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Expectation properties of generalized order statistics based on the Gompertz-G family of distributions

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Abstract

Gompertz-G family of distributions has been considered. The moment properties of generalized order statistics were studied and characterization results have been presented. Further, several examples and special cases were discussed. The results can be applied to many known distributions included in this family.

Keywords: *generalized order statistics, expectation identities, Gompertz-G family of distributions, characterization*

1. Introduction

The order statistics and related general models of ordered random variables are important in statistical theory and its applications. Kamps [19] introduced the generalized order statistics (GOS) and showed that all well-known models of ordered random variables such as record values, order statistics, Pfeifer's records, progressive type II censored order statistics, etc. are the sub-models of GOS in the distributional and theoretical sense. There is no doubt that GOS and different models of ordered random variables will continue to arouse the interest of many researchers working in the fields of theoretical statistics, applications, and statistical mathematics.

Recurrence relations for moments of GOS and characterization through it for various distributions have been investigated by several authors. For a detailed review of the topics, see Keseling [21], Cramer and Kamps [15], Kamps and Cramer [20], Pawlas and Szynal [29], Saran and Pandey [30], Ahmad and Fawzy [1], Athar and Islam [7], Al-Hussaini et al. [3] , Anwar et al. [5], Khan et al. [23], Khwaja et al. [25], Khan and Zia [24], Athar and Nayabuddin [8], Khan and Khan [22], Nayabuddin and Athar [26], Singh et al. [31], Zarrin et al. [32], Athar et al. [9–12] and references therein.

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1.1. Definition of GOS

Let $n \geq 2$ be a given integer and $\tilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \mathbb{R}^{n-1}$, $k \geq 1$ be the parameters such that

$$\gamma_i = k + n - i + \sum_{j=i}^{n-1} m_j \geq 0 \quad \text{for} \quad 1 \leq i \leq n-1$$

The random variables $X_{1,n,\tilde{m},k}, X_{2,n,\tilde{m},k}, \ldots, X_{n,n,\tilde{m},k}$ are said to be GOS from an continuous population with cumulative distribution function (CDF) $F()$ and probability density function (PDF) $f()$, if their joint PDF is of the form

$$k\Big(\prod_{j=1}^{n-1} \gamma_j\Big)\Big(\prod_{i=1}^{n-1} \big(1 - F(x_i)\big)^{m_i} f(x_i)\Big)\big(1 - F(x_n)\big)^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq \ldots \leq x_n < F^{-1}(1)$.

1.2. Sub-model of GOS

The particular cases of model (1) are:

- If $m_1 = m_2 = \cdots = m_{n-1} = 0$ and $k = 1$, then $\gamma_r = n - r + 1, 1 \leq r \leq n-1$. In this case, model (1) reduces to the joint density of order statistics. For more details about order statistics see [16].
- By choosing $n = m$, $m_i = R_i$ for $i = 1, 2, \ldots, m-1$ and $k = R_m + 1$, then $\gamma_r = m - r + 1 + \sum_{i=r}^{m} R_i, 1 \leq r \leq m$ where R_i is a set of prefixed integer that shows R_i random removal at ith failure from surviving items of an experiment. In this case, model (1) reduces to the joint density based on progressively type-II censored order statistics [13].
- Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed random variables with CDF $F(x)$. Let $Y_n = \max\{X_1, X_2, \ldots, X_n\}, n \geq 1$. Then, we say X_j is an upper record values of sequence $\{X_n, n \geq 1\}$, if $Y_j \geq Y_{j-1}, j \geq 1$. Now if we put $m_1 = m_2 = \ldots = m_{n-1} = -1$ and $k = 1$ in (1), then $\gamma_r = 1, 1 \leq r \leq n-1$. In this case, model (1) reduces to the joint density of upper record values. For more details on record values see [2] and [6].
- If $m_i = (n - i + 1)\alpha_i - (n - i)\alpha_{i+1} - 1$ and $k = \alpha_n, \alpha \in R^+, i = 1, 2, \ldots, n-1$ then $\gamma_r = (n - r + 1)\alpha_r, 1 \leq r \leq n-1$. In this case, model (1) reduces to the joint density of sequential order statistics. For more details see [15].

Here we may consider two cases:

Case I. $\gamma_i \neq \gamma_j$, $i, j = 1, 2, \ldots, n-1, i \neq j$.

In view of (1), the PDF of rth GOS $X_{r,n,\tilde{m},k}$ is given as [20]

$$f_{r,n,\tilde{m},k}(x) = C_{r-1} f(x) \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i - 1} \tag{2}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + n - i + \sum_{j=1}^{n-1} m_j > 0$$

and

$$a_i(r) = \prod_{\substack{j=1 \\ j \neq i}}^{r} \frac{1}{(\gamma_j - \gamma_i)}, \quad 1 \leq i \leq r \leq n$$

The joint PDF of $X_{r,\,n,\,\tilde{m},\,k}$ and $X_{s,\,n,\,\tilde{m},\,k}$, $1 \leq r < s \leq n$, is given as [20]

$$f_{r,\,s,\,n,\,\tilde{m},\,k}(x,y) = C_{s-1} \sum_{j=r+1}^{s} a_j^{(r)}(s) \left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}$$
$$\times \left(\sum_{i=1}^{r} a_i(r)(\bar{F}(x))^{\gamma_i}\right) \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}, \quad x < y \tag{3}$$

where

$$a_j^{(r)}(s) = \prod_{\substack{t=r+1 \\ t \neq j}}^{s} \frac{1}{(\gamma_t - \gamma_j)}, \quad r+1 \leq j \leq s \leq n$$

Case II. $m_i = m$, $i = 1,\, 2, \ldots, n-1$

The PDF of rth GOS $X_{r,\,n,\,m,\,k}$ is given as [19]

$$f_{r,\,n,\,m,\,k}(x) \;=\; \frac{C_{r-1}}{(r-1)!} \big(\bar{F}(x)\big)^{\gamma_r - 1} f(x) g_m^{r-1}\big(F(x)\big) \tag{4}$$

where

$$C_{r-1} \;=\; \prod_{i=1}^{r} \gamma_i, \quad \gamma_i = k + (n-i)(m+1)$$

$$h_m(x) \;=\; \begin{cases} -\dfrac{1}{m+1}(1-x)^{m+1}, & m \neq -1 \\[2ex] \log\left(\dfrac{1}{1-x}\right), & m = -1 \end{cases}$$

and

$$g_m(x) \;=\; h_m(x) - h_m(0) = \int_0^x (1-t)^m dt, \quad x \in [0, 1)$$

The joint PDF of $X_{r,\,n,\,m,\,k}$ and $X_{s,\,n,\,m,\,k}$, $1 \leq r < s \leq n$, is given as [29]

$$f_{r,\,s,\,n,\,m,\,k}(x,y) \;=\; \frac{C_{s-1}}{(r-1)!\,(s-r-1)!} \big(\bar{F}(x)\big)^m g_m^{r-1}(F(x))$$
$$\times \big(h_m(F(y)) - h_m(F(x)\big)^{s-r-1} \big(\bar{F}(y)\big)^{\gamma_s - 1} f(x) f(y), \quad -\infty \leq x < y \leq \infty \tag{5}$$

1.3. Gompertz-G family of distributions

Alizadeh et al. [4] proposed the Gompertz generalized (Gompertz-G) family of distributions. It makes kurtosis more flexible compared to baseline models and produces skewness for symmetrical distributions. Thus, it provides greater flexibility in the modeling of real data sets. Some of the distributions belonging to this family like Gompertz–Frechet, Gompertz inverse exponential, Gompertz–Weibull–Frechet, and Gompertz alpha power inverted exponential distributions are separately studied by Oguntunde et al. [27, 28], Bodhisuwan and Aryuyuen [14], Eghwerido et al. [17], respectively. The CDF of the Gompertz-G family of distribution is given by

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1 - (1 - G(x))^{-\beta}\right)\right), \quad \alpha,\, \beta > 0 \tag{6}$$

and the corresponding PDF for this family is

$$f(x) = \alpha g(x)\,(1 - G(x))^{-\beta - 1} \exp\left(\frac{\alpha}{\beta}\left(1 - (1 - G(x))^{-\beta}\right)\right) \tag{7}$$

where $G(x)$ and $g(x)$ refer the CDF and PDF of the base distribution.

In view of (6) and (7), we get the relation between survival function (SF) and PDF as below

$$\bar{F}(x) = \frac{(1 - G(x))^{\beta + 1}}{\alpha g(x)} f(x) \tag{8}$$

Equation (8) can also be expressed as

$$\bar{F}(x) = \frac{1}{\alpha} \sum_{l=0}^{[\beta+1]} (-1)^l \binom{[\beta+1]}{l} \frac{G^l(x)}{g(x)} f(x) \tag{9}$$

where $\bar{F}(x) = 1 - F(x)$ is the survival function (SF) and [.] is an integer.

The paper is organized as follows. Section 2 presents the single moment of GOS for the family of distributions in (6). In addition, some examples and special cases are demonstrated. Section 3 discusses the properties of product moments, whereas the characterization results are studied in Section 4. Finally, the conclusion is given in Section 5.

2. Single moment

Before coming to the main result, we shall reproduce the lemma given by Athar and Islam [7]

Lemma 1. For Case I with PDF given in (2) and $2 \leq r \leq n, n \geq 1, p = 1, 2 \ldots$

$$E\left(X^p_{r,n,\tilde{m},k}\right) - E\left(X^p_{r-1,n,\tilde{m},k}\right) = pC_{r-2} \int_{-\infty}^{\infty} x^{p-1} \sum_{i=1}^{r} a_i(r) \left(\bar{F}(x)\right)^{\gamma_i} dx \tag{10}$$

Proof. For $\gamma_i \neq \gamma_j$, $i,\, j = 1, 2, \ldots, n-1, i \neq j$, Athar and Islam [7] have shown that

$$E\left(\xi(X_{r,n,\tilde{m},k})\right) - E\left(\xi(X_{r-1,n,\tilde{m},k})\right) = C_{r-2} \int_{-\infty}^{\infty} \xi'(x) \sum_{i=1}^{r} a_i(r) \left(\bar{F}(x)\right)^{\gamma_i} dx \tag{11}$$

where $\xi(x)$ is a Borel measurable function of $x \in (-\infty, \infty)$.

Let $\xi(x) = x^p$, then Lemma 1 can be established in view of (11). □

Theorem 1. Assume Case I is satisfied. For the Gompertz-G family of distributions as given in (6) and $n \in N, \tilde{m} \in \mathbb{R}, k > 0, 1 \leq r \leq n, p = 1, 2, \ldots$

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p}{\alpha\gamma_r} E\big(A(X_{r,n,\tilde{m},k})\big) \tag{12}$$

and subsequently

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p}{\alpha\gamma_r} \sum_{l=0}^{[\beta+1]} (-1)^l \binom{[\beta+1]}{l} E\big(B(X^l_{r,n,\tilde{m},k})\big) \tag{13}$$

where $A(x) = x^{p-1}\frac{(1-G(x))^{\beta+1}}{g(x)}$ and $B^l(x) = x^{p-1}\frac{(G(x))^l}{g(x)}$

Proof. In view of (8) and (10), we have

$$\begin{aligned} E\big(X^p_{r,n,\tilde{m},k}\big) &- E\big(X^p_{r-1,n,\tilde{m},k}\big) \\ &= \frac{pC_{r-1}}{\gamma_r} \int_{-\infty}^{\infty} x^{p-1} \sum_{i=1}^{r} a_i(r) \left(\bar{F}(x)\right)^{\gamma_i - 1} \left(\frac{(1-G(x))^{\beta+1}}{\alpha g(x)} f(x) \right) dx \\ &= \frac{pC_{r-1}}{\alpha\gamma_r} \int_{-\infty}^{\infty} A(x) \sum_{i=1}^{r} a_i(r) \left(\bar{F}(x)\right)^{\gamma_i - 1} f(x) dx \end{aligned}$$

This yields (12). In view of (9) and following the same steps, the relation (13) can be obtained. Hence, the proof of Theorem (1) is completed. □

Corollary 1. For Case II and the condition as stated in Theorem 1

$$E\big(X^p_{r,n,m,k}\big) - E\big(X^p_{r-1,,n,m,k}\big) = \frac{p}{\alpha\gamma_r} E\big(A(X_{r,n,m,k})\big) \tag{14}$$

Proof. Since for $\gamma_i \neq \gamma_j; i \neq j = 1, 2, \ldots, n-1$ but $m_i = m$

$$a_i(r) = \frac{1}{(m+1)^{r-1}} (-1)^{r-i} \frac{1}{(i-1)!(r-i)!}$$

Therefore, PDF given in (1.2) reduces to (1.4). Thus, relation (14) can be established by replacing $\tilde{m}$ with m in (12). □

Remark 1. If $m_i = 0; i = 1, 2, \ldots, n-1$ and $k = 1$, then the relation for single moment of order statistics for Gompertz-G family of distribution is given by

$$E\big(X^p_{r:n}\big) - E\big(X^p_{r-1:n}\big) = \frac{p}{\alpha(n-r+1)} E\big(A(X_{r:n})\big)$$

where $E(X^p_{r:n})$ is the pth moment of rth order statistic.

Remark 2. Let $m_i \to -1, i = 1, 2, \ldots, n-1$, then single moment of kth upper record values is given as

$$E(X^p_{U^{(k)}(n)}) - E(X^p_{U^{(k)}(n-1)}) = \frac{p}{\alpha k} E\left(A(X_{U^{(k)}(n)})\right)$$

where $E(X^p_{U^{(k)}(n)})$, is the pth moment of kth upper record values.

2.1. Examples

In this section, we present some special models of Gompertz-G family by considering the baseline distributions like power function, Pareto, exponential, inverse exponential, Lomax, alpha power inverted exponential, and Frechet. The CDF and PDF of these baseline distributions are listed in Table 1.

Table 1. The CDF and PDF of the baseline distributions for the considered examples

Model	$G(x)$	$g(x)$	Parameters
Power function	$\lambda^{-\theta}x^{\theta}$	$\theta x^{\theta-1}\lambda^{-\theta}$	$0 \le x \le \lambda; \lambda, \theta > 0$
Pareto	$1 - \lambda^{\nu}x^{-\nu}$	$\nu\lambda^{\nu}x^{-(\nu+1)}$	$\lambda \le x \le \infty; \lambda, \nu \ge 0$
Exponential	$1 - e^{-\lambda x}$	$\lambda e^{-\lambda x}$	$x > 0; \lambda > 0$
Inverse exponential	$e^{-\theta/x}$	$\frac{\theta}{x^2}e^{-\theta/x}$	$x > 0; \theta > 0$
Lomax	$1 - (1+\delta x)^{-\theta}$	$\theta\delta\,(1+\delta x)^{-(\theta+1)}$	$x > 0; \delta, \theta > 0$
Alpha power inverted exponential	$\frac{1}{\alpha-1}\left(\alpha^{\exp(-c/x)} - 1\right)$	$\frac{c\log\alpha}{x^2(\alpha-1)}\exp(-c/x)\alpha^{\exp(-c/x)}$	$\alpha > 0, x > 0$
Frechet	$\exp\left(-(\alpha/x)^{\beta}\right)$	$\beta\alpha^{\beta}x^{-[\beta+1]}\exp\left(-(\alpha/x)^{\beta}\right)$	$x > 0; \alpha, \beta > 0$

Now using the above base distributions the CDF, PDF and recurrence relations for the single moment of GOS for some of the distributions of Gompertz-G family are presented below.

2.1.1. Gompertz power function distribution (GOPOW)

The CDF and PDF of GOPOW distribution are given as

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1 - (1 - \lambda^{-\theta}x^{\theta})^{-\beta}\right)\right), \quad 0 \le x \le \lambda; \alpha, \beta > 0 \tag{15}$$

and

$$f(x) = \alpha\theta\lambda^{-\theta}x^{\theta-1}(1 - \lambda^{-\theta}x^{\theta})^{-\beta-1}\exp\left(\frac{\alpha}{\beta}\left(1 - (1 - \lambda^{-\theta}x^{\theta})^{-\beta}\right)\right) \tag{16}$$

Now, it is easy to see that

$$B^l(x) = x^{j-1}\frac{\lambda^{-\theta l}x^{\theta l}}{\theta x^{\theta-1}\lambda^{-\theta}} = \frac{1}{\theta}\lambda^{\theta(1-l)}x^{j+\theta(l-1)}$$

Thus, using (13), we get

$$E\left(X^p_{r,n,\tilde{m},k}\right) - E\left(X^p_{r-1,n,\tilde{m},k}\right) = \frac{p}{\theta\alpha\gamma_r}\sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}\lambda^{\theta(1-l)}E\left(X^{p+\theta(l-1)}_{r,n,\tilde{m},k}\right).$$

2.1.2. Gompertz–Pareto distribution (GOPAR)

The CDF and PDF of GOPAR distribution are given as

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1 - \lambda^{-\nu\beta}x^{\nu\beta}\right)\right), \quad \lambda \le x < \infty; \alpha, \beta > 0 \tag{17}$$

and

$$f(x) = \alpha\nu\lambda^{-\nu\beta}x^{\nu\beta-1}\exp\left(\frac{\alpha}{\beta}\left(1 - \lambda^{-\nu\beta}x^{\nu\beta}\right)\right) \tag{18}$$

Further, we have

$$A(x) = x^{p-1}\frac{(1-G(x))^{\beta+1}}{g(x)} = x^{p-1}\frac{(\lambda^{\nu}x^{-\nu})^{\beta+1}}{\nu\lambda^{\nu}x^{-(\nu+1)}} = \frac{\lambda^{\nu\beta}}{\nu}x^{p-\nu\beta}$$

Thus, in the view of (12), we get

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p\lambda^{\nu\beta}}{\nu\alpha\gamma_r}E\big(X^{p-\nu\beta}_{r,n,\tilde{m},k}\big)$$

2.1.3. Gompertz exponential distribution (GOEXP)

The CDF and PDF of GOEXP distribution are given, respectively, by

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1 - \mathrm{e}^{\lambda\beta x}\right)\right), \quad x > 0,\ \alpha,\ \beta,\ \lambda > 0 \tag{19}$$

and

$$f(x) = \alpha\lambda\exp\left(\frac{\alpha}{\beta}\left(1 - \mathrm{e}^{\lambda\beta x}\right) + \lambda\beta x\right), \quad x > 0 \tag{20}$$

Now we compute $A(x)$ as follows

$$A(x) = x^{p-1}\frac{(1-G(x))^{\beta+1}}{g(x)} = x^{p-1}\frac{\mathrm{e}^{-\lambda x(\beta+1)}}{\lambda\mathrm{e}^{-\lambda x}} = \frac{1}{\lambda}x^{p-1}\mathrm{e}^{-\lambda\beta x} = \sum_{u=0}^{\infty}(-1)^u\frac{\lambda^{u-1}\beta^u}{u!}x^{u+p-1}$$

Therefore using (12), we get

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p}{\alpha\gamma_r}\sum_{u=0}^{\infty}(-1)^u\frac{\lambda^{u-1}\beta^u}{u!}E\big(X^{u+p-1}_{r,n,\tilde{m},k}\big)$$

2.1.4. Gompertz inverse exponential distribution (GOIEX)

The CDF and PDF of GOIEX distribution are given, respectively, by

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1 - (1 - \mathrm{e}^{-\frac{\theta}{x}})^{-\beta}\right)\right), \quad x > 0,\ \alpha,\ \beta,\ \theta > 0 \tag{21}$$

and

$$f(x) = \alpha\frac{\theta}{x^2}\mathrm{e}^{-\frac{\theta}{x}}\left(1 - \exp\big(-\frac{\theta}{x}\big)\right)^{-\beta-1}\exp\left(\frac{\alpha}{\beta}\left(1 - (1 - \mathrm{e}^{-\frac{\theta}{x}})^{-\beta}\right)\right), \quad x > 0 \tag{22}$$

Furthermore, one can find that

$$B^l(x) = x^{p-1}\frac{(G(x))^l}{g(x)} = x^{p-1}\frac{\left(e^{\frac{-\theta}{x}}\right)^l}{\frac{\theta}{x^2}e^{\frac{-\theta}{x}}} = \frac{1}{\theta}x^{p+1}e^{-\theta x^{-1}(l-1)}$$
$$= \frac{1}{\theta}x^{p+1}\sum_{u=0}^{\infty}(-1)^u\frac{(\theta(l-1)x^{-1})^u}{u!} = \frac{1}{\theta}\sum_{u=0}^{\infty}(-1)^u\frac{\theta^u}{u!}(l-1)^u x^{p-u+1}$$

Thus from (13) we obtain

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p}{\alpha\theta\gamma_r}\sum_{l=0}^{[\beta+1]}\sum_{u=0}^{\infty}(-1)^{l+u}\binom{[\beta+1]}{l}\frac{\theta^u(l-1)^u}{u!}E\big(X^{p-u+1}_{r,n,\tilde{m},k}\big)$$

2.1.5. Gompertz–Lomax distribution (GOLOM)

The CDF and PDF of GOLOM distribution are given, respectively, by

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\left(1-(1+\delta x)^{\theta\beta}\right)\right), \quad x>0,\ \alpha,\ \beta>0 \tag{23}$$

and

$$f(x) = \alpha\delta(1+x\delta)^{\beta\theta-1}\exp\left(\frac{\alpha}{\beta}\left(1-(1+\delta x)^{\theta\beta}\right)\right), \quad x>0,\ \alpha,\ \beta>0 \tag{24}$$

Now, it can be seen that

$$A(x) = x^{p-1}\frac{(1-G(x))^{\beta+1}}{g(x)} = x^{p-1}\frac{(1+\delta x)^{-\theta(\beta+1)}}{\theta\delta\,(1+\delta x)^{-(\theta+1)}} = \frac{x^{p-1}}{\theta\delta}(1+\delta x)^{1-\theta\beta}$$
$$= \frac{x^{p-1}}{\theta\delta}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}(\delta x)^{1-\theta\beta-t} = \frac{1}{\theta}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}\delta^{-(t+\theta\beta)}x^{p-\theta\beta-t}$$

Thus, in the view of (12), we get

$$E\big(X^p_{r,n,\tilde{m},k}\big) - E\big(X^p_{r-1,n,\tilde{m},k}\big) = \frac{p}{\alpha\theta\gamma_r}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}\delta^{-(t+\theta\beta)}E\big(X^{p-\theta\beta-t}_{r,n,\tilde{m},k}\big)$$

3. Product moments

Lemma 2. For Case I with PDF as given in (2) and $1 \le r < s \le n,\ n \ge 1,\ k>0, p,\ q = 1,\ 2,\ldots$

$$E\big(X^p_{r,n,\tilde{m},k}X^q_{s,n,\tilde{m},k}\big) - E\big(X^p_{r,n,\tilde{m},k}X^q_{s-1,n,\tilde{m},k}\big) = qC_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty}x^p y^{q-1}$$
$$\times\left(\sum_{j=r+1}^{s}a_j^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\right)\left(\sum_{i=1}^{r}a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}dydx \tag{25}$$

Proof. Athar and Islam [7] have shown that

$$E\big(\xi\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\big) - E\big(\xi\{X_{r,n,\tilde{m},k}, X_{s-1,n,\tilde{m},k}\}\big)$$

$$= C_{s-2} \int_{-\infty}^{\infty}\int_{x}^{\infty} \frac{\partial}{\partial y}\xi(x,y)\left(\sum_{j=r+1}^{s} a_j^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\right)\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}dydx \tag{26}$$

Lemma can be established by letting $\xi(x,y) = x^p y^q$ in (26). □

Theorem 2. Let Case I be satisfied. For the Gompertz-G family of distributions as given in (6) and $n \in N$, $\tilde{m} \in \mathbb{R}$, $k > 0$, $1 \leq r < s \leq n$, $p, q = 1, 2, \ldots$

$$E\big(X^p_{r,n,\tilde{m},k} X^q_{s,n,\tilde{m},k}\big) - E\big(X^p_{r,n,\tilde{m},k} X^q_{s-1,n,\tilde{m},k}\big) = \frac{q}{\alpha\gamma_s} E\big(A\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\big) \tag{27}$$

and subsequently

$$\begin{aligned} &E\big[X^p_{r,n,\tilde{m},k}.X^q_{s,n,\tilde{m},k}\big] - E\big[X^p_{r,n,\tilde{m},k}.X^q_{s-1,n,\tilde{m},k}\big] \\ &= \frac{q}{\alpha\,\gamma_s}\sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}E\big[B^l\{X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k}\}\big] \end{aligned} \tag{28}$$

where $A(x,y) = x^p y^{q-1}\frac{(1-G(y))^{\beta+1}}{g(y)}$ and $B^l(x,y) = x^p y^{q-1}\frac{(G(y))^l}{g(y)}$.

Proof. In view of (8) and (25), we obtain

$$\begin{aligned} &E\big(X^p_{r,n,\tilde{m},k} X^q_{s,n,\tilde{m},k}\big) - E\big(X^p_{r,n,\tilde{m},k} X^q_{s-1,n,\tilde{m},k}\big) \\ &= \frac{qC_{s-1}}{\alpha\gamma_s}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^p y^{q-1}\left(\sum_{j=r+1}^{s} a_j^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\right)\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right) \\ &\times \frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}\frac{(1-G(y))^{\beta+1}}{g(y)}dydx = \frac{qC_{s-1}}{\alpha\gamma_s}\int_{-\infty}^{\infty}\int_{x}^{\infty} A(x,y)\left(\sum_{j=r+1}^{s} a_j^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_j}\right) \\ &\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx \end{aligned}$$

This gives (27). Proceeding on the same lines and using (9), we can get (28). Thus, the proof of Theorem (2) is completed. □

Corollary 2. For Case II and the condition as stated in Theorem 2

$$E\big(X^p_{r,n,m,k} X^q_{s,n,m,k}\big) - E\big(X^p_{r,n,m,k} X^q_{s-1,n,m,k}\big) = \frac{q}{\alpha\gamma_s} E\big(A(X_{r,n,m,k}, X_{s,n,m,k})\big) \tag{29}$$

Proof. Since for $\gamma_i \neq \gamma_j$, $i \neq j = 1, 2, \ldots, n-1$ but $m_i = m$

$$a_i^{(r)}(s) = \frac{1}{(m+1)^{s-r-1}}(-1)^{s-i}\frac{1}{(i-r-1)!(s-i)!}.$$

Therefore, joint PDF of $X_{r,n,\tilde{m},k}$ and $X_{s,n,\tilde{m},k}$ given in (3) reduces to (5).

Thus, relation (29) can be established by replacing $\tilde{m}$ with m in (27). □

Remark 3. If $m_i = 0, i = 1, 2, \ldots, n-1$ and $k = 1$, then the relation for product moment of order statistics for Gompertz-G family of distribution is given by

$$E\left(X_{r:n}^p X_{s:n}^q\right) - E\left(X_{r:n}^p X_{s-1:n}^q\right) = \frac{q}{\alpha(n-s+1)} E\left(A(X_{r:n}, X_{s:n})\right)$$

Remark 4. Let $m_i \to -1, i = 1, 2, \ldots, n-1$, then product moment of kth upper record values is

$$E(X_{U^{(k)}(n)}^p X_{U^{(k)}(m)}^q) - E(X_{U^{(k)}(n)}^p X_{U^{(k)}(m-1)}^q) = \frac{q}{\alpha k} E\left(A(X_{U^{(k)}(n)}, X_{U^{(k)}(m)})\right)$$

3.1. Examples

3.1.1. Gompertz power function distribution (GOPOW)

For the given CDF in (15), we get

$$B^l(x,y) = x^p y^{q-1} \frac{(G(y))^l}{g(y)} = x^p y^{q-1} \frac{\lambda^{-\theta l} y^{\theta l}}{\theta y^{\theta-1} \lambda^{-\theta}} = \frac{1}{\theta} \lambda^{\theta(1-l)} y^{q+\theta(l-1)} x^p$$

Therefore, in view of (28), it is easy to see that

$$\begin{aligned} &E\left(X_{r,n,\tilde{m},k}^p X_{s,n,\tilde{m},k}^q\right) - E\left(X_{r,n,\tilde{m},k}^p X_{s-1,n,\tilde{m},k}^q\right) \\ &\qquad = \frac{q}{\alpha\theta\gamma_s} \sum_{l=0}^{[\beta+1]} (-1)^l \lambda^{\theta(1-l)} \binom{[\beta+1]}{l} E\left(X_{r,n,\tilde{m},k}^p X_{s,n,\tilde{m},k}^{q+\theta(l-1)}\right). \end{aligned}$$

3.1.2. Gompertz–Pareto distribution (GOPAR)

Using the CDF given in (17), we have

$$A(x,y) = x^p y^{q-1} \frac{(1-G(y))^{\beta+1}}{g(y)} = x^p y^{q-1} \frac{(\lambda^\nu y^{-\nu})^{\beta+1}}{\nu \lambda^\nu y^{-(\nu+1)}} = \frac{\lambda^{\nu\beta}}{\nu} y^{q-\nu\beta} x^p$$

Thus, using (27), we obtain

$$E\left(X_{r,n,\tilde{m},k}^p X_{s,n,\tilde{m},k}^q\right) - E\left(X_{r,n,\tilde{m},k}^p X_{s-1,n,\tilde{m},k}^q\right) = \frac{q\lambda^{\nu\beta}}{\nu\alpha\gamma_s} E\left(X_{r,n,\tilde{m},k}^p X_{s,n,\tilde{m},k}^{q-\nu\beta}\right).$$

3.1.3. Gompertz exponential distribution (GOEXP)

For the CDF given in (19), we have

$$A(x,y) = x^p y^{q-1} \frac{(1-G(y))^{\beta+1}}{g(y)} = x^p y^{q-1} \frac{\mathrm{e}^{-\lambda y(\beta+1)}}{\lambda \mathrm{e}^{-\lambda y}} = \frac{1}{\lambda} x^p y^{q-1} \mathrm{e}^{-\lambda\beta y} = x^p \sum_{w=0}^{\infty} (-1)^w \frac{\lambda^{w-1}\beta^w}{w!} y^{w+q-1}$$

Now, in view of (27), it is easy to see that

$$E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s,n,\tilde{m},k}\right)-E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s-1,n,\tilde{m},k}\right)=\frac{q}{\alpha\gamma_s}\sum_{w=0}^{\infty}\frac{(-1)^w}{w!}\lambda^{p-1}\beta^{w}E\left(X^{p}_{r,n,\tilde{m},k}X^{q+w-1}_{s,n,\tilde{m},k}\right)$$

3.1.4. Gompertz inverse exponential distribution (GOIEX)

For the given CDF in (21), we have

$$\begin{aligned}B^{l}(x,y)&=x^{p}y^{q-1}\frac{(G(y))^{l}}{g(y)}=x^{p}y^{q-1}\frac{\left(\mathrm{e}^{\frac{-\theta}{y}}\right)^{l}}{\frac{\theta}{y^2}\mathrm{e}^{\frac{-\theta}{y}}}=\frac{x^p}{\theta}y^{q+1}\mathrm{e}^{-\theta y^{-1}(l-1)}\\&=\frac{x^p}{\theta}y^{q+1}\sum_{u=0}^{\infty}(-1)^{u}\frac{(\theta(l-1)y^{-1})^{u}}{u!}=\frac{x^p}{\theta}\sum_{u=0}^{\infty}(-1)^{u}\frac{\theta^{u}}{u!}(l-1)^{u}y^{q-u+1}\end{aligned}$$

Therefore, in view of (28), we obtain

$$\begin{aligned}&E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s,n,\tilde{m},k}\right)-E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s-1,n,\tilde{m},k}\right)\\&\quad=\frac{q}{\alpha\theta\gamma_s}\sum_{l=0}^{[\beta+1]}\sum_{u=0}^{\infty}(-1)^{l+u}\binom{[\beta+1]}{l}\frac{\theta^{u}(l-1)^{u}}{u!}E\left(X^{p}_{r,n,\tilde{m},k}X^{q-u+1}_{s,n,\tilde{m},k}\right)\end{aligned}$$

3.1.5. Gompertz–Lomax distribution (GOLOM)

Using the CDF in (23), one can show that

$$\begin{aligned}A(x,y)&=x^{p}y^{q-1}\frac{(1-G(y))^{\beta+1}}{g(y)}=x^{p}y^{q-1}\frac{(1+\delta y)^{-\theta(\beta+1)}}{\theta\delta\,(1+\delta y)^{-(\theta+1)}}=\frac{x^{p}y^{q-1}}{\theta\delta}(1+\delta y)^{1-\theta\beta}\\&=\frac{x^{p}y^{q-1}}{\theta\delta}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}(\delta y)^{1-\theta\beta-t}=\frac{x^p}{\theta}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}\delta^{-(t+\theta\beta)}y^{q-\theta\beta-t}\end{aligned}$$

Now, in the view of (27), we get

$$\begin{aligned}&E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s,n,\tilde{m},k}\right)-E\left(X^{p}_{r,n,\tilde{m},k}X^{q}_{s-1,n,\tilde{m},k}\right)\\&\quad=\frac{q}{\alpha\theta\gamma_s}\sum_{t=0}^{[1-\theta\beta]}\binom{[1-\theta\beta]}{t}\delta^{-(t+\theta\beta)}E\left(X^{p}_{r,n,\tilde{m},k}X^{q-\theta\beta-t}_{s,n,\tilde{m},k}\right)\end{aligned}$$

4. Characterization

In this section, the characterization of Gompertz-G family of distributions define in (6), is presented through recurrence relations between the moments of GOS.

Theorem 3. Fix a positive integer k and let p be a non-negative integer. A necessary and sufficient condition for a random variable X to be distributed with PDF given in (7) is that

$$E\left(X_{r,n,\tilde{m},k}^{p}\right) - E\left(X_{r-1,n,\tilde{m},k}^{p}\right) = \frac{p}{\alpha\gamma_r}\sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}E\left(B(X_{r,n,\tilde{m},k}^{l})\right) \tag{30}$$

where $B^l(x) = x^{p-1}\dfrac{(G(x))^l}{g(x)}$.

Proof. Necessary part follows from (13). To prove the sufficiency part, suppose the relation in (30) is true. Now, using (2) and (10) in (30), we get

$$pC_{r-2}\int_{-\infty}^{\infty} x^{p-1}\sum_{i=1}^{r} a_i(r)\left(\bar{F}(x)\right)^{\gamma_i} dx = \frac{pC_{r-1}}{\alpha\gamma_r}\sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}$$
$$\times\int_{-\infty}^{\infty} x^{p-1}\left(\frac{(G(x))^l}{g(x)}\sum_{i=1}^{r} a_i(r)\left(\bar{F}(x)\right)^{\gamma_i-1} f(x)\right)dx$$

This implies

$$\frac{pC_{r-1}}{\alpha\gamma_r}\int_{-\infty}^{\infty} x^{p-1}\sum_{i=1}^{r} a_i(r)\left(\bar{F}(x)\right)^{\gamma_i-1}\left(\alpha\bar{F}(x) - \sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}\frac{(G(x))^l}{g(x)}f(x)\right)dx = 0 \tag{31}$$

Applying the extension of Müntz–Szász theorem (see, e.g., [18]) to (31), we get

$$\frac{\bar{F}(x)}{f(x)} = \frac{1}{\alpha g(x)}\sum_{l=0}^{[\beta+1]}(-1)^l\binom{[\beta+1]}{l}G^l(x)$$

Thus, $f(x)$ is the PDF as given in (7). Hence, Theorem (3) holds. □

Theorem 4. Fix a positive integer k and let p, q are non-negative integers. A necessary and sufficient condition for random variables X, Y to be distributed with PDF given in (7) is that

$$\begin{aligned} &E\left(X_{r,n,\tilde{m},k}^{p}X_{s,n,\tilde{m},k}^{q}\right) - E\left(X_{r,n,\tilde{m},k}^{p}X_{s-1,n,\tilde{m},k}^{q}\right) \\ &\qquad = \frac{q}{\alpha\gamma_s}\sum_{t=0}^{[\beta+1]}(-1)^t\binom{[\beta+1]}{t}E\left(B^t(X_{r,n,\tilde{m},k}, X_{s,n,\tilde{m},k})\right) \end{aligned} \tag{32}$$

where $B^t(x,y) = x^i y^{j-1}\frac{(G(y))^t}{g(y)}$.

Proof. If part follows from (28). To prove only if part, suppose the relation in (32) holds. Now, in view of (3) and (25), we have

$$qC_{s-2}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^p y^{q-1}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right)\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}dydx$$

$$= \frac{qC_{s-1}}{\alpha\gamma_s}\sum_{t=0}^{[\beta+1]}(-1)^t\binom{[\beta+1]}{t}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^p y^{q-1}\frac{G^t(y)}{g(y)}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right)$$

$$\times\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}dydx$$

This implies

$$\frac{qC_{s-1}}{\alpha\beta\gamma_s}\int_{-\infty}^{\infty}\int_{x}^{\infty} x^p y^{q-1}\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i}\right)\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{\gamma_i}\right)\frac{f(x)}{\bar{F}(x)}$$
$$\times\left(\alpha\beta-\sum_{t=0}^{[\beta+1]}(-1)^t\binom{[\beta+1]}{t}\frac{G^t(y)}{g(y)}\frac{f(y)}{\bar{F}(y)}\right)dydx = 0 \tag{33}$$

Applying the extension of Müntz–Szász theorem (see, e.g., [18]) to (33), we get

$$\frac{\bar{F}(y)}{f(y)} = \frac{1}{\alpha g(y)}\sum_{t=0}^{[\beta+1]}(-1)^t\binom{[\beta+1]}{t}G^t(y).$$

Thus, $f(y)$ is a PDF as given in (7). Thus, the proof of Theorem (4) is completed. □

5. Conclusions

The Gompertz-G family of distributions with two additional shape parameters has been proposed by Alizadeh et al. [4]. It includes a wide family of continuous distributions and gives a better fit to generated distributions. The GOS is a unified approach for several ordered random variables, like order statistics, record values, sequential order statistics, etc. The main purpose of this study is to demonstrate moments of generalized order statistics for several continuous distributions belonging to this class. Moreover, characterizing a probability distribution plays an important role in statistical studies and has significant applications in natural and applied sciences. Thus, the characterization of this general class of distribution is also carried out using moment properties.

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