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ON AN UPPER GAIN BOUND FOR STRATEGIES WITH CONSTANT AND PROPORTIONAL NUMBER OF ASSETS TRADED[1]

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Abstract: We introduce general formulas for the upper bound of gain obtained from any finite-time trading strategy in discrete and continuous time models. We consider strategies with constant number of assets traded and strategies with proportional number of assets traded. Unfortunately, the estimates obtained in the discrete case become trivial in the continuous case, hence we introduce transaction costs. This leads to the interesting estimates in terms of the so called truncated variation of the price series. We apply the obtained estimates in specific cases of financial time series.

Keywords: trading strategy, transaction costs, truncated variation, AR(1) process, Wiener process, Ornstein-Uhlenbeck process, random walk, the Black-Scholes model

INTRODUCTION

In [Łochowski 2010] we considered the following investment problem: let $P(n)$ and $Q(n), n=0,1,2,\dots$ be two non-stationary time series representing the evolution of the prices of futures contracts for two commodities P and Q. Assuming that the prices of P and Q are cointegrated, such that for some positive

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α and β the process αP-βQ is stationary, we considered the following long-run investment strategy: buy the combination αP-βQ when its value fails below some threshold $-a$ and sell it when the value of the combination exceeds threshold a. Buying the combination physically means entering into α long positions in commodity P contracts and entering into β short positions in commodity Q contracts. Similarly, selling the combination physically means entering into α short positions in commodity P contracts and entering into β long positions in commodity Q contracts.

Naturally, similar problem may be considered for larger number of cointegrated commodity contract prices.

The natural question arises whether the strategy described gives the best possible gains or, at least, to compare it with some upper bound for the best possible gain. To do so, in this article we introduce very general formulas for the upper bound for gain obtained from any finite-time trading strategy in discrete and continuous time. We consider two types of strategies:

1. Strategies with constant number of contracts or assets traded. In these models one always buys the same number of contracts or assets.
2. Strategies with proportional number of contracts or assets traded. In these models one always invests all money earned in the previous trading.

The bounds obtained are closely related to the path variation of the price time series (which, on the other hand, is closely related to volatility). Unfortunately, the bounds obtained in the discrete case become trivial in the continuous case, hence we introduce (constant or proportional) transaction costs. This leads to interesting bounds in terms of the so called truncated variation of the price series.

We apply the obtained bounds in specific cases. In the models with constant number of contracts traded we assume the AR(1) structure of the cointegrated price series and in the models with proportional number of assets traded we assume exponential random walk structure of the price series. The bounds obtained for the maximal gain in both cases reveal quite strong boundedness properties – they have finite moments of all orders.

UPPER BOUND FOR GAIN IN MODELS WITH CONSTANT NUMBER OF CONTRACTS TRADED

Discrete case

Let $R(n)$ denote the value of the linear combination αP-βQ of α long positions in commodity P contracts and β short positions in commodity Q contracts (or the value of linear combination of greater number of contracts, under the condition that it is stationary) at the moment $n=0,1,2,\ldots$. Buying this combination at moments $0\le b_1<b_2<\ldots<b_n<T$ and selling it (i.e. closing all long and short positions) at moments $0<s_1<s_2<\ldots<s_n\le T$ such that $b_1<s_1<b_2<s_2<\ldots$ we obtain the following gain

$$G=R(s_1)-R(b_1)+R(s_2)-R(b_2)+\ldots+R(s_n)-R(b_n) \tag{1}$$

(note that G may be negative). The immediate upper bound for G reads as

$$G\le sup_n sup_{0\le t_0<t_1<\ldots<t_n\le T}\sum_{i=1}^{n}\left|R(t_i)-R(t_{i-1})\right|. \tag{2}$$

The right-hand side of Eq. (2) is simply the total path variation of the time series $R(n)$ and we will denote it as

$$TV\left(R,\left[0;T\right]\right)=sup_n sup_{0\le t_0<t_1<\ldots<t_n\le T}\sum_{i=1}^{n}\left|R(t_i)-R(t_{i-1})\right|. \tag{3}$$

Due to the triangle inequality

$$\left|a-c\right|\le\left|a-b\right|+\left|b-c\right|$$

we simply obtain

$$\left|R(t_i)-R(t_{i-1})\right|\le\left|R(t_i)-R(t_i-1)\right|+\left|R(t_i-1)-R(t_i-2)\right|+\ldots+\left|R(t_i-\left(t_i-t_{i-1}\right)+1)-R(t_{i-1})\right|$$

$$=\sum_{i=t_{i-1}+1}^{t_i}\left|R(i)-R(i-1)\right|.$$

Hence

$$G\le sup_n sup_{0\le t_0<t_1<\ldots<t_n\le T}\sum_{i=1}^{n}\left|R(t_i)-R(t_{i-1})\right|\le\sum_{i=1}^{T}\left|R(i)-R(i-1)\right|. \tag{4}$$

(On the other hand, the opposite equality:

$$sup_n sup_{0\le t_0<t_1<\ldots<t_n\le T}\sum_{i=1}^{n}\left|R(t_i)-R(t_{i-1})\right|\ge\sum_{i=1}^{T}\left|R(i)-R(i-1)\right|$$

is also true and we have $TV\left(R,\left[0;T\right]\right)=\sum_{i=1}^{T}\left|R(i)-R(i-1)\right|$.)Knowing the specific structure of the series $R(n)$ we may calculate the distribution of the random variable $\sum_{i=1}^{T}\left|R(i)-R(i-1)\right|$ or e.g. certain characteristics of this distribution (like the expected value).

Remark. Reasoning similarly it is easy to obtain more accurate bound for G -the positive path variation of the time series $R(n)$ - which may be calculated with the following formula

$$UTV(R,[0;T])=\sum_{i=1}^{T} max\{R(i)-R(i-1),0\}$$

To illustrate the possible application of the obtained bound in a specific case, let usassume as in [Łochowski 2010] that $R(n)$ is a stationary, mean zeroAR(1) process, such that for some $\gamma\in(-1;1)$ and the sequence $Z(0),Z(1),Z(2),\ldots$ of independent random variables with normal $N(0,\sigma^2)$ distribution we have

$$R(n+1)=\gamma R(n)+Z(n). \tag{5}$$

Knowing that $R(0)$ and $Z(0),Z(1),Z(2),\ldots$ are independent, we obtain that (cf. [Łochowski 2010]) $R(n), n=0,1,\ldots$ has normal distribution $N(0,\sigma^2/(1-\gamma^2))$. Hence

$$\begin{aligned} R(i)-R(i-1)&=(\gamma-1)R(i-1)+Z(n)\sim N\left(0,\frac{(1-\gamma)^2\sigma^2}{1-\gamma^2}+\sigma^2\right) \\ &\sim N\left(0,\frac{2\sigma^2}{1+\gamma}\right). \end{aligned} \tag{6}$$

From (6) we easily obtain the expected value of the variable $TV(R,[0;T])$.

$$\begin{aligned} \mathrm{E}G&\le \mathrm{E}\sum_{i=1}^{T}|R(i)-R(i-1)|=T\mathrm{E}|R(1)-R(0)| \\ &=T\frac{\sqrt{2}\sigma}{\sqrt{1+\gamma}}\mathrm{E}|Y|=T\frac{\sqrt{2}\sigma}{\sqrt{1+\gamma}}\frac{\sqrt{2}}{\sqrt{\pi}}=\frac{2}{\sqrt{\pi}}\frac{\sigma}{\sqrt{1+\gamma}}T, \end{aligned}$$

where Y is a standard normal random variable and $\pi\approx 3.1415926.$

Continuous case with constant transaction costs

Now let us turn to the situation when the price process is observed in continuous time and we may sell or buy the combination of the contracts at any time between the moments 0 and T. As it was already noticed in [Łochowski 2010], for $\gamma\in(0;1)$ the continuous counterpart of the AR(1) process given by the recursion (5) is the Ornstein-Uhlenbeck process given by the following sde (stochastic differential equation):

$$dR(t)=ln(\gamma)R(t)dt+\sigma\sqrt{\frac{2ln(\gamma)}{\gamma^2-1}}dW(t), \tag{7}$$

where $W(t), t\geq 0,$ is a standard Brownian motion.
It is well known that the total variation of any process being the solution of any sde driven by a standard Brownian motion, $dR(t)=\mu(t,R(t))dt+\sigma(t,R(t))dW(t),$ satisfying some mild regularity conditions (e.g. the continuity of the functions μ,σ and $\sigma\neq 0$) has infinite total variation, given by the right-hand side of Eq. (2) and Eq. (3) (see [Revuz and Yor 2005, Chapt. IV Proposition 1.2]),

$$TV(R,[0;T])=+\infty.$$

Thus, our estimate (2) becomes trivial. Notice however, that it is no longer the case, when we introduce constant transaction costs.
Let $c/2>0$ be the value of a constant commission, paid for every transaction (regardless of the transaction value). In this setting, the right-hand side of Eq. (1) shall be replaced with

$$\begin{aligned} G&=R(s_1)-c/2-R(b_1)-c/2+R(s_2)-c/2-R(b_2)-c/2+\ldots+R(s_n)-c/2-R(b_n)-c/2 \\ &=R(s_1)-R(b_1)-c+R(s_2)-R(b_2)-c+\ldots+R(s_n)-R(b_n)-c \end{aligned} \tag{8}$$

and the estimate (1) becomes

$$\begin{aligned} G&\leq sup_n sup_{0\leq t_0<t_1<\ldots<t_n\leq T}\sum_{i=1}^{n}\left(|R(t_i)-R(t_{i-1})|-c\right) \\ &\leq sup_n sup_{0\leq t_0<t_1<\ldots<t_n\leq T}\sum_{i=1}^{n}max\left(|R(t_i)-R(t_{i-1})|-c,0\right). \end{aligned} \tag{9}$$

The last estimate we will call truncated variation of the process $R(t), t\geq 0,$ and we will denote it as

$$TV^c(R,[0;T])=sup_n sup_{0\leq t_0<t_1<\ldots<t_n\leq T}\sum_{i=1}^{n}max\left(|R(t_i)-R(t_{i-1})|-c,0\right) \tag{10}$$

Remark. Again, more accurate bound for G in continuous time setting with constant transaction costs is the upward truncated variation of the process $R(t), t\geq 0,$ which is defined with the following formula

$$UTV^c(R,[0;T])=sup_n sup_{0\leq t_0<t_1<\ldots<t_n\leq T}\sum_{i=1}^{n}max\{R(t_i)-R(t_{i-1})-c,0\}$$

It is possible to prove that the truncated variation is always finite for any process $R(t), t\geq 0,$ with continuous (cf. [Łochowski 2011]), càdlàg (cf. [Łochowski 2012]) or even regulated paths (cf. [Ghomrasni and Łochowski 2013]). By a càdlàg path we mean a path which is right continuous, $lim_{t\downarrow t_0} R(t)=R(t_0),$ and its left limits, $lim_{t\uparrow t_0} R(t)=R(t_0-),$ exist but may not coincide with the right limit limits.A regulated path is a path with left $lim_{t\uparrow t_0} R(t)=R(t_0-)$ and right limits $lim_{t\downarrow t_0} R(t)=R(t_0+)$, which may not coincide with the value $R(t_0)$.

The properties of the truncated variation as the function of parameters c and T are well known for broad class of stochastic processes (see [Łochowski 2012], [Bednorz and Łochowski 2012]). In particular, for the process given by Eq. (7) we have the following estimate of the exponential moments of $TV^c(R,[0;T])$, stemming from [Bednorz and Łochowski 2012, Theorem 2]:

$$\mathrm{E}exp\left(\lambda TV^c(R,[0;T])\right)\le 2exp\left(\lambda^2 T\alpha(\gamma,\sigma)+\lambda Tc^{-1}\beta(\gamma,\sigma)+\lambda\theta(T;\gamma,\sigma)\right)\times$$
$$\times\left(1+8\lambda\eta(T;\gamma,\sigma)exp\left(\lambda^2\eta(T;\gamma,\sigma)\right)\right).$$

Here, $\alpha(\gamma,\sigma)$ and $\beta(\gamma,\sigma)$ are constants depending on γ and σ only and $\theta(T;\gamma,\sigma)$ and $\eta(T;\gamma,\sigma)$ are constants depending on $T;\gamma$ and σ.

UPPER BOUND FOR GAIN IN MODELS WITH PROPORTIONAL NUMBER OF ASSETS TRADED

Discrete case

Now let us turn to another situation, when we buy assets (or portfolio of assets) but we exclude the possibility of a short sale. Let $S(n)=S(0)exp(V(n)), n=0,1,2,\ldots$ denotes the price of the asset at the moment n. Again, we assume that we buy this asset at moments $0\le b_1<b_2<\ldots<b_n<T$ and sell it at moments $0<s_1<s_2<\ldots<s_n\le T$ such that $b_1<s_1<b_2<s_2<\ldots$ but contrary to the previous strategy, where we were buying constant amount of the combination of contracts, we invest all money available. To make it clear, we will calculate the return from this strategy. The return from buying at the moment b_1 and selling at the moment s_1 reads as

$$\frac{S(s_1)}{S(b_1)}-1.$$

Similarly, the return from buying at the moment b_2 and selling at the moment s_2 reads as

$$\frac{S(s_2)}{S(b_2)}-1.$$

When we invest all the money obtained from selling the asset at the moment s_1 to buy the asset at the moment b_2, the return from all four operations reads as

$$\frac{S(s_1)}{S(b_1)}\frac{S(s_2)}{S(b_2)}-1.$$

Similarly, when we always invest all the money earned in the previous trading to buy the asset again, the return from buying the asset at moments $0\le b_1<b_2<\ldots<b_n<T$ and selling it at moments $0<s_1<s_2<\ldots<s_n\le T$ reads as

$$R=\frac{S(s_1)}{S(b_1)}\frac{S(s_2)}{S(b_2)}\cdots\frac{S(s_n)}{S(b_n)}-1$$
$$=exp(V(s_1)-V(b_1)+V(s_2)-V(b_2)+\ldots+V(s_n)-V(b_n))-1.$$

Reasoning similarly as in the preceding section we obtain the following upper bound

$$R=\frac{S(s_1)}{S(b_1)}\frac{S(s_2)}{S(b_2)}\cdots\frac{S(s_n)}{S(b_n)}-1$$
$$=exp(V(s_1)-V(b_1)+V(s_2)-V(b_2)+\ldots+V(s_n)-V(b_n))-1 \qquad (11)$$
$$\le exp(TV(V,[0;T]))-1=exp\left(\sum_{i=1}^{T}|V(i)-V(i-1)|\right)-1.$$

Remark. Similarly as for models with constant transaction costs, more accurate bound for G is expressed with the exponent of the positive path variation of the time series $R(n)$ and may be calculated with the following formula

$$exp(UTV(R,[0;T]))-1=exp\left(\sum_{i=1}^{T}max\{R(i)-R(i-1),0\}\right)-1.$$

Assuming the specific structure of the series $V(n), n=0,1,\ldots,$ we may again calculate the distribution or characteristics of the upper bound obtained. The simple yet widely used model assumes that the series $V(n), n=0,1,\ldots,$ is a random walk, i.e.

$$V(n)=X(1)+X(2)+\ldots+X(n), \qquad (12)$$

where $X(n), n=0,1,\ldots,$ are i.i.d. (independent and identically distributed). In particular, assuming that $X(1),X(2),\ldots\sim N(\mu,\sigma^2),$ we may calculate

$$\mathrm{E}R\le \mathrm{E}exp\left(\sum_{i=1}^{T}|V(i)-V(i-1)|\right)-1\le \mathrm{E}exp\left(\sum_{i=1}^{T}|X(i)|\right)-1$$
$$=(\mathrm{E}exp(|X(1)|))^T-1=\left(\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}exp\left(|x|-\frac{(x-\mu)^2}{2\sigma^2}\right)dx\right)^T-1.$$

Continuous case with proportional transaction costs

Similarly as before, let us now turn to the situation when the price process is observed in continuous time and we may sell or buy the assets at any time between the moments 0 and T. The continuous counterpart of the discrete random walk (12) with normally distributed increments $X(1), X(2), \ldots \sim N(\mu, \sigma^2)$ is the classical Black-Scholes model of the evolution of stock prices, given by the following sde:

$$dS(t) = \left(\mu - \frac{\sigma^2}{2}\right) S(t) dt + \sigma S(t) dW(t), \quad (13)$$

where $W(t), t \geq 0,$ denotes, as before, the standard Brownian motion. Under the assumption that $W(t), t \geq 0,$ is independent from $S(0)$ the solution of Eq. (13) reads as:

$$S(t) = S(0) exp(\mu t + \sigma W(t)) \quad (14)$$

and the process $V(t), t \geq 0,$ may be written as

$$V(t) = \mu t + \sigma W(t). \quad (15)$$

(From the properties of the standard Brownian motion we immediately obtain that

$$X(i) = V(i) - V(i\text{-}1) = \mu + \sigma(W(i) - W(i\text{-}1)) \sim N(\mu, \sigma^2)$$

are for $i = 1, 2, \ldots,$ i.i.d. random variables.)

Again, in the continuous case, for $V(t)$ given e.g. by Eq. (15), the upper bound given by Eq. (11) becomes trivial, since

$$TV(V, [0;T]) = +\infty.$$

Thus, similarly as in the previous section, let us introduce transaction costs. In the present case the transaction costs shall not be constant but rather proportional to the transaction value. Let $\delta \in (0;1)$ denote the ratio of every transaction value paid as a commission. Now, the return from buying at the moment b_1 and selling at the moment s_1 reads as

$$\frac{S(s_1)(1-\delta)}{S(b_1)(1+\delta)} - 1.$$

Similarly, the return from buying the asset at moments $0\le b_1<b_2<\ldots<b_n<T$ and selling it at moments $0<s_1<s_2<\ldots<s_n\le T$ reads as

$$R=\frac{S(s_1)}{S(b_1)}\frac{1-\delta}{1+\delta}\frac{S(s_2)}{S(b_2)}\frac{1-\delta}{1+\delta}\cdots\frac{S(s_n)}{S(b_n)}\frac{1-\delta}{1+\delta}-1$$

$$=exp\left(V(s_1)-V(b_1)-ln\frac{1+\delta}{1-\delta}+V(s_2)-V(b_2)-ln\frac{1+\delta}{1-\delta}+\ldots+V(s_n)-V(b_n)-ln\frac{1+\delta}{1-\delta}\right)-1.$$

Denoting $c=ln\frac{1+\delta}{1-\delta}>0$ we obtain the following estimate

$$R=\frac{S(s_1)}{S(b_1)}\frac{1-\delta}{1+\delta}\frac{S(s_2)}{S(b_2)}\frac{1-\delta}{1+\delta}\cdots\frac{S(s_n)}{S(b_n)}\frac{1-\delta}{1+\delta}-1$$

$$=exp\left(V(s_1)-V(b_1)-c+V(s_2)-V(b_2)-c+\ldots+V(s_n)-V(b_n)-c\right)-1 \text{ Remark}$$

$$\le exp\left(TV^c(V,[0;T])\right)-1<+\infty.$$

Again, more accurate bound for G in continuous time setting with proportional transaction costs is expressed with the exponent of the upward truncated variation of the process $V(t),\ t\ge 0$,

$$R\le exp\left(UTV^c(V,[0;T])\right)-1$$

$$=exp\left(sup_n sup_{0\le t_0<t_1<\ldots<t_n\le T}\sum_{i=1}^{n}max\{R(t_i)-R(t_{i-1})-c,0\}\right)-1.$$

By results of [Łochowski 2011, Sect. 2] it follows that for V being the Wiener process with drift, given by Eq. (15), for any real p we have $\mathrm{E}R^p<+\infty$, and from the results of [Bednorz and Łochowski 2012, Theorem 2]) we get more precise estimate of the form

$$\mathrm{E}R^p\le \mathrm{E}exp\left(pTV^c(V,[0;T])\right)\le 2exp\left(p^2T\alpha(\sigma)+pTc^{-1}\beta(\sigma)+p|\mu|\right).$$

Here $\alpha(\sigma)$ and $\beta(\sigma)$ are constants depending on σ only.

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