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Determination of EOQ in terms of optimum degrees of horizontal and vertical cooperation at a node of supply chain

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Abstract

In a complex supply chain network, the production nodes, seller nodes, and buyers are connected randomly. We assume a process of joining two random nodes leading to the bivariate Poisson probability mass function. There exist two types of links - one is horizontal (H) and the other is vertical (V), which support the continuous flow of commodities through the supply chain. This induces competition among workers at a node to manage these two types of links within fixed constraints and creates bargaining to decide the optimal degree of both types of links at a node. We use the Nash security point to obtain the bargaining solution describing the optimal links. We reduce the carrying cost and ordering cost of inventory, which are contrary in their nature by introducing horizontal and vertical links, respectively. We modify the total cost function and establish a new economic order quantity (EOQ), optimal shortage quantity, and total optimal cost in terms of the optimal degree of H and V cooperation.

Keywords: horizontal cooperation, vertical cooperation, carrying cost, ordering cost, economic order quantity (EOQ)

1. Introduction

Cooperation and coordination among agents are instrumental in establishing successful trades in both national and international markets. Cooperation can be defined as the process of coordinating the goals and actions of agents under some binding agreements. The number of agents cooperating with a node determines its bargaining power in a complex network. Hence, in the era of networking, it is considered very important and relevant to decide the optimal degree of cooperation at a node of the supply chain according to the cost of investing in a link. This creates a new area of research. In this paper, we investigate how we can reduce the total cost of inventory and improve customer service by determining the economic order quantity (EOQ) based on the degree of cooperation at a node of the supply chain.

Ashkan et al. [10] introduce a cooperative aggregate production plan to decrease operating costs. When the production plans of several plants are integrated, they can exchange workspace and product inventory. Thus, the demand for products can be satisfied at a lower cost. Hendalianpour et al. [14]

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investigate a two-echelon supply chain model with two manufacturers and two retailers. They develop a competitive structure with Grey stochastic demand and present Grey optimization and analysis of coordination within the chain. The management of inventory deals with the coordination of materials, production, and information flows among suppliers, distributors, retailers, and consumers. This management is not possible without cooperative links among supply chain nodes. A measure of the optimal degree of cooperation (e.g. number of cooperative links) at a node is necessary to predict the EOQ and also to know about the optimal shortage quantity to appropriately create goodwill from the customers. Fiestras-Janeiro et al. [8] survey the applications of cooperative game theory in the management of centralized inventory systems. In a lot size cooperative game, the cooperation links are of two types, "horizontal" and "vertical". Yang et al. [28] suggest that cooperation is more powerful in comparison to non-cooperation in reducing operational costs in pickup and delivery services.

The cooperative links are further divided into horizontal buyer-buyer or seller-seller links and vertical buyer and seller links. Horizontal links refer to the shared service centers, horizontal alliances, and horizontal cooperative purchasing. Lozano et.al. [20] argue that cooperation among firms has paramount importance, as it not only reduces the transportation (or machine running) costs, but also improves the performance of each participating firm. Vertical links apply to situations such as co-makership, and public-private partnerships. Hendalianpour [12] adopts Double Interval Grey Numbers to more accurately formulate consumer behavior and enhance the quality of the analytical results in practical decision making. In this work [12], a game-theoretic model is proposed for the joint decisions made on pricing and lot-sizing by retailers of perishable goods. Peide Liu et al. [18] use the Interval Valued Fuzzy Rough Number-BWM approach to address decision-making in selecting competent green suppliers in the supply chain.

We get the motivation of our work from a similar article by Julia [5] which shows that cooperation through horizontal links reduces the carrying and ordering costs through the vertical links. The players enhance the degrees of horizontal and vertical cooperation through investing in links in the supply chain network. Carrying and ordering costs play a major role in determining the EOQ. Buckley and Casson [3] identify cooperation as a special type of coordination. Peide Liu and Hendalianpour [17] presents a model, which aims to integrate physical and material dimensions to maximize net corporate profits through inbound and outbound financial flows. One factor influencing the intensity of competition is the bargaining power of suppliers. Schotanus [23] mentions that many cooperative organizations consider horizontal and vertical co-operation to have significant differences, because the number of partners in vertical technical alliances is rather low and co-operation is at a higher level. The focus of vertical co-operation is more on technical capabilities, mainly regarding purchasing (quantity and outcome), as well as on process improvements and new technologies. Thun [24] identifies vertical cooperation as the basis of supply chain management, because it includes companies, industries, and corporates at different levels of the supply chain. Brandenburger and Harborne [2] show that horizontal co-operation differs from its vertical counterpart, where companies complement each other. Hendalianpour et.al. [13] optimize a multi-product, multi-level Omni-channel distribution network and the shipping flow of products within a network under uncertain conditions. They develop a combined algorithm based on Benders Decomposition (BD) and Lagrangian Relaxation. We refer to Schotanus [23] for an in-depth survey of the literature on such models of co-operation.

This paper answers the following research questions, under the assumption of the model:

1. What would be the optimal degree of horizontal and vertical cooperation at a node of the supply chain?
2. How does the formation of horizontal and vertical cooperative links affect EOQ and optimum shortage quantity?

The reduced total optimal cost of EOQ with shortage is an ensuing result in terms of the optimal degrees of both types of cooperation. Other studies made in this area are by Johnson [15], Essig [7], Heijboer [11], Cruijssen et al. [4], etc. In all of these works, results on the potential benefits of horizontal cooperation under logistics service providers are presented. Kawamura [16] gives a theorem regarding the sum of n independent bivariate vectors. Shenle Pan et al. [22] study, as a tool to explain horizontal cooperation, cooperative games under precedence constraints and obtain the Shapley value for these games as a suitable solution concept. Small and medium-sized companies have more interest in horizontal cooperation as it enhances their bargaining power in comparison to their suppliers. Horizontal cooperation plays a very important role in the reduction of carrying costs (e.g. transportation, running logistic, holding and setup costs). Fiske and Bai [25] explore difference in status (respect, prestige) as vertical inequality and unequal power sharing (resource control) as horizontal inequality. However, power-sharing is mutual cooperation of the warm. It is also known as horizontal equality. Peide Liu et al. [19] use Bertrand cooperation and Stackelberg competition games based on Double Interval Grey numbers and also specify the price according to various demand functions and power structures between the manufacturers and common retailers. For further reference, the interested reader can look into, Christian and Gerald [27], Herbert et al. [6], Mishra et al. [21], Vasnani et al. [26], Yu et al. [29], and Binmore [1], etc. It is to be noted that one of the most prominent reasons for horizontal cooperation is to use synergistic effects in a supply chain (viz., economies of scale, scope, speed, etc.) that reduce transaction costs due to decreasing the number of transactions and strengthening competitiveness. For instance, if there is a fixed cost per order, agents will pay less when they order simultaneously as a group, as opposed to making their orders separately. Michel et al. [9] propose an allocation problem that determines how these savings should be divided among the agents.

In a complex supply chain network, the linkage costs of horizontal and vertical cooperation are decisive variables. We develop a new way to compute the optimal degree of horizontal and vertical cooperation by employing the concept of a Nash security point at a node of a complex supply chain network. Moreover, we further intend to optimize the order quantity of inventory flow by determining the Nash security point of a game involving horizontal and vertical cooperation where horizontal links help reduce the carrying costs and vertical links help reduce the ordering costs simultaneously. Under the assumptions given in Sections 2 and 6.1, we also derive a formula for the optimal shortage quantity and total optimal inventory cost at a node in terms of the degrees of both types of co-operation.

2. Definitions and assumptions

Kazutomo - Kawamura [16] defines the bivariate Poisson distribution. We use an analogous Poisson distribution theorem in our model. Let H and V denote a horizontal cooperation variate and a vertical cooperation variate, respectively, in a supply chain network.

Theorem 1. Consider the sum of independent bivariate horizontal and vertical cooperation (H, V) vectors (H_1, V_1), (H_2, V_2),..., (H_n, V_n) in the case where the probability of links of each type are p_{00}, p_{01} and p_{10}, p_{11}, and $np_{00} = \lambda_{00}$, $np_{01} = \lambda_{01}$, $np_{10} = \lambda_{10}$ and $np_{11} = \lambda_{11}$ are the expected numbers of each type of horizontal and vertical cooperation pairs. The limiting distribution of the sum of the n vectors is given by

$$P\left(H = \sum_{i=1}^{n} H_i = m, V = \sum_{i=1}^{n} V_i = n\right) = \sum_{i=0}^{\min(m,n)} \frac{{\lambda_{10}}^{m-i}\ \lambda_{01}^{n-i}\ \lambda_{11}^{i}\ e^{-(\lambda_{10}+\lambda_{01}+\lambda_{11})}}{(k-i)!\ \ (l-i)!\ \ i!}$$

Definition 1 (Degree of Horizontal Co-operation). The degree of horizontal cooperation at a node is the sum of all the weights of linked companies that belong to the same stage in the supply chain, i.e.

$$\sum_{i=1}^{n} H_i = m$$

Definition 2 (Degree of Vertical Cooperation). The degree of vertical cooperation at a node is the sum of all the weights of linked companies that belong to different stages of the supply chain, i.e.

$$\sum_{i=1}^{n} V_i = n$$

Definition 3 (Successful meetings). A meeting by one node is said to be successful when the node forms a link to cooperate with other nodes with probability one. For example, in the case of horizontal cooperation, this random variate may take the value 0 or 1. If $H_i = 1$, it implies that the i-th node attempts to make a horizontal link which is successful, but if $H_i = 0$, it shows that horizontal meeting is unsuccessful. Similarly, in the case of vertical cooperation, $V_i = 1$ implies that the ith node makes a successful attempt to form a vertical link. But $V_i = 0$ shows that vertical meeting is unsuccessful.

The following assumptions are made in the development of the model.

(i) Inventory management is based mainly on two costs. One is the carrying cost which includes all the costs involving the stock and transactions, and the second is the ordering cost that includes the set up cost. It works as a mechanism to supply the inventory from one node to another node. These two separate costs are managed by two cooperative teams of players at a node. This co-operation is split into horizontal and vertical.

(ii) Horizontal cooperation occurs among the nodes involved in the same stage of the supply chain and it normally produces/trades the same products among the nodes.

(iii) Vertical cooperation takes place through the formation of coalitions between the seller and the buyer from one stage to the next stage of the supply chain. This type of cooperation is responsible for reducing the ordering cost (viz., advertisement, stationery costs, communication costs, etc.).

(iv) At a node of the supply chain, there are two types of cooperation, some workers are involved in horizontal cooperation and others in vertical cooperation.

(v) Bargaining power is proportional to the number of horizontal or vertical cooperation links in the supply chain network.

(vi) The ordering cost is inversely proportional to the degree of vertical cooperation and the carrying cost is inversely proportional to the degree of horizontal cooperation.

(vii) The costs resulting from horizontal cooperation are sub-additive i.e., disjoint coalitions among nodes that belong to the same stage of the supply chain have an incentive to cooperate, since such co-operation will reduce costs in comparison to individual action.

3. Notation

The following notation is used to develop the paper.

- λ_{01} = average number of successful buyer-seller links (average number of successful meetings to initiate vertical cooperation),
- λ_{10} = average number of successful buyer-buyer or seller-seller links (average number of successful meetings to initiate horizontal cooperation),
- λ_{11} = average number of successful buyer-buyer or seller-seller links, as well as buyer-seller links (average number of successful meetings to initiate horizontal & vertical cooperation),
- λ_{00} = average number of links without successful meetings to initiate co-operation,
- h^* = status quo point of the degree of horizontal co-operation,
- v^* = status quo point of the degree of vertical cooperation,

- h = maximum degree of horizontal cooperation,
- v = maximum degree of vertical cooperation,
- D = total demand,
- Q = quantity supplied per order,
- S = quantity shortage in a cycle,
- t = shortage time as a fraction of a cycle, i.e., $0 \leq t \leq 1$,
- C_h = holding cost per unit item,
- C_0 = ordering cost per order,
- C_s = shortage cost,
- C_H^{link} = cost of horizontal cooperation link per node,
- C_V^{link} = cost of vertical cooperation link per node.

4. The mathematical model

4.1. Non-cooperative game between horizontal and vertical cooperative teams

According to our assumptions, vertical and horizontal cooperation leads to the evolution of two distinct teams of players at a particular node of the supply chain network. One team acts to increase the H links of horizontal cooperation and the second team acts to increase the V links of vertical cooperation. Let H be the horizontal degree and V the vertical degree. Each node has a fixed capacity for horizontal and vertical co-operation holding at most h and v degrees, respectively. We search for an equilibrium point of (H, V). Thus, at every node in the supply chain, there exists a non-cooperative game whose strategies are H and V.

For example, let a node of the supply chain be a wholeseller. The wholeseller has two teams of players to enhance cooperation in the market for reducing the cost of the supplied commodity and making more impact on the market. The wholeseller utilizes successful meetings of the type buyer-buyer (initiating horizontal cooperation λ_{10}) and successful meetings of the type buyer-seller (initiating vertical cooperation λ_{01}), but also utilizes cooperation between the buyer part of a node and seller part of the same node (initiating horizontal and vertical cooperation simultaneously λ_{11}). The wholeseller can also utilize links where co-operation has not been initiated (where the is no declaration of horizontal or vertical cooperation λ_{00}).

4.2. Pure strategic game

We consider competitive games in which one team of players gets a reward according to the penalty of another team. We observe a dilemma at a node of the supply chain which is analogous to a prisoner's dilemma. Two teams with conflicting interests are working to enhance the degree of cooperation with other nodes of the supply chain. The dilemma faced by these teams is to declare or not declare cooperation with other nodes. When both teams of players P_1 and P_2 declare cooperation, they both obtain a reward of R. Player P_1 has two strategies: declare cooperation (s_1) or not to declare (s_2). Player B also has two strategies: declare cooperation (t_1) or not to declare (t_2).

	t_1	t_2
s_1	(R, R)	(T, S)
s_2	(S, T)	(S, S)

R is a reward when both cooperate, T denotes the temptation from exploiting a cooperator and S is the sucker payoff from being exploited.

Theorem 2. For any non-cooperative two-person, non-zero-sum matrix game there exists at least one strategy pair in equilibrium.

Consider two teams (P_1, P_2) having conflicting objectives and using mixed strategy vectors $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, respectively, such that $x_1 + x_2 = 1$, $y_1 + y_2 = 1$. We apply this to the following problem. Let us consider the payoff tableau of the prisoner's dilemma game at a supply chain node, played by players P_1 and P_2 working to enhance the number of degrees in terms of horizontal and vertical cooperation, respectively. Player P_1 has two strategies: attempt to initiate a horizontal coalition (s_1) and not attempt to initiate a horizontal coalition (s_2) . Player P_2 also has two strategies: attempt to initiate a vertical coalition (t_1) and not to initiate a vertical coalition (t_2).

	t_1	t_2
s_1	$(\lambda_{11}, \lambda_{11})$	$(\lambda_{10}, 0)$
s_2	$(0, \lambda_{01})$	$(\lambda_{00}, \lambda_{00})$

The payoff matrices of players P_1 and P_2 are $A_H = \begin{bmatrix} \lambda_{11} & \lambda_{10} \\ 0 & \lambda_{00} \end{bmatrix}$ and $B_v^T = \begin{bmatrix} \lambda_{11} & \lambda_{01} \\ 0 & \lambda_{00} \end{bmatrix}$, respectively. In both cases, when $\lambda_{11} \geq \lambda_{10} \geq \lambda_{01} \geq \lambda_{00}$ and $\lambda_{11} \geq \lambda_{01} \geq \lambda_{10} \geq \lambda_{00}$, we find that the strategy s_1 dominates s_2 and t_1 dominates t_2. In this case, according to two basic principles of the game: (i) each player acts to maximize his or her own security level and (ii) the players use strategy pairs $((1, 0), (1, 0))$, which leads to a unique equilibrium point $(\lambda_{11}, \lambda_{11})$. Consequently, it is clear that when the average number of meetings of horizontal and vertical cooperation (meeting among buyer and seller simultaneously) is more than the average number of meetings of type buyer-buyer (only horizontal cooperation) and seller-seller (only vertical cooperation), then the unique equilibrium is to declare cooperation.

Consequently, we can say that when any node of the supply chain wants to make a link with a node using both ways of cooperation i.e. horizontally as well as vertically, then a cooperative node of the supply chain will be at equilibrium.

4.3. Mixed strategic game between two conflicting teams regarding horizontal and vertical cooperation at a node

4.3.1. Mixed strategic game

The expected payoff for a game with payoff matrix $A = (a_{ij})$, $1 \leq i \leq 2$, $1 \leq j \leq 2$, in which P_1 uses strategy $X = (x_1, x_2)$ and P_2 uses strategy $Y = (y_1, y_2)$, is

$$XAY^T = \sum_{1 \leq i \leq m} \sum_{1 \leq j \leq n} x_i a_{ij} y_j$$

4.3.2. Optimal security levels (status quo point)

Definition 4. The status quo payoff point (otherwise known as the safety point or security point) is the pair of payoffs (h^*, v^*) that each player can ensure if there is no cooperation between two teams of players.

In other words, we can say that for a game with payoff matrix A, define P_1's and P_2's optimal security levels, denoted by h^* and v^*, respectively, where $h^* = \max_{X \in S} \min_{Y \in T} XAY^T$ and $v^* = \min_{Y \in T} \max_{X \in S} XAY^T$. This is also known as the Nash security point (h^*, v^*). According to this definition, the minimum and maximum are to be taken over the finite sets T and S, respectively.

A mixed strategy for the horizontal cooperative player is denoted by a vector $X = (x_1, x_2)$ and for the vertical cooperative player $Y = (y_1, y_2)$. This game is a two-person non-zero-sum game. The payoff matrix of this game is defined as

	y_1	y_2
x_1	$(\lambda_{11}, \lambda_{11})$	$(\lambda_{10}, 0)$
x_2	$(0, \lambda_{01})$	$(\lambda_{00}, \lambda_{00})$

The payoff matrix is $\begin{bmatrix}(\lambda_{11}, \lambda_{11}) & (\lambda_{10}, 0)\\ (0, \lambda_{01}) & (\lambda_{00}, \lambda_{00})\end{bmatrix}$, where, the payoff matrix for the horizontal cooperation team $A_H = \begin{bmatrix}\lambda_{11} & \lambda_{10}\\ 0 & \lambda_{00}\end{bmatrix}$ and the payoff matrix for the vertical cooperation team $B_v^T = \begin{bmatrix}\lambda_{11} & \lambda_{01}\\ 0 & \lambda_{00}\end{bmatrix}$. We have to find the security point for the above payoff matrices. In this bi-matrix game, both players P_1 and P_2 have mixed strategies vectors denoted by $X = (x_1, x_2)$ and $Y = (y_1, y_2)$, respectively.

For a game with payoff matrix A_H define P_1's security level, denoted by h_1^*, as follows

$$\begin{aligned} h_1^* &= \max_{X\in S}\ \min_{Y\in T} XAY^T = \max_{(x_1,x_2)\in S}\ \min_{1\le j\le 2} XA_H^{(j)} \\ &= \max_{(x_1,x_2)\in S} \min\left[x_1, x_2\right]\begin{bmatrix}\lambda_{11} & \lambda_{10}\\ 0 & \lambda_{00}\end{bmatrix} = \max_{(x_1,x_2)\in S} \min\left\{x_1\lambda_{11}, \lambda_{10}x_1 + \lambda_{00}x_2\right\} \\ &= \max_{0\le x_1\le 1} \min\left\{x_1\lambda_{11}, (\lambda_{10}-\lambda_{00})x_1 + \lambda_{00}\right\}, \text{ since } x_2 = (1-x_1) \end{aligned}$$

The maximin value lies at the intersection of the two line segments $z = x_1\lambda_{11}$ and $z = (\lambda_{10}-\lambda_{00})x_1+\lambda_{00}$. Hence

$$\begin{aligned} & x_1\lambda_{11} = (\lambda_{10}-\lambda_{00})x_1 + \lambda_{00} \\ & x_1 = \frac{\lambda_{00}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})},\ x_2 = \frac{(\lambda_{11}-\lambda_{10})}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} \\ & X = \begin{bmatrix}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} & \frac{(\lambda_{11}-\lambda_{10})}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix} \end{aligned}$$

Similarly, for a game with payoff matrix A_H define P_2's security level, denoted by h_2^*, as follows

$$\begin{aligned} h_2^* &= \min_{(y_1,y_2)\in T}\ \max_{1\le i\le 2} A_{(i)}Y^T = \min_{(y_1,y_2)\in T}\ \max_{1\le i\le 2}\{\lambda_{11}y_1 + \lambda_{10}y_2, \lambda_{00}y_2\} \\ &= \min_{0\le y_1\le 1}\ \max_{1\le i\le 2}\{(\lambda_{11}-\lambda_{10})y_1 + \lambda_{10}, (\lambda_{00}-\lambda_{00}y_1)\} \end{aligned}$$

We find $Y = \begin{bmatrix}\frac{\lambda_{00}-\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} & \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix}$ is the vector defining the security strategy of P_2. The expected payoff of the horizontal cooperation team is

$$h^* = XA_HY^T$$

where

$$\begin{aligned} h_1^* = h_2^* = h^* &= \begin{bmatrix}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} & \frac{(\lambda_{11}-\lambda_{10})}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix}\begin{bmatrix}\lambda_{11} & \lambda_{10}\\ 0 & \lambda_{00}\end{bmatrix}\begin{bmatrix}\frac{\lambda_{00}-\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\\ \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix} \\ &= \begin{bmatrix}\frac{\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} & \frac{\lambda_{00}\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} + \frac{(\lambda_{11}-\lambda_{10})\lambda_{00}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix}\begin{bmatrix}\frac{\lambda_{00}-\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\\ \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix} \\ &= \begin{bmatrix}\frac{\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\cdot\frac{\lambda_{00}-\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} + \left(\frac{\lambda_{00}\lambda_{10}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})} + \frac{(\lambda_{11}-\lambda_{10})\lambda_{00}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\right)\frac{\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})}\end{bmatrix} \end{aligned}$$

Then

$$\begin{aligned}h^* &= \text{Expected payoff of } (A_H)\\ &= \frac{\lambda_{00}\lambda_{11}(\lambda_{00}-\lambda_{10})+\lambda_{00}\lambda_{10}\lambda_{11}+(\lambda_{11}-\lambda_{10})\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{10}+\lambda_{00})^2}\\ &= \frac{\lambda_{00}\lambda_{11}}{\lambda_{11}-\lambda_{10}+\lambda_{00}}\end{aligned}$$

Similarly, for a game with payoff matrix B_v define P_1's security level denoted by v_1^* as follows

$$\begin{aligned}v_1^* &= \max_{X\in S}\min_{Y\in T} XB_vY^T = \max_{(x_1,x_2)\in S}\min_{1\le j\le 2} XB_v^{(j)} = \max_{(x_1,x_2)\in S}\min\begin{bmatrix}x_1, x_2\end{bmatrix}\begin{bmatrix}\lambda_{11} & 0\\ \lambda_{01} & \lambda_{00}\end{bmatrix}\\ &= \max_{(x_1,x_2)\in S}\min\{\lambda_{11}x_1+\lambda_{01}x_2, x_2\lambda_{00}\}\\ &= \max_{0\le x_1\le 1}\min\{\lambda_{11}x_1+\lambda_{01}(1-x_1),(1-x_1)\lambda_{00}\}, \qquad \text{since } x_2=(1-x_1)\\ &= \max_{0\le x_1\le 1}\min\{(\lambda_{11}-\lambda_{01})x_1+\lambda_{01},(-\lambda_{00}x_1+\lambda_{00})\},\\ X &= \begin{bmatrix}\frac{\lambda_{00}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} & \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\end{aligned}$$

For a game with payoff matrix B_v define P_2's security level denoted by v_2^* as follows

$$\begin{aligned}v_2^* &= \min_{(y_1,y_2)\in T}\max_{1\le i\le 2} B_{v(i)}Y^T = \min_{(y_1,y_2)\in T}\max_{1\le i\le 2}\left\{\lambda_{11}y_1, \lambda_{01}y_1+\lambda_{00}y_2\right\}\\ &= \min_{0\le y_1\le 1}\max_{1\le i\le 2}\left\{\lambda_{11}y_1,(\lambda_{01}-\lambda_{00})y_1+\lambda_{00}\right\}\end{aligned}$$

$$y_1 = \frac{\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}, y_2 = \frac{\lambda_{11}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}$$

The expected payoff of the vertical co-operation team $v^* = XB_vY^T$, where $v^* = v_1^* = v_2^*$,

$$B_v = \begin{bmatrix}\lambda_{11} & 0\\ \lambda_{01} & \lambda_{00}\end{bmatrix}, X = \begin{bmatrix}\frac{\lambda_{00}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} & \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}, Y = \begin{bmatrix}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} & \frac{\lambda_{11}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}$$

Thus

$$\begin{aligned}v^* &= \begin{bmatrix}\frac{\lambda_{00}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} & \frac{\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\begin{bmatrix}\lambda_{11} & 0\\ \lambda_{01} & \lambda_{00}\end{bmatrix}\begin{bmatrix}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\\ \frac{\lambda_{11}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\\ &= \begin{bmatrix}\frac{\lambda_{11}(\lambda_{00}-\lambda_{01})}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}+\frac{\lambda_{11}\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} & \frac{\lambda_{11}\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\begin{bmatrix}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\\ \frac{\lambda_{11}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\\ &= \begin{bmatrix}\left\{\frac{\lambda_{11}(\lambda_{00}-\lambda_{01})}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}+\frac{\lambda_{11}\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\right\}\frac{\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}+\frac{\lambda_{11}\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\frac{\lambda_{11}-\lambda_{01}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{bmatrix}\\ &= \frac{\lambda_{00}^2\lambda_{11}-\lambda_{01}\lambda_{00}\lambda_{11}+\lambda_{11}^2\lambda_{00}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})^2}\\ &= \frac{\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}\end{aligned}$$

So, $h^* = \frac{\lambda_{00}\lambda_{11}}{\lambda_{11}-\lambda_{10}+\lambda_{00}}$ and $v^* = \frac{\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})}$ is the security point of horizontal and vertical degrees at a node of the supply chain.

The capacity of a node to deal with the costs of links in the network is fixed. Hence, two teams of players may bargain to decide the horizontal degree and vertical degree of a particular node. This can be formulated as a Nash bargaining problem where the disagreement point is defined as the status quo point (h^*, v^*). These degrees play an important role in reducing the carrying costs and ordering costs within the supply chain. The Nash bargaining solution to this problem satisfies the following six axioms.

5. Axioms defining the Nash bargaining solution

Let (h, v) denote the Nash bargaining payoff

Axiom 1. We must have $h \geq h^*$ and $v \geq v^*$, i.e. each player must get at least the relevant status quo payoff.

Axiom 2. The point $(h, v) \in N \times N$, that is, it must be a feasible point.

Axiom 3. If (h, v) is any point in $N \times N$, so that $h \geq h^*$, $v \geq v^*$, then it must be the case that $h = h^*$, $v = v^*$. In other words, there is no point in $N \times N$, where both players receive more. Hence, the Nash bargaining solution is Pareto-optimal.

Axiom 4. If $(h, v) \in T \subset S \subset N \times N$ and $(h, v) = f(T, h^*, v^*)$ is the solution to the bargaining problem with the feasible set T, then for the larger feasible set S either $(h^o, v^o) = f(S, h^*, v^*)$ is the bargaining solution for S or the actual bargaining solution for S is in $S - T$. We assume here that the security point is the same for T and S. So, if we have more alternatives, the new negotiated position cannot change to one of the old possibilities.

Axiom 5. If T is an affine transformation of S, $T = aS + b = \varphi(S)$ and $(\bar{h}, \bar{v}) = f(S, h^*, v^*)$ is the bargaining solution of S with security point (h^*, v^*), then $(a\bar{h} + b, a\bar{v} + b) = f(T, ah^* + b, av^* + b)$ is the bargaining solution associated with T and security point $(ah^* + b, av^* + b)$. This states that the solution is independent of the scale or units used to measure payoffs.

Axiom 6. If the game is symmetric with respect to the players, then the bargaining solution treats the players symmetrically. In other words, if $(h, v) = f(S, h^*, v^*)$ and (i) If $h^* = v^*$,and (ii) $(h, v) \in S \implies (v, h) \in S$, then $\bar{h} = \bar{v}$. So, if the players are essentially interchangeable they should get the same negotiated payoff. Binmore [1] defined the bargaining problem and its solution as below.

5.1. Nash bargaining problem

A Nash bargaining problem is simply a pair **(X, d)** in which **X** represents the set of feasible payoff pairs **(X, d)** and **d** is a point in **X** representing the consequences of disagreement. Only feasible sets that satisfy the following conditions will be considered: the set **X** is convex, closed and bounded above and free disposal is allowed. In this paper, the optimal security level provides the security point (h^*, v^*) which thus may be treated as the point of disagreement between horizontal and vertical degrees at an individual node. This point belongs to the feasible set of payoffs. The horizontal cooperation team and vertical cooperation team both negotiate to enhance the degree of cooperation above its security level. Now, our objective is to obtain the optimal degree of horizontal and vertical cooperation which maximises both degrees of cooperation. This is obtained by maximizing a non-linear expression subject to given constraints (5), (6) and (7).

$$\begin{aligned} &\text{Maximize} \quad g(h, v) = (h - h^*)(v - v^*) \\ &\text{Subject to} \quad (h, v) \in S, h \geq h^*, v \geq v^*, h + v \leq 2\lambda_{11} \end{aligned}$$

This can be expressed as

$$\text{Maximize} \quad (h - \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{10} + \lambda_{00}})(v - \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{01} + \lambda_{00}}) \tag{4}$$

$$\text{Subject to} \quad h \geq \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{10} + \lambda_{00}} \tag{5}$$

$$v \geq \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{01} + \lambda_{00}} \tag{6}$$

$$h + v \leq 2\lambda_{11} \tag{7}$$

An alternative method to solve the above non-linear problem is described below

$$\begin{aligned} f(h) &= \left(h - \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{10} + \lambda_{00}}\right)\left(-h + 2\lambda_{11} - \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{01} + \lambda_{00}}\right) \\ &= \left(h - \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{10} + \lambda_{00}}\right)\left(-h + \frac{2\lambda_{11}(\lambda_{11} - \lambda_{01} + \lambda_{00}) - \lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{01} + \lambda_{00}}\right) \end{aligned} \tag{8}$$

For $\frac{\partial F(h)}{\partial h} = 0$,we get the following optimal values for the degrees of horizontal cooperation h and vertical cooperation v:

$$h = \frac{1}{2}\left(\frac{2\lambda_{11}^3 + (4\lambda_{00} - \lambda_{01} - 2\lambda_{10})\lambda_{11}^2 + (\lambda_{00}^2 + \lambda_{01}\lambda_{10} - \lambda_{10}\lambda_{00} - \lambda_{01}\lambda_{00})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right) \tag{9}$$

$$v = \frac{1}{2}\left(\frac{2\lambda_{11}^3 + \lambda_{11}^2(4\lambda_{00} - 3\lambda_{01} - 2\lambda_{10}) + (3\lambda_{00}^2 - 3\lambda_{01}\lambda_{00} - 3\lambda_{00}\lambda_{10} + 3\lambda_{10}\lambda_{01})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right) \tag{10}$$

After substituting the values from equations (9) and (10) into equations (5) and (6), we conclude that the parameters $\lambda_{00}, \lambda_{10}, \lambda_{01}$ and λ_{11} should satisfy the following conditions to obtain a feasible solution for the optimal degree of horizontal and vertical cooperation.

Theorem 3. If h, v are optimal degrees of horizontal and vertical cooperation, then the parameters $\lambda_{00}, \lambda_{10}, \lambda_{01}$ and λ_{11}, which measure the average number of each type of link with a node, must satisfy the following two conditions for horizontal and vertical cooperation, respectively.

(a) $2{\lambda_{11}}^3 + (4\lambda_{00} - \lambda_{01} - 4\lambda_{10}){\lambda_{11}}^2 + (2\lambda_{10}\lambda_{01} + {\lambda_{00}}^2 + 2\lambda_{01}\lambda_{10} + 2{\lambda_{10}}^2 - 5\lambda_{00}\lambda_{10})\lambda_{11} + (\lambda_{00}{\lambda_{10}}^2 - \lambda_{01}{\lambda_{10}}^2 - {\lambda_{00}}^3 + \lambda_{01}{\lambda_{00}}^2) \geq 0$

(b) $2\lambda_{11}^3 + (4\lambda_{00} - 5\lambda_{01} - 2\lambda_{10})\lambda_{11}^2 + (3{\lambda_{00}}^2 - 8\lambda_{01}\lambda_{00} - 3\lambda_{00}\lambda_{10} + 5\lambda_{10}\lambda_{01} + 3{\lambda_{01}}^2)\lambda_{11} + ({\lambda_{00}}^3 - 4\lambda_{01}{\lambda_{00}}^2 - {\lambda_{00}}^2\lambda_{10} + 3\lambda_{00}{\lambda_{01}}^2 - 3{\lambda_{01}}^2\lambda_{10} + 4\lambda_{00}\lambda_{10}\lambda_{01}) \geq 0$

Proof. From the axioms for the Nash bargaining solution, if $(h, v) \in N \times N$, then it must satisfy $h \geq h^*, v \geq v^*$. Substituting in the values of h and h^*, v and v^*, we obtain

$$\frac{1}{2}\left(\frac{2\lambda_{11}^3 + (4\lambda_{00} - \lambda_{01} - 2\lambda_{10})\lambda_{11}^2 + (\lambda_{00}^2 + \lambda_{01}\lambda_{10} - \lambda_{10}\lambda_{00} - \lambda_{01}\lambda_{00})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right) \geq \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{10} + \lambda_{00}} \tag{11}$$

$$\frac{1}{2}\left(\frac{2\lambda_{11}^3 + \lambda_{11}^2(4\lambda_{00} - 3\lambda_{01} - 2\lambda_{10}) + (3\lambda_{00}^2 - 3\lambda_{01}\lambda_{00} - 3\lambda_{00}\lambda_{10} + 3\lambda_{10}\lambda_{01})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right) \geq \frac{\lambda_{00}\lambda_{11}}{\lambda_{11} - \lambda_{01} + \lambda_{00}} \tag{12}$$

After solving (11) and (12), we obtain the two conditions (a) and (b) of the theorem. □

5.2. Nash bargaining solutions

A bargaining solution is a function $F : B \longrightarrow \mathbb{R}^2$ with the property that $F(X, d)$, where $d = (d_1, d_2)$ is in the set X. One interprets $F(X, d)$ as the payoff pair on which rational players will agree when confronted with the bargaining problem (X, d). For an extension of the Nash bargaining solution, take $\alpha \geq 0$ and $\beta \geq 0$ with $\alpha + \beta = 1$. The function $G : B \longrightarrow \mathbb{R}^2$ which maximizes the expression $(x - d_1)^\alpha (y - d_2)^\beta$ corresponds to the generalized Nash Bargaining solution with the players' bargaining powers being α and β, respectively.

6. Application of the Nash bargaining solution to an EOQ model

Suppose that the cost of forming a horizontal link is C_H^{link} and the cost of forming a vertical link is C_V^{link}. Hence, the total linking costs (to initiate both horizontal and vertical cooperation) is $(hC_H^{link} + vC_V^{link})$. In order for links to be stable, $C_V^{link} < C_0$ and $C_H^{link} < C_h$. We know that if the i^{th} node cooperates with h nodes at the same level, then the carrying costs (which include holding costs, transportation costs etc.) are reduced and if the i^{th} node cooperates with v nodes at a different level, then the ordering costs (which include advertisement costs, setup costs, communication costs etc.) are also reduced.

The status quo payoff point (h, v) has a significant effect in reducing the total inventory costs. We have taken as an example the total cost function in an inventory model with shortage. In a supply chain, the optimal amount of a good passing through a particular node and the mean time for which stock is held at that node, are important when managing the flow of a commodity at that node.

6.1. Assumptions of EOQ model

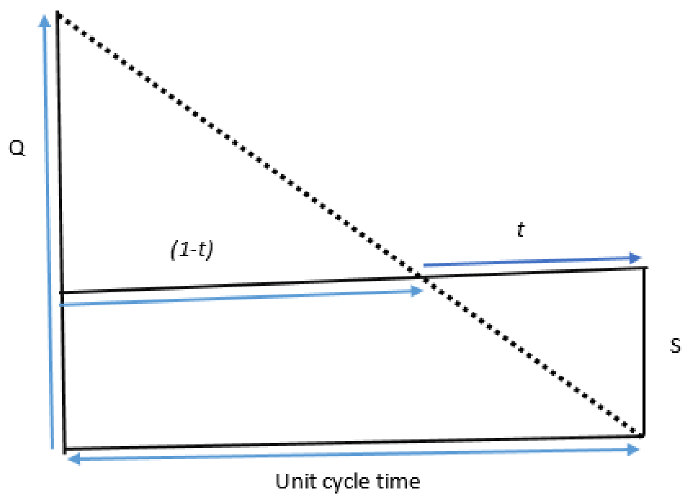

Figure 1. Illustration of a cycle: D– quantity demanded, Q–instantaneous replenishment quantity, S–shortage in a cycle

Our EOQ model, as illustrated in Figure 1, deals with the shortage of stock at a node. Shortage time t is measured as a fraction of the cycle length, i.e., $0 \leq t \leq 1$. The total number of orders in one cycle is $\frac{D}{Q}$. The average amount of the commodity held is $\frac{(Q-S)^2}{2Q}$. The average shortage of the commodity is $\frac{S^2}{2Q}$. Hence, according to the well-known formula of the total cost function, when the node has no cooperation in terms of degrees, the total cost is given below

$$TC = C_0 \frac{D}{Q} + C_h \frac{(Q - S)^2}{2Q} + C_s \frac{S^2}{2Q} \tag{13}$$

However, when the concepts of horizontal and vertical cooperation are used to manage the commodity at an individual node of the supply chain, we obtain the following reduced form of the cost function

$$TC = \frac{C_0}{v}\frac{D}{Q} + \frac{C_h}{h}\frac{(Q-S)^2}{2Q} + \left(hC_H^{link} + vC_V^{link}\right) + \frac{C_s}{(h+v)}\frac{S^2}{2Q} \tag{14}$$

In equation (14), the ordering cost is uniformly distributed according to the degree of vertical cooperation, v. So, $\frac{C_0}{v}$ becomes the ordering cost per unit order. Similarly, the cost for holding the commodity is also uniformly distributed according to the degree of horizontal cooperation, h. So, $\frac{C_h}{h}$ becomes the holding cost per unit commodity. The shortage of the commodity occurs at a supply chain node that possesses a total degree of cooperation $(h+v)$. It is assumed that the shortage costs are uniformly distributed over the $(h+v)$ degrees of cooperation. Hence, $\frac{C_s}{(h+v)}$ is the shortage cost of the commodity per unit link.

To obtain the stationary point of the total cost function, we have to take partial derivatives of TC with respect to Q and S and equate them to zero, as described in the following equations:

$$\begin{aligned}
\frac{\partial TC}{\partial Q} &= -\frac{C_0 D}{vQ^2} + \frac{C_h(2Q(Q-S)-(Q-S)^2)}{2hQ^2} - \frac{C_s S^2}{2(h+v)Q^2} = 0 \\
&\Rightarrow \frac{C_0 D}{v} + \frac{C_s S^2}{2(h+v)} = \frac{C_h(Q^2-S^2)}{2h} \\
&\Rightarrow \frac{C_0 D}{v} + \left(\frac{C_s}{2(h+v)} + \frac{C_h}{2h}\right) S^2 = \frac{C_h Q^2}{2h} \\
&\Rightarrow \frac{C_0 D}{v} = \frac{C_h Q^2}{2h} - \left(\frac{C_s}{2(h+v)} + \frac{C_h}{2h}\right) S^2 \\
\frac{\partial TC}{\partial S} &= -\frac{2C_h}{h}\cdot\frac{(Q-S)}{2Q} + \frac{2SC_s}{2Q(h+v)} = 0 \\
&\Rightarrow S = \frac{\frac{C_h Q}{h}}{\frac{C_h}{h} + \frac{C_s}{(h+v)}} = \frac{C_h Q(h+v)}{(C_h(h+v)+C_s h)}
\end{aligned} \tag{15}$$

Substituting the appropriate value of S in equation (15), we obtain

$$\frac{C_0 D}{v} = \frac{C_h Q^2}{2h} - \left(\frac{C_s}{2(h+v)} + \frac{C_h}{2h}\right)\left(\frac{C_h Q(h+v)}{\{C_h(h+v)+C_s h\}}\right)^2$$

$$\Rightarrow \quad \frac{C_0 D}{v} = Q^2\left[\frac{C_h}{2h} - \left(\frac{C_s}{2(h+v)} + \frac{C_h}{2h}\right)\left(\frac{C_h(h+v)}{C_h(h+v)+C_s h}\right)^2\right]$$

$$\Rightarrow \quad Q = \sqrt{\frac{\frac{C_0 D}{v}}{\frac{C_h}{2h} - \left(\frac{C_s}{2(h+v)} + \frac{C_h}{2h}\right)\left(\frac{C_h(h+v)}{C_h(h+v)+C_s h}\right)^2}} = \sqrt{\frac{\frac{C_0 D}{v}}{\frac{C_h(h+v)(C_h(h+v)+C_s h)^2 - (C_s h + {C_h}^2(h+v)^2)}{2h(h+v)(C_h(h+v)+C_s h)^2}}}$$

$$\Rightarrow \quad Q = \sqrt{\frac{(C_h(h+v)+C_s h)2hC_0 D}{vC_h[(C_h(h+v)+C_s h) - C_h(h+v)]}} = \sqrt{\frac{2C_0 D(C_h(h+v)+C_s h)}{vC_h C_s}}$$

Therefore,

$$Q^* = \sqrt{\frac{2C_0 D(C_h(h+v)+C_s h)}{vC_h C_s}} \tag{16}$$

and

$$S^* = \frac{C_h Q^*(h+v)}{(C_h(h+v)+C_s h)} = (h+v)\sqrt{\frac{2C_0 D C_h}{C_s v(C_h(h+v)+C_s h)}} \tag{17}$$

Substituting the appropriate values of h and v from equations (9) and (10) into (16) and (17), the optimal total cost is

$$TC^* = \frac{C_0}{v}\frac{D}{Q^*} + \frac{C_h}{h}\frac{(Q^*-S^*)^2}{2Q^*} + \left(hC_H^{link} + vC_V^{link}\right) + \frac{C_s}{(h+v)}\frac{S^{*2}}{2Q^*} \tag{18}$$

6.2. Example

In a complex supply chain network, suppose that a node satisfies all the assumptions of Sections 2 and 6.1. This node faces a demand of $D = 100$ units of a good, where the ordering cost per item is $C_o =$ Rs. 100 and the holding cost per item is $C_h =$ Rs. 90. The shortage cost per item $C_s =$ Rs. 200 is a penalty cost that takes into account a loss of goodwill among customers. The cost of forming a horizontal link is $C_H^{link} =$ Rs. 10 and the cost of forming a vertical link $C_V^{link} =$ Rs. 5. In the general interaction between the node considered and other nodes, the average number of successful meetings to form horizontal cooperation with nodes at the same level ($\lambda_{10} = 6$) and the average number of successful meetings to form vertical cooperation between nodes at different levels ($\lambda_{01} = 3$). The average number of links where no effort was made to initiate cooperation is $\lambda_{00} = 2$ nodes. The average number of successful meetings to initiate both horizontal and vertical cooperation is $\lambda_{11} = 9$.

(i) Derive the security point for the degree of horizontal and vertical cooperation.

(ii) Calculate the optimal degree of horizontal and vertical cooperation under these fixed linking costs.

(iii) Find the optimal number of units of the good to be ordered by the node per cycle.

(iv) Calculate the optimal shortage per cycle for this node.

(v) Derive the Economic Order Quantity at the node.

(vi) Evaluate the optimal total cost of inventory at the node.

6.2.1. Solution

The parameters considered are $C_0 =$ Rs. 100, $C_h =$ Rs. 90, $C_s =$ Rs. 200, $C_H^{link} =$ Rs. 10, $C_v^{link} =$ Rs. 5, $D = 100$ units, $\lambda_{11} = 9$, $\lambda_{00} = 2$, $\lambda_{10} = 6$ and $\lambda_{01} = 3$. First, we have to check the following conditions ensuring the feasibility of the optimal solution, (h, v):

(a) $2{\lambda_{11}}^3 + (4\lambda_{00} - \lambda_{01} - 4\lambda_{10}){\lambda_{11}}^2 + (2\lambda_{10}\lambda_{01} + {\lambda_{00}}^2 + 2\lambda_{01}\lambda_{10} + 2{\lambda_{10}}^2 - 5\lambda_{00}\lambda_{10})\lambda_{11} + (\lambda_{00}{\lambda_{10}}^2 - \lambda_{01}{\lambda_{10}}^2 - {\lambda_{00}}^3 + \lambda_{01}{\lambda_{00}}^2) = 679 \geq 0$

(b) $2\lambda_{11}^3 + (4\lambda_{00} - 5\lambda_{01} - 2\lambda_{10})\lambda_{11}^2 + (3{\lambda_{00}}^2 - 8\lambda_{01}\lambda_{00} - 3\lambda_{00}\lambda_{10} + 5\lambda_{10}\lambda_{01} + 3{\lambda_{01}}^2)\lambda_{11} + ({\lambda_{00}}^3 - 4\lambda_{01}{\lambda_{00}}^2 - {\lambda_{00}}^2\lambda_{10} + 3\lambda_{00}{\lambda_{01}}^2 - 3{\lambda_{01}}^2\lambda_{10} + 4\lambda_{00}\lambda_{10}\lambda_{01}) = 296 \geq 0$

We used a program written in C^{++} language to show that the set $\lambda_{11} = 9$, $\lambda_{00} = 2$, $\lambda_{10} = 6$, $\lambda_{01} = 3$ satisfy the above conditions. We then proceed to answer the questions formulated above.

(i) The security point of the degrees of horizontal and vertical cooperation (h^*, v^*) is calculated as $h^* = \frac{\lambda_{00}\lambda_{11}}{\lambda_{11}-\lambda_{10}+\lambda_{00}} = 3.6$ degrees, $v^* = \frac{\lambda_{00}\lambda_{11}}{(\lambda_{11}-\lambda_{01}+\lambda_{00})} = 2.25$ degrees.

(ii)

$$h = \frac{1}{2}\left(\frac{2\lambda_{11}^3 + (4\lambda_{00} - \lambda_{01} - 2\lambda_{10})\lambda_{11}^2 + (\lambda_{00}^2 + \lambda_{01}\lambda_{10} - \lambda_{10}\lambda_{00} - \lambda_{01}\lambda_{00})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right)$$
$$= 11.59 \text{ degrees},$$
$$v = \frac{1}{2}\left(\frac{2\lambda_{11}^3 + \lambda_{11}^2(4\lambda_{00} - 3\lambda_{01} - 2\lambda_{10}) + (3\lambda_{00}^2 - 3\lambda_{01}\lambda_{00} - 3\lambda_{00}\lambda_{10} + 3\lambda_{10}\lambda_{01})\lambda_{11}}{(\lambda_{11} + \lambda_{00})^2 - (\lambda_{01} + \lambda_{10})(\lambda_{11} + \lambda_{00}) + \lambda_{10}\lambda_{01}}\right)$$
$$= 6.41 \text{ degrees}.$$

(iii) $Q^* = \sqrt{\frac{2C_0D(C_h(h+v)+C_sh)}{vC_hC_s}} = 26.12$ units.

(iv) $S^* = (h+v)\sqrt{\frac{2C_0DC_h}{C_sv(C_h(h+v)+C_sh)}} = 10.75$ units.

(v) $TC^* = \frac{C_0}{v}\frac{D}{Q^*} + \frac{C_h}{h}\frac{(Q^*-S^*)^2}{2Q^*} + \left(hC_H^{link} + vC_V^{link}\right) + \frac{C_s}{(h+v)}\frac{S^{*2}}{2Q^*} =$ Rs. 267.34.

6.2.2. Sensitivity analysis

Sensitivity to λ_{10}. $C_0 =$ Rs. 100, $C_h =$ Rs. 90, $C_s =$ Rs. 200 and $C_H^{link} =$ Rs. 10 , $C_V^{link} =$ Rs. 5, $D = 100$ units, $\lambda_{00} = 100$ links, $\lambda_{01} = 500$ links, $\lambda_{11} = 1000$ links.

Table 1. Sensitivity to λ_{10}

λ_{10}	TC^*	Q^*	S^*	h^*	v^*	h	v
150	17106.42	29.85	11.57	105.26	166.67	1421.05	578.95
300	17188.67	30.39	11.70	125.00	166.67	1437.50	562.50
450	17308.88	31.22	11.90	153.85	166.67	1461.54	538.46
600	17501.22	32.66	12.25	200.00	166.67	1500.00	500.00
750	17858.45	35.80	13.04	285.71	166.67	1571.43	428.57

From Table 1, we may conclude that λ_{10} has a positive effect on TC^*, Q^*, S^* and h. However, it has a negative effect on the optimal degree of vertical cooperation v.

Sensitivity to λ_{01}. $C_0 =$ Rs. 100, $C_h =$ Rs. 90, $C_s =$ Rs. 200 and $C_H^{link} =$ Rs. 10, $C_V^{link} =$ Rs. 5, $D = 100$ units, $\lambda_{00} = 100$ links, $\lambda_{10} = 500$ links, $\lambda_{11} = 1000$ links.

Table 2. Sensitivity to λ_{01}

λ_{01}	TC^*	Q^*	S^*	h^*	v^*	h	v
150	15571.18	22.48	10.04	166.67	105.26	1114.04	885.96
300	16146.88	24.76	10.47	166.67	125.00	1229.17	770.83
450	16988.32	29.11	11.40	166.67	153.85	1397.44	602.56
600	18334.78	41.37	14.50	166.67	200.00	1666.67	333.33
625	18641.94	46.34	15.87	166.67	210.53	1728.07	271.93
650	$h \geq h^*$ is true but $v \geq v^*$ is not true						

From Table 2, TC^*, Q^*, S^*, h and v^* are increasing in λ_{01}, but the optimal degree of vertical cooperation v is decreasing in λ_{01}.

Sensitivity to λ_{00}. C_0 = Rs. 100, C_h = Rs. 90, C_s = Rs. 200 and C_H^{link} = Rs. 10, C_V^{link} = Rs. 5, $D = 100$ units, $\lambda_{01} = 400$ links, $\lambda_{10} = 600$ links, $\lambda_{11} = 1000$ links.

Table 3. Sensitivity to λ_{00}

λ_{00}	TC^*	Q^*	S^*	h^*	v^*	h	v
100	16786.83	27.93	11.14	200.00	142.86	1357.14	642.86
150	16743.53	27.69	11.08	272.73	200.00	1348.48	651.52
200	16667.77	27.28	10.99	333.33	250.00	1333.33	666.67
250	16573.49	26.79	10.89	384.62	294.12	1312.48	685.52
300	16469.32	26.27	10.78	428.57	333.33	1293.33	706.35

From Table 3, we conclude that TC^*, Q^*, S^*, h are decreasing in λ_{00}. However, h^*, v^* and v are increasing in λ_{00}.

Sensitivity to λ_{11}. C_0 = Rs. 100, C_h = Rs. 90, C_s = Rs. 200 and C_H^{link} = Rs. 10, C_V^{link} = Rs. 5, $D = 100$ units, $\lambda_{00} = 100$ links, $\lambda_{01} = 400$ links, $\lambda_{10} = 600$ links.

Table 4. Sensitivity to λ_{11}

λ_{11}	TC^*	Q^*	S^*	h^*	v^*	h	v
800	14268.29	35.57	12.98	266.67	160.00	1253.33	346.67
900	15470.05	30.39	11.70	225.00	150.00	1293.75	506.25
1000	16786.83	27.93	11.14	200.00	142.86	1333.33	642.86
1098	18134.54	26.50	10.83	183.61	137.59	1430.71	765.29
1099	$h \geq h^*$, $v \geq v^*$ are correct but $(h+v) \leq 2\lambda_{11}$ is incorrect						
1100	$h \geq h^*$, $v \geq v^*$ are correct but $(h+v) \leq 2\lambda_{11}$ is incorrect						
1101	$h \geq h^*$, $v \geq v^*$ are correct but $(h+v) \leq 2\lambda_{11}$ is incorrect						
1102	18190.36	26.46	10.82	183.06	137.41	1433.86	770.12
1200	19572.31	25.51	10.62	171.43	133.33	1514.29	885.71

From Table 4, we conclude that TC^*, h and v are increasing in λ_{11}. However, Q^*, S^*, h^* and v^* are decreasing in λ_{11}.

7. Conclusion

The whole effort of this paper has been to establish the relationship between the cost and the degree of cooperation at a node of a supply chain. We have also been able to solve the problem of how much of a commodity to order at a node in terms of the degree of horizontal and vertical co-operation. The bargaining problem induced at a node between two intrinsic teams of players regarding the degrees of horizontal and vertical cooperation has been solved and this paves a way to find an optimal degree. The innovative concept of the optimal degree of horizontal and vertical co-operation has been employed to determine the economic order quantity and its corresponding optimal cost. The main observations of the sensitivity analysis are that λ_{01}, λ_{10} are positively related to TC^*, Q^*, S^* and h, but has the opposite effect on the optimal degree of vertical cooperation v. However, λ_{11} is negatively related to Q^*, S^*, h^* and v^*, while having a positive effect on TC^*, h and v. Finally, λ_{00} has a negative effect on TC^*, Q^*, S^*, h but has a positive effect on h^*, v^* and v . These results are very helpful when taking managerial decisions at a node of a complex supply chain network.

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