
307 f3(c2,x) = c2 | P(c2,f3(c2,x)) | P(c2,c2). [resolve(195,c,35,a)].
323 f3(c2,x) = c2 | P(c2,c2).   [resolve(307,b,39,a),merge(c)].
337 f3(c2,x) = c2 | P(c2,f3(c2,y)).    [resolve(323,b,41,b)].
340 f3(c2,x) = c2 | P(f3(c2,y),c2).    [resolve(323,b,37,b)].
360 f3(c2,x) = c2 | -P(f3(c2,y),c2) | f3(c2,y) = c2.   [resolve(337,b,33,b),flip(c)].
361 f3(c2,x) = c2 | -P(f3(c2,x),c2).   [factor(360,a,c)].
431 f3(c2,x) = c2 | f3(c2,y) = c2. [resolve(361,b,340,b)].
432 f3(c2,x) = c2.  [factor(431,a,b)].
437 -C(c2,f3(x,c6)).   [para(432(a,1),115(a,1))].
449 C(c2,f3(x,c4)).    [para(432(a,1),173(a,1))].
525 -C(c2,c2).  [para(432(a,1),437(a,2))].
563 $F.    [para(432(a,1),449(a,2)),unit_del(a,525)].
```

Proof of theorem {6, 7, 9, 15, 13} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))    [assumption].
5 (exists x exists y C(x,y))    [assumption].
6 (exists x exists y -C(x,y))   [assumption].
7 (all x all y (P(x,y) -> -P(y,x)))    [assumption].
8 (all x all y all z (P(y,x) & P(z,x) & y != z -> P(y,z) | P(z,y)))    [assumption].
12 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
13 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
17 S(f3(x,y),x).    [clausify(3)].
19 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
20 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
21 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
22 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
26 E(f3(x,y),y).    [clausify(3)].
27 P(c1,c2).    [clausify(4)].
28 C(c3,c4).    [clausify(5)].
29 -C(c5,c6).   [clausify(6)].
30 -P(x,y) | -P(y,x).   [clausify(7)].
31 -P(x,y) | -P(z,y) | z = x | P(x,z) | P(z,x).    [clausify(8)].
46 -P(f3(x,y),z) | P(x,z).  [resolve(17,a,12,a)].
47 P(f3(x,y),z) | -P(x,z).  [resolve(17,a,13,a)].
64 -C(x,f3(y,z)) | C(x,z).  [resolve(26,a,19,a)].
65 C(x,f3(y,z)) | -C(x,z).  [resolve(26,a,20,a)].
66 -C(f3(x,y),z) | C(y,z).  [resolve(26,a,21,a)].
67 C(f3(x,y),z) | -C(y,z).  [resolve(26,a,22,a)].
68 -P(x,x).    [factor(30,a,b)].
70 -P(x,c2) | c1 = x | P(c1,x) | P(x,c1).  [resolve(31,a,27,a),flip(b)].
84 P(f3(c1,x),c2).  [resolve(47,b,27,a)].
101 -C(c5,f3(x,c6)).   [ur(64,b,29,a)].
102 C(c3,f3(x,c4)).    [resolve(65,b,28,a)].
105 -P(x,f3(x,y)).  [ur(47,a,68,a)].
106 -P(f3(x,y),x).  [ur(46,b,68,a)].
109 -C(f3(x,c5),f3(y,c6)).  [ur(66,b,101,a)].
179 f3(c1,x) = c1.  [resolve(70,a,84,a),flip(a),unit_del(b,105),unit_del(c,106)].
189 C(c3,c1).   [para(179(a,1),102(a,2))].
191 -C(c1,f3(x,c6)).   [para(179(a,1),109(a,1))].
195 C(f3(x,c3),c1).    [resolve(189,a,67,b)].
249 C(c1,c1).   [para(179(a,1),195(a,1))].
291 $F.    [para(179(a,1),191(a,2)),unit_del(a,249)].
```

Proof of theorem {6, 7, 9, 15, 12} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))    [assumption].
5 (exists x exists y C(x,y))    [assumption].
```

```
6 (exists x exists y -C(x,y))   [assumption].
7 (all x all y all z (P(y,x) & P(z,x) & y != z -> P(y,z) | P(z,y)))    [assumption].
8 (all x all y (P(x,y) & P(y,x) -> x = y)) [assumption].
10 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
11 S(f3(x,y),x).    [clausify(3)].
13 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
15 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
19 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
20 E(f3(x,y),y).    [clausify(3)].
22 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
24 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
28 P(c1,c2).    [clausify(4)].
29 C(c3,c4).    [clausify(5)].
31 -C(c5,c6).   [clausify(6)].
33 -P(x,y) | -P(y,x) | y = x.   [clausify(8)].
34 -P(x,y) | -P(z,y) | z = x | P(x,z) | P(z,x).    [clausify(7)].
35 -P(f3(x,y),z) | P(x,z).  [resolve(10,a,11,a)].
37 P(f3(x,y),z) | -P(x,z).  [resolve(13,a,11,a)].
39 -P(x,f3(y,z)) | P(x,y).  [resolve(14,a,11,a)].
41 P(x,f3(y,z)) | -P(x,y).  [resolve(15,a,11,a)].
55 -C(f3(x,y),z) | C(y,z).  [resolve(19,a,20,a)].
57 C(f3(x,y),z) | -C(y,z).  [resolve(22,a,20,a)].
59 -C(x,f3(y,z)) | C(x,z).  [resolve(23,a,20,a)].
61 C(x,f3(y,z)) | -C(x,z).  [resolve(24,a,20,a)].
76 -P(x,c2) | c1 = x | P(c1,x) | P(x,c1).   [resolve(34,a,28,a),flip(b)].
78 P(f3(c1,x),c2).  [resolve(37,b,28,a)].
95 -C(f3(x,c5),c6).     [ur(55,b,31,a)].
97 C(f3(x,c3),c4).  [resolve(57,b,29,a)].
115 -C(f3(x,c5),f3(y,c6)).  [ur(59,b,95,a)].
173 C(f3(x,c3),f3(y,c4)).   [resolve(97,a,61,b)].
195 f3(c1,x) = c1 | P(c1,f3(c1,x)) | P(f3(c1,x),c1).   [resolve(76,a,78,a),flip(a)].
308 f3(c1,x) = c1 | P(c1,f3(c1,x)) | P(c1,c1). [resolve(195,c,35,a)].
324 f3(c1,x) = c1 | P(c1,c1).   [resolve(308,b,39,a),merge(c)].
338 f3(c1,x) = c1 | P(c1,f3(c1,y)).     [resolve(324,b,41,b)].
341 f3(c1,x) = c1 | P(f3(c1,y),c1).     [resolve(324,b,37,b)].
361 f3(c1,x) = c1 | -P(f3(c1,y),c1) | f3(c1,y) = c1.   [resolve(338,b,33,b),flip(c)].
362 f3(c1,x) = c1 | -P(f3(c1,x),c1).    [factor(361,a,c)].
433 f3(c1,x) = c1 | f3(c1,y) = c1. [resolve(362,b,341,b)].
434 f3(c1,x) = c1.  [factor(433,a,b)].
439 -C(c1,f3(x,c6)).    [para(434(a,1),115(a,1))].
451 C(c1,f3(x,c4)).     [para(434(a,1),173(a,1))].
530 -C(c1,c1).  [para(434(a,1),439(a,2))].
568 $F.    [para(434(a,1),451(a,2)),unit_del(a,530)].
```

Proof of theorem {7, 9, 16, 13} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y C(x,y))    [assumption].
5 (exists x exists y -C(x,y))   [assumption].
6 (all x all y (P(x,y) -> -P(y,x)))     [assumption].
7 (all x all y (P(x,y) | P(y,x) | x = y))   [assumption].
11 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
12 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
16 S(f3(x,y),x).    [clausify(3)].
18 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
25 E(f3(x,y),y).    [clausify(3)].
26 C(c1,c2).    [clausify(4)].
27 -C(c3,c4).   [clausify(5)].
28 -P(x,y) | -P(y,x).   [clausify(6)].
29 P(x,y) | P(y,x) | y = x.     [clausify(7)].
44 -P(f3(x,y),z) | P(x,z).  [resolve(16,a,11,a)].
45 P(f3(x,y),z) | -P(x,z).  [resolve(16,a,12,a)].
62 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,18,a)].
64 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,20,a)].
65 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,21,a)].
66 -P(x,x).     [factor(28,a,b)].
129 -C(c3,f3(x,c4)).    [ur(62,b,27,a)].
132 C(f3(x,c1),c2).     [resolve(65,b,26,a)].
133 -P(x,f3(x,y)).  [ur(45,a,66,a)].
134 -P(f3(x,y),x).  [ur(44,b,66,a)].
280 -C(f3(x,c3),f3(y,c4)).  [ur(64,b,129,a)].
285 f3(x,y) = x.    [resolve(133,a,29,b),flip(b),unit_del(a,134)].
```

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288 -C(x,y).    [back_rewrite(280),rewrite([285(2),285(2)])].
289 $F.    [resolve(288,a,132,a)].
```

Proof of theorem {7, 9, 16, 12}$\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z)))))) #
label(non_clause). 2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))
# label(non_clause). 3 (all x all y exists z (S(z,x) & E(z,y))) # label(non_clause).   [assumption].
4 (exists x exists y C(x,y)) # label(non_clause).  [assumption].
5 (exists x exists y -C(x,y)) # label(non_clause). [assumption].
6 (all x all y (P(x,y) & P(y,x) -> x = y)) # label(non_clause).[assumption].
7 (all x all y (P(x,y) | P(y,x) | x = y)) # label(non_clause). [assumption].
9 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
10 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
11 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
12 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
16 S(f3(x,y),x).   [clausify(3)].
18 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
25 E(f3(x,y),y).   [clausify(3)].
26 C(c1,c2).   [clausify(4)].
27 -C(c3,c4).  [clausify(5)].
28 -P(x,y) | -P(y,x) | y = x.  [clausify(6)].
29 P(x,y) | P(y,x) | y = x.    [clausify(7)].
46 -P(x,f3(y,z)) | P(x,y).  [resolve(16,a,9,a)].
47 P(x,f3(y,z)) | -P(x,y).  [resolve(16,a,10,a)].
48 -P(f3(x,y),z) | P(x,z).  [resolve(16,a,11,a)].
49 P(f3(x,y),z) | -P(x,z).  [resolve(16,a,12,a)].
66 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,18,a)].
68 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,20,a)].
69 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,21,a)].
119 P(x,f3(y,z)) | P(y,x) | x = y. [resolve(47,b,29,b)].
140 -C(c3,f3(x,c4)).   [ur(66,b,27,a)].
143 C(f3(x,c1),c2).    [resolve(69,b,26,a)].
319 -C(f3(x,c3),f3(y,c4)). [ur(68,b,140,a)].
322 -C(f3(x,f3(y,c3)),f3(z,c4)).   [ur(68,b,319,a)].
343 C(f3(x,f3(y,c1)),c2).  [resolve(143,a,69,b)].
514 -C(f3(x,f3(y,f3(z,c3))),f3(u,c4)). [ur(68,b,322,a)].
518 C(f3(x,f3(y,f3(z,c1))),c2).    [resolve(343,a,69,b)].
758 -C(f3(x,f3(y,f3(z,c3))),f3(u,f3(w,c4))).   [ur(66,b,514,a)].
761 C(f3(x,f3(y,f3(z,f3(u,c1)))),c2).  [resolve(518,a,69,b)].
779 P(x,f3(y,z)) | f3(y,z) = x | P(y,f3(x,u)). [resolve(119,a,48,a)].
824 P(x,f3(x,y)) | f3(x,y) = x.    [factor(779,a,c)].
829 f3(x,y) = x | P(x,x).  [resolve(824,a,46,a)].
846 f3(x,y) = x | -P(f3(x,y),x).   [resolve(824,a,28,b),flip(c),merge(c)].
847 f3(x,y) = x | P(f3(x,z),x).    [resolve(829,b,49,b)].
982 -C(f3(x,f3(y,f3(z,c3))),f3(u,f3(w,f3(v5,c4)))).    [ur(66,b,758,a)].
983 f3(x,y) = x | f3(x,z) = x. [resolve(847,b,846,b)].
1003 f3(x,y) = x.  [factor(983,a,b)].
1004 -C(x,y).  [back_rewrite(982),rewrite([1003(2),1003(1),1003(1),1003(2),1003(1),1003(1)])].
1005 $F.   [resolve(1004,a,761,a)].
```

Proof of theorem {7, 9, 16, 12}$\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y C(x,y))   [assumption].
5 (exists x exists y -C(x,y))  [assumption].
6 (all x all y (P(x,y) & P(y,x) -> x = y)) [assumption].
7 (all x all y (P(x,y) | P(y,x) | x = y))  [assumption].
9 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
10 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
11 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
12 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
16 S(f3(x,y),x).   [clausify(3)].
18 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
25 E(f3(x,y),y).   [clausify(3)].
26 C(c1,c2).   [clausify(4)].
27 -C(c3,c4).  [clausify(5)].
28 -P(x,y) | -P(y,x) | y = x.  [clausify(6)].
29 P(x,y) | P(y,x) | y = x.    [clausify(7)].
46 -P(x,f3(y,z)) | P(x,y). [resolve(16,a,9,a)].
47 P(x,f3(y,z)) | -P(x,y). [resolve(16,a,10,a)].
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48 -P(f3(x,y),z) | P(x,z).  [resolve(16,a,11,a)].
49 P(f3(x,y),z) | -P(x,z).  [resolve(16,a,12,a)].
66 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,18,a)].
68 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,20,a)].
69 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,21,a)].
119 P(x,f3(y,z)) | P(y,x) | x = y.  [resolve(47,b,29,b)].
140 -C(c3,f3(x,c4)).    [ur(66,b,27,a)].
143 C(f3(x,c1),c2).     [resolve(69,b,26,a)].
319 -C(f3(x,c3),f3(y,c4)).  [ur(68,b,140,a)].
322 -C(f3(x,f3(y,c3)),f3(z,c4)).    [ur(68,b,319,a)].
343 C(f3(x,f3(y,c1)),c2).  [resolve(143,a,69,b)].
514 -C(f3(x,f3(y,f3(z,c3))),f3(u,c4)).  [ur(68,b,322,a)].
518 C(f3(x,f3(y,f3(z,c1))),c2).    [resolve(343,a,69,b)].
758 -C(f3(x,f3(y,f3(z,c3))),f3(u,f3(w,c4))).    [ur(66,b,514,a)].
761 C(f3(x,f3(y,f3(z,f3(u,c1)))),c2).  [resolve(518,a,69,b)].
779 P(x,f3(y,z)) | f3(y,z) = x | P(y,f3(x,u)).  [resolve(119,a,48,a)].
824 P(x,f3(x,y)) | f3(x,y) = x.     [factor(779,a,c)].
829 f3(x,y) = x | P(x,x).  [resolve(824,a,46,a)].
846 f3(x,y) = x | -P(f3(x,y),x).    [resolve(824,a,28,b),flip(c),merge(c)].
847 f3(x,y) = x | P(f3(x,z),x).     [resolve(829,b,49,b)].
982 -C(f3(x,f3(y,f3(z,c3))),f3(u,f3(w,f3(v5,c4)))).    [ur(66,b,758,a)].
983 f3(x,y) = x | f3(x,z) = x.  [resolve(847,b,846,b)].
1003 f3(x,y) = x.  [factor(983,a,b)].
1004 -C(x,y).  [back_rewrite(982),rewrite( [1003(2),1003(1),1003(1),1003(2),1003(1),1003(1)])].
1005 $F.   [resolve(1004,a,761,a)].
```

Proof of theorem {7, 9, 17, 13}$\vdash_{\mathbb{M}} \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y C(x,y))    [assumption].
5 (exists x exists y -C(x,y))   [assumption].
6 (all x all y (P(x,y) -> -P(y,x)))     [assumption].
7 (exists x all y (x != y -> P(x,y)))   [assumption].
11 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
16 S(f3(x,y),x).    [clausify(3)].
18 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
19 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
20 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
21 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
25 E(f3(x,y),y).    [clausify(3)].
26 C(c1,c2).    [clausify(4)].
27 -C(c3,c4).   [clausify(5)].
28 -P(x,y) | -P(y,x).   [clausify(6)].
29 x = c5 | P(c5,x).    [clausify(7)].
30 c5 = x | P(c5,x).    [copy(29),flip(a)].
45 -P(x,f3(y,z)) | P(x,y).  [resolve(16,a,11,a)].
63 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,18,a)].
64 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,19,a)].
65 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,20,a)].
66 C(x,f3(y,z)) | -C(x,z).  [resolve(25,a,21,a)].
67 -P(x,x).     [factor(28,a,b)].
116 -C(f3(x,c3),c4).    [ur(63,b,27,a)].
117 C(f3(x,c1),c2).     [resolve(64,b,26,a)].
121 -P(x,f3(x,y)).  [ur(45,b,67,a)].
127 f3(c5,x) = c5.  [resolve(121,a,30,b),flip(a)].
166 -C(c5,c4).  [para(127(a,1),116(a,1))].
168 C(c5,c2).   [para(127(a,1),117(a,1))].
170 C(c5,f3(x,c2)).     [resolve(168,a,66,b)].
307 -C(c5,f3(x,c4)).    [ur(65,b,166,a)].
313 -C(c5,c5).  [para(127(a,1),307(a,2))].
352 $F.    [para(127(a,1),170(a,2)),unit_del(a,313)].
```

Proof of theorem {7, 9, 17, 12}$\vdash_{\mathbb{M}} \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
5 (exists x exists y C(x,y))    [assumption].
6 (exists x exists y -C(x,y))   [assumption].
7 (exists x all y (x != y -> P(x,y)))   [assumption].
8 (all x all y (P(x,y) & P(y,x) -> x = y)) [assumption].
11 S(f3(x,y),x).    [clausify(3)].
13 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
19 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
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20 E(f3(x,y),y).    [clausify(3)].
22 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
24 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
29 C(c3,c4).    [clausify(5)].
30 x = c7 | P(c7,x).    [clausify(7)].
31 c7 = x | P(c7,x).    [copy(30),flip(a)].
33 -C(c5,c6).   [clausify(6)].
35 -P(x,y) | -P(y,x) | y = x.  [clausify(8)].
38 P(f3(x,y),z) | -P(x,z).  [resolve(13,a,11,a)].
40 -P(x,f3(y,z)) | P(x,y).  [resolve(14,a,11,a)].
56 -C(f3(x,y),z) | C(y,z).  [resolve(19,a,20,a)].
58 C(f3(x,y),z) | -C(y,z).  [resolve(22,a,20,a)].
60 -C(x,f3(y,z)) | C(x,z).  [resolve(23,a,20,a)].
62 C(x,f3(y,z)) | -C(x,z).  [resolve(24,a,20,a)].
76 -P(x,c7) | c7 = x.   [resolve(35,a,31,b),flip(b),merge(c)].
82 P(f3(c7,x),y) | c7 = y.  [resolve(38,b,31,b)].
131 -C(f3(x,c5),c6).    [ur(56,b,33,a)].
133 C(f3(x,c3),c4).     [resolve(58,b,29,a)].
151 -C(f3(x,c5),f3(y,c6)).  [ur(60,b,131,a)].
207 C(f3(x,c3),f3(y,c4)).   [resolve(133,a,62,b)].
514 f3(x,y) = c7 | P(f3(c7,z),x).   [resolve(82,a,40,a),flip(a)].
826 f3(c7,x) = c7 | f3(c7,y) = c7.  [resolve(514,b,76,a),flip(b)].
855 f3(c7,x) = c7.  [factor(826,a,b)].
860 -C(c7,f3(x,c6)).    [para(855(a,1),151(a,1))].
872 C(c7,f3(x,c4)).     [para(855(a,1),207(a,1))].
936 -C(c7,c7).  [para(855(a,1),860(a,2))].
991 $F.     [para(855(a,1),872(a,2)),unit_del(a,936)].
```

Proof of theorem {7, 9, 18, 13} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y C(x,y))    [assumption].
5 (exists x exists y -C(x,y))   [assumption].
6 (all x all y (P(x,y) -> -P(y,x)))     [assumption].
7 (exists x all y (x != y -> P(y,x)))   [assumption].
11 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
16 S(f3(x,y),x).    [clausify(3)].
18 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
19 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
20 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
21 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
25 E(f3(x,y),y).    [clausify(3)].
26 C(c1,c2).    [clausify(4)].
27 -C(c3,c4).   [clausify(5)].
28 -P(x,y) | -P(y,x).   [clausify(6)].
29 c5 = x | P(x,c5).    [clausify(7)].
44 -P(f3(x,y),z) | P(x,z).  [resolve(16,a,11,a)].
62 -C(x,f3(y,z)) | C(x,z).  [resolve(25,a,18,a)].
63 C(x,f3(y,z)) | -C(x,z).  [resolve(25,a,19,a)].
64 -C(f3(x,y),z) | C(y,z).  [resolve(25,a,20,a)].
65 C(f3(x,y),z) | -C(y,z).  [resolve(25,a,21,a)].
66 -P(x,x).     [factor(28,a,b)].
115 -C(c3,f3(x,c4)).    [ur(62,b,27,a)].
116 C(c1,f3(x,c2)).     [resolve(63,b,26,a)].
120 -P(f3(x,y),x).  [ur(44,b,66,a)].
126 f3(c5,x) = c5.  [resolve(120,a,29,b),flip(a)].
165 -C(c3,c5).  [para(126(a,1),115(a,2))].
167 C(c1,c5).   [para(126(a,1),116(a,2))].
169 C(f3(x,c1),c5).     [resolve(167,a,65,b)].
302 -C(f3(x,c3),c5).    [ur(64,b,165,a)].
308 -C(c5,c5).  [para(126(a,1),302(a,1))].
347 $F.     [para(126(a,1),169(a,1)),unit_del(a,308)].
```

Proof of theorem {7, 9, 18, 12} $\vdash_{\mathbb{M}} \perp$

```
1 (all x all y (S(x,y) <-> (all z ((P(x,z) <-> P(y,z)) & (P(z,x) <-> P(z,y))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(x,z) <-> C(y,z)) & (C(z,x) <-> C(z,y))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
5 (exists x exists y C(x,y))    [assumption].
6 (exists x exists y -C(x,y))   [assumption].
7 (exists x all y (x != y -> P(y,x)))   [assumption].
8 (all x all y (P(x,y) & P(y,x) -> x = y)) [assumption].
10 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
11 S(f3(x,y),x).    [clausify(3)].
```

```
15 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
19 -E(x,y) | -C(x,z) | C(y,z).  [clausify(2)].
20 E(f3(x,y),y).   [clausify(3)].
22 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 -E(x,y) | -C(z,x) | C(z,y).  [clausify(2)].
24 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
29 C(c3,c4).   [clausify(5)].
30 x = c7 | P(x,c7).   [clausify(7)].
31 c7 = x | P(x,c7).   [copy(30),flip(a)].
33 -C(c5,c6).   [clausify(6)].
35 -P(x,y) | -P(y,x) | y = x.  [clausify(8)].
36 -P(f3(x,y),z) | P(x,z).  [resolve(10,a,11,a)].
42 P(x,f3(y,z)) | -P(x,y).  [resolve(15,a,11,a)].
56 -C(f3(x,y),z) | C(y,z).  [resolve(19,a,20,a)].
58 C(f3(x,y),z) | -C(y,z).  [resolve(22,a,20,a)].
60 -C(x,f3(y,z)) | C(x,z).  [resolve(23,a,20,a)].
62 C(x,f3(y,z)) | -C(x,z).  [resolve(24,a,20,a)].
76 -P(c7,x) | c7 = x.  [resolve(35,a,31,b),merge(c)].
78 P(x,c7) | f3(x,y) = c7.  [resolve(36,a,31,b),flip(b)].
141 -C(f3(x,c5),c6).   [ur(56,b,33,a)].
143 C(f3(x,c3),c4).    [resolve(58,b,29,a)].
161 -C(f3(x,c5),f3(y,c6)).  [ur(60,b,141,a)].
217 C(f3(x,c3),f3(y,c4)).  [resolve(143,a,62,b)].
251 f3(x,y) = c7 | P(x,f3(c7,z)).  [resolve(78,a,42,b)].
490 f3(c7,x) = c7 | f3(c7,y) = c7.  [resolve(251,b,76,a),flip(b)].
527 f3(c7,x) = c7.  [factor(490,a,b)].
532 -C(c7,f3(x,c6)).   [para(527(a,1),161(a,1))].
544 C(c7,f3(x,c4)).    [para(527(a,1),217(a,1))].
618 -C(c7,c7).  [para(527(a,1),532(a,2))].
656 $F.    [para(527(a,1),544(a,2)),unit_del(a,618)].
```

Proof of theorem {3, 19} $\vdash_{\mathbb{M}} \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y)) & (exists x exists y C(x,y))  [assumption].
5 (all x all y (P(x,y) -> -C(x,y)))    [assumption].
8 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f3(x,y),x).   [clausify(3)].
17 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
19 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 E(f3(x,y),y).   [clausify(3)].
24 P(c1,c2).   [clausify(4)].
25 C(c3,c4).   [clausify(4)].
26 -P(x,y) | -C(x,y).  [clausify(5)].
44 P(x,f3(y,z)) | -P(x,y).  [resolve(14,a,8,a)].
46 P(f3(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
64 C(x,f3(y,z)) | -C(x,z).  [resolve(23,a,17,a)].
66 C(f3(x,y),z) | -C(y,z).  [resolve(23,a,19,a)].
85 P(c1,f3(c2,x)).  [resolve(44,b,24,a)].
103 C(c3,f3(x,c4)).    [resolve(64,b,25,a)].
114 P(f3(c1,x),f3(c2,y)).  [resolve(85,a,46,b)].
151 C(f3(x,c3),f3(y,c4)).  [resolve(103,a,66,b)].
255 -C(f3(c1,x),f3(c2,y)).  [resolve(114,a,26,a)].
256 $F.    [resolve(255,a,151,a)].
```

Proof of theorem {19} $\vdash_{\mathbb{M}}$ **4**

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))    [assumption].
5 (exists x exists y C(x,y))    [assumption].
6 (exists x exists y (P(x,y) & C(y,x)))     [goal].
9 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
11 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
15 S(f3(x,y),x).   [clausify(3)].
18 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
20 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
24 E(f3(x,y),y).   [clausify(3)].
25 P(c1,c2).   [clausify(4)].
26 C(c3,c4).   [clausify(5)].
27 -P(x,y) | -C(y,x).  [deny(6)].
45 P(x,f3(y,z)) | -P(x,y).  [resolve(15,a,9,a)].
47 P(f3(x,y),z) | -P(x,z).  [resolve(15,a,11,a)].
```

```
65 C(x,f3(y,z)) | -C(x,z).  [resolve(24,a,18,a)].
67 C(f3(x,y),z) | -C(y,z).  [resolve(24,a,20,a)].
86 P(c1,f3(c2,x)).  [resolve(45,b,25,a)].
104 C(c3,f3(x,c4)).     [resolve(65,b,26,a)].
115 P(f3(c1,x),f3(c2,y)).   [resolve(86,a,47,b)].
152 C(f3(x,c3),f3(y,c4)).   [resolve(104,a,67,b)].
256 -C(f3(c2,x),f3(c1,y)).  [resolve(115,a,27,a)].
257 $F.     [resolve(256,a,152,a)].
```

Proof of theorem {19}$\vdash_{\mathbb{M}}$??

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(z,x) <-> C(z,y)) & (C(x,z) <-> C(y,z))))))   [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y)) & (exists x exists y C(x,y))  [assumption].
5 (exists x exists y (P(x,y) & C(x,y)))     [goal].
8 -S(x,y) | P(z,x) | -P(z,y).   [clausify(1)].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f3(x,y),x).    [clausify(3)].
17 -E(x,y) | C(z,x) | -C(z,y).  [clausify(2)].
19 -E(x,y) | C(x,z) | -C(y,z).  [clausify(2)].
23 E(f3(x,y),y).    [clausify(3)].
24 P(c1,c2).    [clausify(4)].
25 C(c3,c4).    [clausify(4)].
26 -P(x,y) | -C(x,y).   [deny(5)].
44 P(x,f3(y,z)) | -P(x,y).  [resolve(14,a,8,a)].
46 P(f3(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
64 C(x,f3(y,z)) | -C(x,z).  [resolve(23,a,17,a)].
66 C(f3(x,y),z) | -C(y,z).  [resolve(23,a,19,a)].
85 P(c1,f3(c2,x)).  [resolve(44,b,24,a)].
103 C(c3,f3(x,c4)).     [resolve(64,b,25,a)].
114 P(f3(c1,x),f3(c2,y)).   [resolve(85,a,46,b)].
151 C(f3(x,c3),f3(y,c4)).   [resolve(103,a,66,b)].
255 -C(f3(c1,x),f3(c2,y)).  [resolve(114,a,26,a)].
256 $F.     [resolve(255,a,151,a)].
```

Proof of theorem {1, 20, 3, 4, 5}$\vdash \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(y,z) -> C(x,z)) & (C(z,y) -> C(z,x))))))     [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y (P(x,y) & C(y,x)))     [assumption].
5 (all x all y (P(x,y) -> -C(x,y)))     [assumption].
8 -S(x,y) | P(z,x) | -P(z,y).   [clausify(1)].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f3(x,y),x).    [clausify(3)].
16 -E(x,y) | -C(y,z) | C(x,z).  [clausify(2)].
17 -E(x,y) | -C(z,y) | C(z,x).  [clausify(2)].
21 E(f3(x,y),y).    [clausify(3)].
22 P(c1,c2).    [clausify(4)].
23 C(c2,c1).    [clausify(4)].
24 -P(x,y) | -C(x,y).   [clausify(5)].
42 P(x,f3(y,z)) | -P(x,y).  [resolve(14,a,8,a)].
44 P(f3(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
53 -C(x,y) | C(f3(z,x),y).  [resolve(21,a,16,a)].
54 -C(x,y) | C(x,f3(z,y)).  [resolve(21,a,17,a)].
73 P(c1,f3(c2,x)).  [resolve(42,b,22,a)].
83 C(f3(x,c2),c1).  [resolve(53,a,23,a)].
92 P(f3(c1,x),f3(c2,y)).    [resolve(73,a,44,b)].
129 C(f3(x,c2),f3(y,c1)).   [resolve(83,a,54,a)].
219 -C(f3(c1,x),f3(c2,y)).  [ur(24,a,92,a)].
220 $F.     [resolve(219,a,129,a)].
```

Proof of theorem {1, 20, 3, 19, 5}$\vdash \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(y,z) -> C(x,z)) & (C(z,y) -> C(z,x))))))     [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y)) & (exists x exists y C(x,y))  [assumption].
5 (all x all y (P(x,y) -> -C(x,y)))     [assumption].
8 -S(x,y) | P(z,x) | -P(z,y).   [clausify(1)].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f3(x,y),x).    [clausify(3)].
16 -E(x,y) | -C(y,z) | C(x,z).  [clausify(2)].
17 -E(x,y) | -C(z,y) | C(z,x).  [clausify(2)].
21 E(f3(x,y),y).    [clausify(3)].
22 P(c1,c2).    [clausify(4)].
```

```
23 C(c3,c4).    [clausify(4)].
24 -P(x,y) | -C(x,y).   [clausify(5)].
42 P(x,f3(y,z)) | -P(x,y).  [resolve(14,a,8,a)].
44 P(f3(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
53 -C(x,y) | C(f3(z,x),y).  [resolve(21,a,16,a)].
54 -C(x,y) | C(x,f3(z,y)).  [resolve(21,a,17,a)].
73 P(c1,f3(c2,x)).  [resolve(42,b,22,a)].
83 C(f3(x,c3),c4).  [resolve(53,a,23,a)].
92 P(f3(c1,x),f3(c2,y)).    [resolve(73,a,44,b)].
129 C(f3(x,c3),f3(y,c4)).   [resolve(83,a,54,a)].
211 -C(f3(c1,x),f3(c2,y)).  [ur(24,a,92,a)].
212 $F.    [resolve(211,a,129,a)].
```

Proof of theorem {1, 20, 16, 7, 9, 13} $\vdash \bot$

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) <-> (all z ((C(y,z) -> C(x,z)) & (C(z,y) -> C(z,x))))))     [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
5 (exists x exists y C(x,y))    [assumption].
7 (exists x exists y -C(x,y))   [assumption].
8 (all x all y (P(x,y) | P(y,x) | x = y))   [assumption].
9 (all x all y (P(x,y) -> -P(y,x)))     [assumption].
13 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
14 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
18 S(f3(x,y),x).    [clausify(3)].
20 -E(x,y) | -C(y,z) | C(x,z).  [clausify(2)].
21 -E(x,y) | -C(z,y) | C(z,x).  [clausify(2)].
25 E(f3(x,y),y).    [clausify(3)].
27 C(c3,c4).    [clausify(5)].
29 -C(c7,c8).   [clausify(7)].
30 P(x,y) | P(y,x) | y = x.     [clausify(8)].
31 -P(x,y) | -P(y,x).   [clausify(9)].
46 -P(f3(x,y),z) | P(x,z).  [resolve(18,a,13,a)].
47 P(f3(x,y),z) | -P(x,z).  [resolve(18,a,14,a)].
56 -C(x,y) | C(f3(z,x),y).  [resolve(25,a,20,a)].
57 -C(x,y) | C(x,f3(z,y)).  [resolve(25,a,21,a)].
58 -P(x,x).     [factor(31,a,b)].
131 C(f3(x,c3),c4).     [resolve(56,a,27,a)].
133 -P(x,f3(x,y)).  [ur(47,a,58,a)].
134 -P(f3(x,y),x).  [ur(46,b,58,a)].
156 f3(x,y) = x.    [resolve(133,a,30,b),flip(b),unit_del(a,134)].
158 C(x,c4).    [back_rewrite(131),rewrite( [156(2)])].
159 -C(x,y) | C(x,z).   [back_rewrite(57),rewrite( [156(2)])].
161 C(x,y).     [resolve(159,a,158,a)].
162 $F.    [resolve(161,a,29,a)].
```

Proof of theorem {1, 20, 16, 7, 9, 12}

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z)))))) #
label(non_clause). 2 (all x all y (E(x,y) <-> (all z ((C(y,z) -> C(x,z)) & (C(z,y) -> C(z,x)))))) #
label(non_clause).  [3 (all x all y exists z (S(z,x) & E(z,y))) # label(non_clause).   [assumption].
4 (exists x exists y C(x,y)) # label(non_clause).  [assumption].
5 (exists x exists y -C(x,y)) # label(non_clause). [assumption].
6 (all x all y (P(x,y) & P(y,x) -> x = y)) # label(non_clause). [assumption].
7 (all x all y (P(x,y) | P(y,x) | x = y)) # label(non_clause).  [assumption].
9 -S(x,y) | -P(z,x) | P(z,y).  [clausify(1)].
10 -S(x,y) | P(z,x) | -P(z,y).  [clausify(1)].
11 -S(x,y) | -P(x,z) | P(y,z).  [clausify(1)].
12 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
16 S(f3(x,y),x).    [clausify(3)].
18 -E(x,y) | -C(y,z) | C(x,z).  [clausify(2)].
19 -E(x,y) | -C(z,y) | C(z,x).  [clausify(2)].
23 E(f3(x,y),y).    [clausify(3)].
24 C(c1,c2).    [clausify(4)].
25 -C(c3,c4).   [clausify(5)].
26 -P(x,y) | -P(y,x) | y = x.   [clausify(6)].
27 P(x,y) | P(y,x) | y = x.     [clausify(7)].
44 -P(x,f3(y,z)) | P(x,y).  [resolve(16,a,9,a)].
45 P(x,f3(y,z)) | -P(x,y).  [resolve(16,a,10,a)].
46 -P(f3(x,y),z) | P(x,z).  [resolve(16,a,11,a)].
47 P(f3(x,y),z) | -P(x,z).  [resolve(16,a,12,a)].
56 -C(x,y) | C(f3(z,x),y).  [resolve(23,a,18,a)].
57 -C(x,y) | C(x,f3(z,y)).  [resolve(23,a,19,a)].
107 P(x,f3(y,z)) | P(y,x) | x = y.  [resolve(45,b,27,b)].
120 C(f3(x,c1),c2).     [resolve(56,a,24,a)].
297 C(f3(x,c1),f3(y,c2)).   [resolve(120,a,57,a)].
317 C(f3(x,f3(y,c1)),f3(z,c2)).     [resolve(297,a,56,a)].
472 C(f3(x,f3(y,f3(z,c1))),f3(u,c2)).   [resolve(317,a,56,a)].
```

```
669 P(x,f3(y,z)) | f3(y,z) = x | P(y,f3(x,u)). [resolve(107,a,46,a)].
714 P(x,f3(x,y)) | f3(x,y) = x.    [factor(669,a,c)].
719 f3(x,y) = x | P(x,x).  [resolve(714,a,44,a)].
736 f3(x,y) = x | -P(f3(x,y),x).   [resolve(714,a,26,b),flip(c),merge(c)].
737 f3(x,y) = x | P(f3(x,z),x).    [resolve(719,b,47,b)].
868 f3(x,y) = x | f3(x,z) = x.     [resolve(737,b,736,b)].
888 f3(x,y) = x.   [factor(868,a,b)].
915 C(x,y).    [back_rewrite(472),rewrite( [888(2),888(1),888(1),888(2)])].
916 $F.    [resolve(915,a,25,a)].
```

Proof of theorem {1, 21, 3, 4, 5}

```
1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) -> C(x,y))) [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y (P(x,y) & C(y,x)))    [assumption].
5 (all x all y (P(x,y) -> -C(x,y)))    [assumption].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f2(x,y),x).   [clausify(3)].
15 E(f2(x,y),y).   [clausify(3)].
16 -E(x,y) | C(x,y).   [clausify(2)].
17 -P(x,y) | -C(x,y).  [clausify(5)].
19 C(f2(x,y),y).   [resolve(15,a,16,a)].
20 P(c1,c2).   [clausify(4)].
40 P(f2(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
42 -P(f2(x,y),y).  [resolve(19,a,17,b)].
60 P(f2(c1,x),c2).  [resolve(40,b,20,a)].
61 $F.  [resolve(60,a,42,a)].
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Proof of theorem {1, 21, 3, 6, 5}

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1 (all x all y (S(x,y) <-> (all z ((P(z,x) <-> P(z,y)) & (P(x,z) <-> P(y,z))))))   [assumption].
2 (all x all y (E(x,y) -> C(x,y))) [assumption].
3 (all x all y exists z (S(z,x) & E(z,y))) [assumption].
4 (exists x exists y P(x,y))   [assumption].
5 (all x all y (P(x,y) -> -C(x,y)))    [assumption].
10 -S(x,y) | P(x,z) | -P(y,z).  [clausify(1)].
14 S(f2(x,y),x).   [clausify(3)].
15 E(f2(x,y),y).   [clausify(3)].
16 -E(x,y) | C(x,y).   [clausify(2)].
17 C(f2(x,y),y).   [resolve(15,a,16,a)].
18 -P(x,y) | -C(x,y).  [clausify(5)].
19 P(c1,c2).   [clausify(4)].
39 P(f2(x,y),z) | -P(x,z).  [resolve(14,a,10,a)].
40 -P(f2(x,y),y).  [resolve(17,a,18,b)].
58 P(f2(c1,x),c2).  [resolve(39,b,19,a)].
59 $F.  [resolve(58,a,40,a)].
```