STATISTICS IN TRANSITION new series, Autumn 2014
Vol. 15, No. 4, pp. 501–524

A GENERAL CLASS OF MEAN ESTIMATORS USING MIXTURE OF AUXILIARY VARIABLES FOR TWO-PHASE SAMPLING IN THE PRESENCE OF NON-RESPONSE

Zahoor Ahmad[1], Rahma Zubair[2], Ummara Shahid[3]

ABSTRACT

In this paper we have proposed a general class of estimators for two-phase sampling to estimate the population mean in the case when non-responses occur at the first phase. Furthermore, several continuous and categorical auxiliary variable(s) have been simultaneously used while constructing the class. Also, it is assumed that the information on all auxiliary variables is not available for population, which is often the case. The expressions of the mean square error of the suggested class have been derived and several special cases of the proposed class have been identified. The empirical study has also been conducted.

Key words: non-response, multi-auxiliary variables, regression-cum-ratio-exponential estimators, no information case.

1. Introduction

The most common method of data collection in survey research is sending the questionnaire through mail. The reason may be the minimum cost involved in this method. But this method has a major disadvantage that a large rate of non-response may occur, which may result in an unknown bias, while the estimate based only on responding units is representative of both responding and non-responding units.

A personal interview is another method of data collection which generally may result in a complete response, but the cost involved in personal interviews is much higher than the mail questionnaire method. We may conclude from the above discussion that the advantage of one method is the disadvantage of the other and vice versa. Hansen and Hurwitz (1946) combined the advantages of both procedures. They considered the issue of determining the number of mail

[1] Department of Statistics, University of Gujrat, Gujrat, Pakistan. E-mail: zahoor.ahmed@uog.edu.pk.
[2] Department of Statistics, University of Gujrat, Gujrat, Pakistan. E-mail: rahma_stat@ymail.com.
[3] Department of Statistics, University of Gujrat, Gujrat, Pakistan. E-mail: ummara.shahid@uog.edu.pk.

questionnaires along with the number of personal interviews to be carry out given non-response to the mail questionnaire in order to attain the required precision at minimum cost.

Hansen and Hurwitz (1946) discussed the sampling scheme considering non-response and constructed the following unbiased estimator for population mean $\bar{Y}$ of variable of interest y as

$$\bar{y}_1^* = w_1\bar{y}_1 + w_1^{'}\bar{y}_{1r}, \tag{1.1}$$

where $\bar{y}_1$ and $\bar{y}_{1r}$ denote the means for the respondent and re-contacted sample respectively, and further it is assumed that there is no non-response at re-contacted sample. The weights $w_1 = n_{11}/n_1$ and $w_1^{'} = n_{12}/n_1$.

The variance of (1.1) is

$$Var(\bar{y}_1^*) = (N - n_1)(n_1 N)^{-1} S_y^2 + \frac{W_2(k_1 - 1)}{n_1} S_{y_2^*}^2, \tag{1.2}$$

where $S_y^2 = (N-1)^{-1}\sum_{j=1}^{N}(y_j - \bar{Y})^2$ and $S_{y_2^*}^2 = (N_2 - 1)\sum_{j=1}^{N_2}(y_j - \bar{Y}_2)^2$ are population variances for responding and non-responding portions with means $\bar{Y} = (N)^{-1}\sum_{i=1}^{N} y_i$ and $\bar{Y}_2 = (N_2)^{-1}\sum_{i=1}^{N_2} y_i$, $W_2 = N_2(N)^{-1}$.

Singh et al. (2010) emphasized that precision of an estimator can be increased using auxiliary variable in estimation procedure when the study variable y is highly correlated with the auxiliary variable x. In the case of two phase sampling, Wu and Luan (2003) argue that when we take a large first phase sample from the population and a sub-sample from the first phase sample then there is an issue of small sample size and large non-response rate, and as a result the mean square error becomes larger. This effect can be compensated using auxiliary variables that are highly correlated with the study variable in the estimation procedure. The major advantage of using two-phase sampling is the gain in high precision without substantial increase in cost.

The availability of population auxiliary information plays an important role in efficiency of estimators in two-phase sampling. In the case of at least two auxiliary variables, Samiuddin and Hanif (2007) show that auxiliary information can be utilized in three ways depending on the availability of auxiliary information for population. Firstly, No Information Case (NIC): when population information on all auxiliary variables is not available. Secondly, Partial Information Case (PIC): when population information on some auxiliary variables is available. Thirdly, Full Information Case (FIC): when population information on all auxiliary variables is available. Ahmad and Hanif (2010) clarify that case for a specific estimation procedure, - the estimator for FIC will be more efficient then the estimator for PIC and the estimator for PIC will me more efficient then the estimator for NIC.

Ahmad et al. (2009a, 2009b, 2010) and Ahmad and Hanif (2010) developed several univariate and multivariate classes of ratio and regression estimators using multi-auxiliary variables under these three cases of availability of auxiliary information for population.

Many survey statisticians have used the quantitative auxiliary variables for constructing their estimators in two-phase sampling. Furthermore, some authors have used qualitative auxiliary variables for estimating the unknown population parameters (see Jhajj et al. (2006), Shabbir and Gupta (2007), Samiuddin and Hanif (2007), Shahbaz and Hanif (2009), Haq et al. (2009), Hanif et al. (2010)).

As mentioned earlier, Hansen-Hurwitz (1946) dealt with non-response problem for simple random sampling and suggested an estimator without using auxiliary information. Many researchers such as Khair and Srivastava (1993,1995), Singh and Kumar (2008a, 2009a) developed different ratios, product and regression estimators to estimate population mean of study variables in two-phase sampling when non-response occurs at the second phase. Tabasum and Khan (2004) revisited the ratio-type estimator by Khair and Srivastava (1993) and found that the cost of this estimator is lower than the cost gained by Hansen-Hurwitz (1946) estimator. Singh et al. (2010) proposed two exponential-type estimators and/or auxiliary variables when non-response occurs during the study.

Ahmad et al. (2012, 2013a, 2013b) proposed the class of generalized estimators to estimate the population mean using multi-auxiliary quantitative variables in the presence of non-responses at the first phase, second phase and both phases.

After introducing the concept of estimating the mean of study variable using a mixture of auxiliary variables in the presence of non-responses, some important references regarding estimators of population mean in the presence of non-responses in single and two-phase sampling using quantitative and qualitative auxiliary variables have been discussed separately in Section 1. In Section 2 we have proposed a generalized class of regression-cum-ratio-exponential estimators for estimating the mean of study variable using a mixture of auxiliary variables in the presence of non-responses at the first phase and its special cases are also given in this section. A detailed empirical study have been conducted and discussed in Section 3. Some conclusions are provided in Section 4.

2. Generalized class of regression-cum-ratio-exponential estimators in two-phase sampling

Most of the literature is devoted to the case when non-responses occur at the second phase, but in two-phase sampling, when auxiliary information is obtained at the first phase sample that is relatively larger than the second phase sample, the non-response rate will be high as compared to the second phase. The two-phase sampling scheme when non-responses occur at the first phase is discussed as follows.

Consider the total population (denoted by U) of N units is divided into two sections: one is the section (denoted by U_1) of N_1 units, which would be available at the first attempt at the first phase, and the other section (denoted by U_2) of N_2 units, which are not available at the first attempt at the first phase but will be available at the second attempt. From N units, a first phase sample (denoted by u_1) of n_1 units is drawn by simple random sampling without replacement (SRSWOR). At the first phase let m_1' units supply information which is denoted by v_1' and m_2' units refuse to respond, which is denoted by v_2', where $v_1' = u_1 \cap U_1$ and $v_2' = u_1 \cap U_2$. A subsample (denoted by v_{2m}') of r_1 units is randomly taken from the m_2' non-respondents by applying the strategy defined by Hansen and Hurwitz (1946) and this subsample is specified by $r_1 = m_2'/k_1$, $k_1 > 1$. It is assumed that no non-response is observed in this subsample. A second phase sample (denoted by u_2) of n_2 units (i.e. $n_2 < n_1$) is drawn from n_1 by SRSWOR and the variable of interest y is measured at the second phase. The above sampling scheme can be easily understandable from Figure 1.

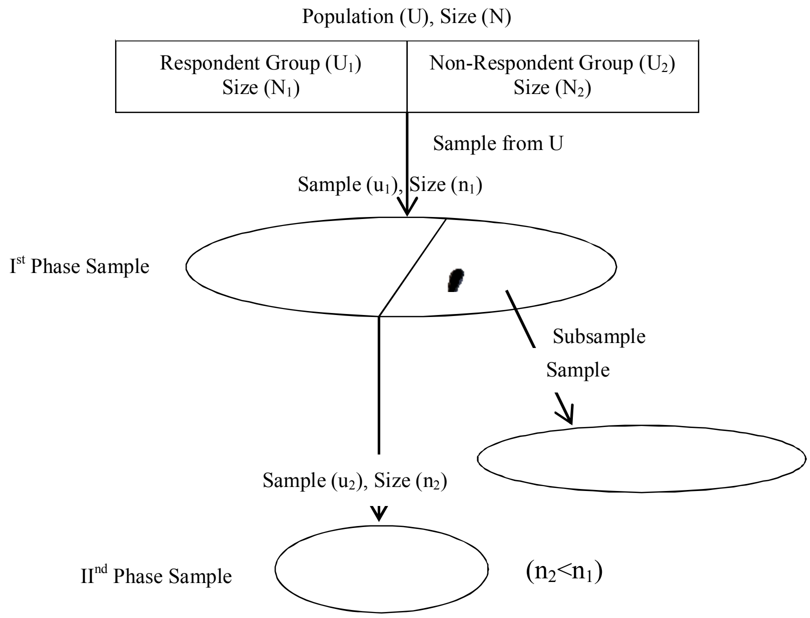

Figure 1. Two-phase sampling scheme when non-responses occur at the first phase

The literature is evident that there is no estimator that can utilize auxiliary information on both quantitative and qualitative variables. But, in sample surveys, the information on both quantitative and qualitative variables is collected either at the first phase and/or the second phase. For example, we want to estimate the average CGPA of a student in BS (Honor). The information on variables like previous degree marks, attendance, number of hours spent in library, if a student is a member of rural or urban area, father's profession, having a laptop or not, having internet facility or not, etc., can be used as auxiliary information to estimate average CGPA with more efficiency. Hence, there is a need to develop an estimator that can utilize auxiliary information on both quantitative and qualitative variables.

For the first time a combination of regression and ratio technique for simple random sampling called regression-cum-ratio estimator was used by Mohanty (1967) to estimate the population mean of study variable. Similarly, the sum of the ratio and exponential components with some suitable weights can be combined with regression component to develop a general class of regression-cum-ratio-exponential estimators. Furthermore, the objective of suggesting such a class is to search for the best member from all members of the class.

We have suggested the general class of estimators for two-phase sampling to estimate the population mean of the study variable in the case when non-response occurs at the first phase. Moreover, several quantitative and qualitative auxiliary variables have been used simultaneously while constructing the class. Also, it is assumed that population information is not available for all auxiliary variables that is the natural case.

The proposed class is

$$t_{mix} = t_1(t_2 + t_3), \tag{2.1}$$

where

$$t_1 = \eta \bar{y}_2 + a\sum_{i=1}^{q_1} \alpha_{1i}\left(\bar{x}^*_{(1)i} - \bar{x}_{(2)i}\right) + b\sum_{i'=1}^{q_2} \alpha_{2i'}\left(\tau^*_{(1)i'} - \tau_{(2)i'}\right),$$

$$t_2 = c\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{d\alpha_{3j}} \prod_{j'=1}^{q_4}\left(\frac{\omega^*_{(1)j'}}{\omega_{(2)j'}}\right)^{h\alpha_{4j'}}\right]$$

and $$t_3 = e\exp\left[\sum_{k=1}^{q_5} f\alpha_{5k}\left(\frac{\bar{w}_{(2)k} - \bar{w}^*_{(1)k}}{\bar{w}_{(2)k} + \bar{w}^*_{(1)k}}\right) + \sum_{k'=1}^{q_6} l\alpha_{6k'}\left(\frac{\phi_{(2)k} - \phi^*_{(1)k}}{\phi_{(2)k} + \phi^*_{(1)k}}\right)\right] \text{ for } c + e = 1,$$

where *a, b, c, d, e, f, h, l* are constants to be chosen for generating members of this class and η & α_i's for all $i = 1,2,...,6$ are unknown constants to be determined by minimizing the mean square error of t_{mix} given in (2.1) and $\sum_{i=1}^{6} q_i = m.$ Where

y : denotes the study variable

x_i : denotes the i^{th} auxiliary quantitative variables for $i = 1,2,3,...,q_1$

$\tau_{i'}$: denotes the i'^{th} auxiliary qualitative variables for $i' = 1,2,3,...,q_2$

z_j : denotes the j^{th} auxiliary quantitative variables for $j = 1,2,3,...,q_3$

$\omega_{j'}$: denotes the j'^{th} auxiliary qualitative variables for $j' = 1,2,3,...,q_4$

w_k : denotes the k^{th} auxiliary quantitative variables for $k = 1,2,3,...,q_5$

$\phi_{k'}$: denotes the k'^{th} auxiliary qualitative variables for $k' = 1,2,3,...,q_6$

$\bar{X}_i = N^{-1}\sum_{t=1}^{N} x_{ti}$: denotes the population mean of i^{th} auxiliary variable

$\Phi_{i'} = N^{-1}\sum_{t=1}^{N} \tau_{ti'}$: denotes the population proportion of i'^{th} auxiliary attribute

$\bar{Z}_j = N^{-1}\sum_{t=1}^{N} z_{tj}$: denotes the population mean of j^{th} auxiliary variable

$\Psi_{j'} = N^{-1}\sum_{t=1}^{N} \omega_{tj'}$: denotes the population proportion of j'^{th} auxiliary attribute

$\bar{W}_k = N^{-1}\sum_{t=1}^{N} w_{tk}$: denotes the population mean of k^{th} auxiliary variable

$\mathrm{E}_{k'} = N^{-1}\sum_{t=1}^{N} \phi_t$: denotes the population proportion of k'^{th} auxiliary attribute

$\bar{x}_{(1)i} = n_1^{-1}\sum_{t=1}^{n_1} x_{ti}$: denotes the first phase sample mean of i^{th} auxiliary variable

$\bar{z}_{(1)j} = n_1^{-1}\sum_{t'=1}^{n_1} z_{t'j}$: denotes the first phase sample mean of j^{th} auxiliary variable

$\bar{w}_{(1)k} = n_1^{-1}\sum_{t''=1}^{n_1} w_{t''k}$: denotes the first phase sample mean of k^{th} auxiliary variable

$\tau_{(1)i'} = n_1^{-1}\sum_{t=1}^{n_1} \tau_{ti'}$: denotes the first phase sample proportion of i'^{th} auxiliary attribute

$\omega_{(1)j'} = n_1^{-1}\sum_{t'=1}^{n_1} \omega_{t'j'}$: denotes the first phase sample proportion of j'^{th} auxiliary attribute

$\phi_{(1)k'} = n_1^{-1}\sum_{t''=1}^{n_1} \phi_{t''k'}$: denotes the first phase sample proportion of k'^{th} auxiliary attribute

$\bar{y}_2$: denotes the mean of the study variable for the second phase sample

$\bar{x}_{(2)i} = n_2^{-1}\sum_{t=1}^{n_2} x_{ti}$: denotes the second phase sample mean of i^{th} auxiliary variable

$\bar{z}_{(2)j} = n_2^{-1}\sum_{t'=1}^{n_2} z_{t'j}$: denotes the second phase sample mean of j^{th} auxiliary variable

$\bar{w}_{(2)k} = n_2^{-1}\sum_{t''=1}^{n_2} w_{t''k}$: denotes the second phase sample mean of k^{th} auxiliary variable

$\tau_{(2)i'} = n_2^{-1}\sum_{t=1}^{n_2} \tau_{ti'}$: denotes the second phase sample proportion of i'^{th} auxiliary attribute

$\omega_{(2)j'} = n_2^{-1}\sum_{t'=1}^{n_2} \omega_{t'j'}$: denotes the second phase sample proportion of j'^{th} auxiliary attribute

$\phi_{(2)k'} = n_2^{-1}\sum_{t''=1}^{n_2} \phi_{t''k'}$: denotes the second phase sample proportion of k'^{th} auxiliary attribute

$\bar{x}_{(r_1)i} = \frac{1}{r_1}\sum_{t=1}^{r_1} x_t$: denotes the first phase subsample mean of i^{th} auxiliary variable

$\bar{z}_{(r_1)j} = \frac{1}{r_1}\sum_{t=1}^{r_1} z_t$: denotes the first phase subsample mean of j^{th} auxiliary variable

$\bar{w}_{(r_1)k} = \frac{1}{r_1}\sum_{t=1}^{r_1} w_t$: denotes the subsample mean of k^{th} auxiliary variable

$\tau_{(r_1)i'} = \frac{1}{r_1}\sum_{t=1}^{r_1} \tau_{ti'}$: denotes the first phase subsample proportion of i'^{th} auxiliary attribute

$\omega_{(r_1)j'} = \frac{1}{r_1}\sum_{t'=1}^{r_1} \omega_{t'j'}$: denotes the first phase subsample proportion of j'^{th} auxiliary attribute

$\phi_{(r_1)k'} = \frac{1}{r_1}\sum_{t''=1}^{r_1} \phi_{t''k'}$: denotes the first phase subsample proportion of k'^{th} auxiliary attribute

$\bar{x}^*_{(1)i} = w_1\bar{x}_{(1)i} + w_2\bar{x}_{(r_1)i}$: denotes the first phase sample mean of i^{th} auxiliary variable considering non-response

$\bar{z}^*_{(1)j} = w_1\bar{z}_{(1)j} + w_2\bar{z}_{(r_1)j}$: denotes the first phase sample mean of j^{th} auxiliary variable considering non-response

$\bar{w}^*_{(1)k} = w_1\bar{w}_{(1)k} + w_2\bar{w}_{(r_1)k}$: denotes the first phase sample mean of k^{th} auxiliary variable considering non-response

$\tau^*_{(1)i'} = w_1\tau_{(1)i'} + w_2\tau_{(r_1)i'}$: denotes the first phase sample proportion of i'^{th} auxiliary variable considering non-response

$\omega^*_{(1)j'} = w_1\omega_{(1)j'} + w_2\omega_{(r_1)j'}$: denotes the first phase sample proportion of j'^{th} auxiliary variable considering non-response

$\phi^*_{(1)k'} = w_1\phi_{(1)k'} + w_2\phi_{(r_1)k'}$: denotes the first phase sample proportion of k'^{th} auxiliary variable considering non-response

First, considering the regression component t_1 of (2.1), let $\bar{e}_{y(2)} = \bar{y}_{(2)} - \bar{Y}$ be the sampling errors of y, $\bar{e}^*_{x(1)i} = \bar{x}^*_{(1)i} - \bar{X}_i$ and $\bar{e}^*_{\tau(1)i'} = \tau^*_{(1)i'} - \Phi_{i'}$ be the sampling errors of i^{th} auxiliary variable and i'^{th} auxiliary attributes respectively in the presence of non-responses at the first phase, and let $\bar{e}_{x(1)i} = \bar{x}_{(1)i} - \bar{X}_i$ and $\bar{e}_{\tau(1)i'} = \tau_{(1)i'} - \Phi_{i'}$ be the sampling errors of i^{th} auxiliary variable and i'^{th} auxiliary attribute respectively at the first phase. Using sampling errors form, and after simplification, t_1 becomes

$$t_1 = \eta\left(\bar{e}_{y(2)} + \bar{Y}\right) + a\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{e}^*_{x(1)i} - \bar{e}_{x(2)i}\right) + b\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\bar{e}^*_{\tau(1)i'} - \bar{e}_{\tau(2)i'}\right),$$

In matrix notation we can write

$$t_1 = \eta\left(\bar{e}_{y(2)} + \bar{Y}\right) - a\boldsymbol{\alpha}_1^{\mathbf{t}}\mathbf{d}_{\mathbf{x}} - b\boldsymbol{\alpha}_2^{\mathbf{t}}\mathbf{d}_{\boldsymbol{\tau}}, \tag{2.2}$$

where $\boldsymbol{\alpha}_1^{\mathbf{t}}$ and $\boldsymbol{\alpha}_2^{\mathbf{t}}$ are vectors of an unknown coefficients, and vectors $\mathbf{d}_{\mathbf{x}}^{\mathbf{t}} = \left[d_i\right]_{1\times q_1}$, with $d_i = \left(\bar{e}_{x(2)i} - \bar{e}^*_{x(1)i}\right)$ and $\mathbf{d}_{\boldsymbol{\tau}}^{\mathbf{t}} = \left[d_{i'}\right]_{1\times q_2}$, with $d_i = \left(\bar{e}_{\tau(2)i'} - \bar{e}^*_{\tau(1)i'}\right)$.

Now, considering the ratio component t_2 of (2.1) let $\bar{e}^*_{z(1)j} = \bar{z}^*_{(1)j} - \bar{Z}_j$, $\bar{e}^*_{\omega(1)j'} = \omega^*_{(1)j'} - \Psi_{j'}$ be the sampling errors of j^{th} auxiliary quantitative variable and j'^{th} auxiliary qualitative variable at the first phase in the presence of non-responses and $\bar{e}_{z(1)j} = \bar{z}_{(1)j} - \bar{Z}_j$, $\bar{e}_{\omega(1)j'} = \omega_{(1)j'} - \Psi_{j'}$ be the sampling errors of j^{th} auxiliary quantitative variable and j'^{th} auxiliary qualitative variable at the first phase. Then, simplifying and using binomial expansion up to the second order terms, t_2 becomes

$$t_2 = c\prod_{j=1}^{q_3}\left[1 + \frac{d\alpha_{3j}}{\bar{Z}_j}\bar{e}^*_{z(1)j} + \frac{d\alpha_{3j}(d\alpha_{3j}-1)}{2\bar{Z}_j^2}\bar{e}^{*2}_{z(1)j}\right]\left[1 - \frac{d\alpha_{3j}}{\bar{Z}_j}\bar{e}_{z(2)j} + \frac{d\alpha_{3j}(d\alpha_{3j}+1)}{2\bar{Z}_j^2}\bar{e}^2_{z(2)j}\right]$$

$$\prod_{j'=1}^{q_4}\left[1+\frac{h\alpha_{4j'}}{\Psi_{j'}}\bar{e}^{*}_{\omega(1)j'}+\frac{h\alpha_{4j'}(h\alpha_{4j'}-1)}{2\Psi^2_{j'}}\bar{e}^{*2}_{\omega(1)j'}\right]\left[1-\frac{h\alpha_{4j'}}{\Psi_{j'}}\bar{e}_{\omega(2)j'}+\frac{h\alpha_{4j'}(h\alpha_{4j'}+1)}{2\Psi^2_{j'}}\bar{e}^{2}_{\omega(2)j'}\right],$$

or

$$t_2=c\left[1-\sum_{j=1}^{q_3}\left\{\frac{d\alpha_{3j}}{\bar{Z}_j}\left(\bar{e}_{z(2)j}-\bar{e}^{*}_{z(1)j}\right)+\frac{d^2\alpha^2_{3j}}{2\bar{Z}^2_j}\left(\bar{e}^2_{z(2)j}+\bar{e}^{2^*}_{z(1)j}\right)+\frac{d\alpha_{3j}}{2\bar{Z}^2_j}\left(\bar{e}^2_{z(2)j}-\bar{e}^{2^*}_{z(1)j}\right)\right\}\right]$$

$$\left[1-\sum_{j'=1}^{q_4}\left\{\frac{h\alpha_{4j}}{\Psi_{j'}}\left(\bar{e}_{\omega(2)j'}-\bar{e}^{*}_{\omega(1)j'}\right)+\frac{h^2\alpha^2_{4j'}}{2\Psi^2_{j'}}\left(\bar{e}^2_{\omega(2)j'}+\bar{e}^{2^*}_{\omega(1)j'}\right)+\frac{h\alpha_{4j'}}{2\Psi^2_{j'}}\left(\bar{e}^2_{\omega(2)j'}-\bar{e}^{2^*}_{\omega(1)j'}\right)\right\}\right].$$

Ignoring the third and higher order terms and writing in matrix notations, we have

$$t_2=c\Big[1-h\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\mathbf{d}_{\boldsymbol{\omega}}-2^{-1}h^2\boldsymbol{\alpha}_4^{\mathbf{2^t}}\boldsymbol{\Psi}^{\mathbf{2}}\mathbf{u}_{\boldsymbol{\omega}}+2^{-1}h\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}^{\mathbf{2}}\mathbf{v}_{\boldsymbol{\omega}}-d\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\mathbf{d}_{\mathbf{z}}$$
$$+hd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\mathbf{d}_{\mathbf{z}}\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\mathbf{d}_{\boldsymbol{\omega}}+2^{-1}d^2\boldsymbol{\alpha}_3^{\mathbf{2^t}}\mathbf{Z}^{\mathbf{2}}\mathbf{u}_{\mathbf{z}}+2^{-1}d\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}^{\mathbf{2}}\mathbf{v}_{\mathbf{z}}\Big], \qquad (2.3)$$

where $\boldsymbol{\alpha}_3^{\mathbf{t}}$ and $\boldsymbol{\alpha}_4^{\mathbf{t}}$ are vectors of unknown coefficient, $\boldsymbol{\Psi}=\left[\Psi^{-1}_{j'}\right]_{q_4\times q_4}$; $j'=1,2,3...q_4$, and $\mathbf{Z}=\left[\bar{Z}^{-1}_j\right]_{q_3\times q_3}$; $j=1,2,3...q_3$ are diagonal matrices and vectors $\mathbf{d}^{\mathbf{t}}_{\mathbf{z}}=\left[d_j\right]_{1\times q_3}$ with $d_j=\left(\bar{e}_{z(2)j}-\bar{e}^{*}_{z(1)j}\right)$, $\mathbf{d}^{\mathbf{t}}_{\boldsymbol{\omega}}=\left[d_{j'}\right]_{1\times q_4}$ with $d_{j'}=\left(\bar{e}_{\omega(2)j'}-\bar{e}^{*}_{\omega(1)j'}\right)$, $\mathbf{u}^{\mathbf{t}}_{\boldsymbol{\omega}}=\left[u_{j'}\right]_{1\times q_4}$ with $u_{j'}=\left(e^2_{\omega_2 j'}+e^{2^*}_{\omega_1 j'}\right)$, $\mathbf{u}^{\mathbf{t}}_{\mathbf{z}}=\left[u_j\right]_{1\times q_3}$ with $u_j=\left(e^2_{z_2 j}+e^{2^*}_{z_1 i}\right)$, $\mathbf{v}^{\mathbf{t}}_{\boldsymbol{\omega}}=\left[v_{j'}\right]_{1\times q_4}$ with $v_{j'}=\left(e^2_{\omega_2 j'}-e^{2^*}_{\omega_1 j'}\right)$ and $\mathbf{v}^{\mathbf{t}}_{\mathbf{z}}=\left[v_j\right]_{1\times q_3}$ with $v_j=\left(e^2_{z_2 i}-e^{2^*}_{z_1 j}\right)$.

Now, considering the exponential component (t_3) of (2.1), let $\bar{e}^{*}_{w(1)k}=\bar{w}^{*}_{(1)k}-\bar{W}_k$, $\bar{e}^{*}_{\phi(1)k'}=\phi^{*}_{(1)k'}-\mathrm{E}_{k'}$ $\bar{e}^{*}_{\phi(2)k'}=\phi^{*}_{(2)k'}-\mathrm{E}_{k'}$ be the sampling errors of k^{th} quantitative auxiliary variable and k'^{th} qualitative auxiliary variable at the first phase with non-response and $\bar{e}_{w(1)k}=\bar{w}_{(1)k}-\bar{W}_k$, $\bar{e}_{\phi(1)k'}=\phi_{(1)k'}-\mathrm{E}_{k'}$ be the sampling errors of k^{th} quantitative auxiliary variable and k'^{th} qualitative auxiliary variable at the first phase. Then, simplifying and using binomial expansion up to the second order term, t_3 becomes

$$t_3=e\exp\left[\sum_{k=1}^{q_5}f\alpha_{5k}\left(\frac{\bar{e}_{\bar{w}_{(2)k}}-\bar{e}^{*}_{\bar{w}_{(1)k}}}{\bar{e}_{\bar{w}_{(2)k}}+\bar{e}^{*}_{\bar{w}_{(1)k}}+2\bar{W}_k}\right)+\sum_{k'=1}^{q_6}l\alpha_{6k'}\left(\frac{\bar{e}_{\phi_{(2)k'}}-\bar{e}^{*}_{\phi_{(1)k'}}}{\bar{e}_{\phi_{(2)k'}}+\bar{e}^{*}_{\phi_{(1)k'}}+2\mathrm{E}_{k'}}\right)\right],$$

or

$$t_3=e\exp\left[\sum_{k=1}^{q_5}\frac{f\alpha_{5k}}{2\bar{W}_k}\left(\bar{e}_{\bar{w}_{(2)k}}-\bar{e}^{*}_{\bar{w}_{(1)k}}\right)\left(1+\frac{\bar{e}_{\bar{w}_{(2)k}}+\bar{e}^{*}_{\bar{w}_{(1)k}}}{2\bar{W}_k}\right)^{-1}+\sum_{k'=1}^{q_6}\frac{l\alpha_{6k'}}{2\mathrm{E}_{k'}}\left(\bar{e}_{\phi_{(2)k'}}-\bar{e}^{*}_{\phi_{(1)k'}}\right)\left(1+\frac{\bar{e}_{\phi_{(2)k'}}+\bar{e}^{*}_{\phi_{(1)k'}}}{2\mathrm{E}_{k'}}\right)^{-1}\right]$$

or
$$t_3 = e\exp\left[\sum_{k=1}^{q_5}\frac{f\alpha_{5k}}{2\overline{W}_k}\left(\overline{e}_{\overline{w}_{(2)k}} - \overline{e}^*_{\overline{w}_{(1)k}}\right)\left(1-\frac{\overline{e}_{\overline{w}_{(2)k}}+\overline{e}^*_{\overline{w}_{(1)k}}}{2\overline{W}_k}+\frac{\left(\overline{e}_{\overline{w}_{(2)k}}+\overline{e}^*_{\overline{w}_{(1)k}}\right)^2}{4\overline{W}_k^2}\right)\right.$$
$$\left.+\sum_{k'=1}^{q_6}\frac{l\alpha_{6k'}}{2\mathrm{E}_{k'}}\left(\overline{e}_{\phi_{(2)k'}} - \overline{e}^*_{\phi_{(1)k'}}\right)\left(1-\frac{\overline{e}_{\phi_{(2)k'}}+\overline{e}^*_{\phi_{(1)k'}}}{2\mathrm{E}_{k'}}+\frac{\left(\overline{e}_{\phi_{(2)k'}}+\overline{e}^*_{\phi_{1k'}}\right)^2}{4\mathrm{E}^2_{k'}}\right)\right]$$

Using exponential series and writing in matrix notation, after ignoring the third and higher order terms

$$t_3 = e + 2^{-1}ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}\mathbf{d}_{\mathbf{w}} - 4^{-1}ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}^2\mathbf{v}_{\mathbf{w}} + 2^{-1}el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\mathbf{d}_{\mathbf{f}} - 4^{-1}el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}^2\mathbf{v}_{\mathbf{f}}, \tag{2.4}$$

where $\boldsymbol{\alpha}_5^{\mathbf{t}}$ and $\boldsymbol{\alpha}_6^{\mathbf{t}}$ are vectors of unknown coefficient, $\mathbf{W} = \left[\overline{W}_k^{-1}\right]_{q_5\times q_5}$; $k = 1,2,3...q_5$ and $\mathbf{E} = \left[\mathrm{E}_{k'}^{-1}\right]_{q_6\times q_6}$; $k' = 1,2,3...q_6$ are diagonal matrices and vectors $\mathbf{d}_{\mathbf{w}}^{\mathbf{t}} = [d_k]_{1\times q_5}$ with $d_k = \left(\overline{e}_{w(2)k} - \overline{e}^*_{w(1)k}\right)$, $\mathbf{d}_{\phi}^{\mathbf{t}} = [d_{k'}]_{1\times q_6}$ with $d_{k'} = \left(\overline{e}_{\phi(2)k'} - \overline{e}^*_{\phi(1)k'}\right)$, $\mathbf{v}_{\mathbf{w}}^{\mathbf{t}} = [v_k]_{1\times q_5}$ with $v_k = \left(e^2_{w_{2k}} - e^{2^*}_{w_{1k}}\right)$ and $\mathbf{v}_{\phi}^{\mathbf{t}} = [v_{k'}]_{1\times q_6}$ with $v_{k'} = \left(e^2_{\phi_{2k'}} - e^{2^*}_{\phi_{1k'}}\right)$.

Substituting the expressions of t_1, t_2 and t_3 from (2.2), (2.3) and (2.4) in (2.1), we get

$$t_{mix} = \left[\eta\left(\overline{e}_{y(2)} + \overline{Y}\right) - a\boldsymbol{\alpha}_1^{\mathbf{t}}\mathbf{d}_{\mathbf{x}} - b\boldsymbol{\alpha}_2^{\mathbf{t}}\mathbf{d}_{\boldsymbol{\tau}}\right]\left[c - ch\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\mathbf{d}_{\boldsymbol{\omega}} - 2^{-1}ch^2\boldsymbol{\alpha}_4^{2^{\mathbf{t}}}\boldsymbol{\Psi}^2\mathbf{u}_{\boldsymbol{\omega}}\right.$$
$$+2^{-1}ch\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}^2\mathbf{v}_{\boldsymbol{\omega}} - cd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\mathbf{d}_{\mathbf{x}} + chd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\mathbf{d}_{\mathbf{x}}\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\mathbf{d}_{\boldsymbol{\omega}}$$
$$+2^{-1}cd^2\boldsymbol{\alpha}_3^{2^{\mathbf{t}}}\mathbf{Z}^2\mathbf{u}_{\mathbf{z}} + 2^{-1}cd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}^2\mathbf{v}_{\mathbf{z}} + e + ef\,2^{-1}\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}\mathbf{d}_{\mathbf{w}}$$
$$\left.-4^{-1}ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}^2\mathbf{v}_{\mathbf{w}} + 2^{-1}el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\mathbf{d}_{\phi} - 4^{-1}el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}^2\mathbf{v}_{\phi}\right]; c+e=1 \tag{2.5}$$

Ignoring the third and higher order terms of the expression given by (2.5) and applying the expectation, we get

$$E\left(t_{mix} - \overline{Y}\right) = (\eta-1)\overline{Y} - \eta ch\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\boldsymbol{\delta}_4 - \eta cd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\boldsymbol{\delta}_3 + 2^{-1}\eta ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}\boldsymbol{\delta}_5$$
$$+2^{-1}\eta el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\boldsymbol{\delta}_6 + 2^{-1}\overline{Y}\eta ch^2\boldsymbol{\alpha}_4^{2^{\mathbf{t}}}\boldsymbol{\Psi}^2\left(\lambda_4\mathbf{S}^2_{\boldsymbol{\omega}} + \theta\mathbf{S}^2_{\boldsymbol{\omega}_{(2)}}\right)$$
$$+\overline{Y}\eta chd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\boldsymbol{\Delta}_{34}\boldsymbol{\Psi}\boldsymbol{\alpha}_4 + 2^{-1}\overline{Y}\eta cd^2\boldsymbol{\alpha}_3^{2^{\mathbf{t}}}\mathbf{Z}^2\left(\lambda_4\mathbf{S}^2_{\mathbf{z}} + \theta\mathbf{S}^2_{\mathbf{z}_{(2)}}\right)$$
$$+2^{-1}\eta\overline{Y}ch\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}^2\left(\lambda_3\mathbf{S}^2_{\boldsymbol{\omega}} + \theta\mathbf{S}^2_{\boldsymbol{\omega}_{(2)}}\right) + 2^{-1}\eta\overline{Y}cd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}^2\left(\lambda_3\mathbf{S}^2_{\mathbf{z}} + \theta\mathbf{S}^2_{\mathbf{z}_{(2)}}\right)$$
$$-4^{-1}\overline{Y}\eta ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}^2\left(\lambda_3\mathbf{S}^2_{\mathbf{w}} + \theta\mathbf{S}^2_{\mathbf{w}_{(2)}}\right) - 4^{-1}\overline{Y}\eta el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\left(\lambda_3\mathbf{S}^2_{\phi} + \theta\mathbf{S}^2_{\phi_{(2)}}\right)$$
$$+ach\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{44}\boldsymbol{\Psi}\boldsymbol{\alpha}_4 + acd\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{13}\mathbf{Z}\boldsymbol{\alpha}_3 - 2^{-1}aef\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{15}\mathbf{W}\boldsymbol{\alpha}_5$$
$$-2^{-1}ael\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{16}\mathbf{E}\boldsymbol{\alpha}_6 + bch\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{44}\boldsymbol{\Psi}\boldsymbol{\alpha}_4 + bcd\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{23}\mathbf{Z}\boldsymbol{\alpha}_3$$
$$-2^{-1}bef\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{25}\mathbf{W}\boldsymbol{\alpha}_5 - 2^{-1}bel\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{26}\mathbf{E}\boldsymbol{\alpha}_6, \tag{2.6}$$

where

$\boldsymbol{\delta_1} = E\left(\mathbf{d_x} e_{\bar{y}_2}\right) = \left[\lambda_3 S_{x_i y}\right]_{q_1 \times 1}$, $\boldsymbol{\delta_2} = E\left(\mathbf{d_\tau} e_{\bar{y}_2}\right) = \left[\lambda_3 S_{\tau_{i'} y}\right]_{q_2 \times 1}$, $\boldsymbol{\delta_3} = E\left(\mathbf{d_z} e_{\bar{y}_2}\right) = \left[\lambda_3 S_{z_j y}\right]_{q_3 \times 1}$

$\boldsymbol{\delta_4} = E\left(\mathbf{d_\omega} e_{\bar{y}_2}\right) = \left[\lambda_3 S_{\omega_{j'} y}\right]_{q_4 \times 1}$, $\boldsymbol{\delta_5} = E\left(\mathbf{d_w} e_{\bar{y}_2}\right) = \left[\lambda_3 S_{w_k y}\right]_{q_5 \times 1}$, $\boldsymbol{\delta_6} = E\left(\mathbf{d}_\phi e_{\bar{y}_2}\right) = \left[\lambda_3 S_{\phi_{k'} y}\right]_{q_6 \times 1}$

$E\left(\mathbf{u_\omega}\right) = \left(\lambda_4 \mathbf{S}^2_{\boldsymbol{\omega}} + \theta \mathbf{S}^2_{\boldsymbol{\omega}_{(2)}}\right)$, $E\left(\mathbf{v_\omega}\right) = \left(\lambda_3 \mathbf{S}^2_{\boldsymbol{\omega}} + \theta \mathbf{S}^2_{\boldsymbol{\omega}_{(2)}}\right)$ $E\left(\mathbf{u_z}\right) = \left(\lambda_4 \mathbf{S}^2_{\mathbf{z}} + \theta \mathbf{S}^2_{\mathbf{z}_{(2)}}\right); \lambda_4 = \left(\lambda_2 + \lambda_1\right)$,

$E\left(\mathbf{v_z}\right) = \left(\lambda_3 \mathbf{S}^2_{\mathbf{z}} + \theta \mathbf{S}^2_{\mathbf{z}_{(2)}}\right)$, $E\left(\mathbf{v_w}\right) = \left(\lambda_3 \mathbf{S}^2_{\mathbf{w}} + \theta \mathbf{S}^2_{\mathbf{w}_{(2)}}\right)$, $E\left(\mathbf{v}_\phi\right) = \left(\lambda_3 \mathbf{S}^2_\phi + \theta \mathbf{S}^2_{\phi_{(2)}}\right); \lambda_3 = \left(\lambda_2 - \lambda_1\right)$,

$\boldsymbol{\Delta_{13}} = E(\mathbf{d_x d_z^t}) = \left[\lambda_3 S_{x_i z_j} + \theta S_{x_i z_{j(2)}}\right]_{q_1 \times q_3}$, $\boldsymbol{\Delta_{15}} = E(\mathbf{d_x d_w^t}) = \left[\lambda_3 S_{x_i w_k} + \theta S_{x_i w_{k(2)}}\right]_{q_1 \times q_5}$,

$\boldsymbol{\Delta_{16}} = E(\mathbf{d_x d}_\phi^{\mathbf{t}}) = \left[\lambda_3 S_{x_i \phi_{k'}} + \theta S_{x_i \phi_{k'(2)}}\right]_{q_1 \times q_6}$, $\boldsymbol{\Delta_{23}} = E(\mathbf{d_\tau d_z^t}) = \left[\lambda_3 S_{\tau_{i'} z_j} + \theta S_{\tau_{i'} z_{j(2)}}\right]_{q_2 \times q_3}$,

$\boldsymbol{\Delta_{25}} = E(\mathbf{d_\tau d_w^t}) = \left[\lambda_3 S_{\tau_{i'} w_k} + \theta S_{\tau_{i'} w_{k(2)}}\right]_{q_2 \times q_5}$, $\boldsymbol{\Delta_{26}} = E(\mathbf{d_\tau d}_\phi^{\mathbf{t}}) = \left[\lambda_3 S_{\tau_{i'} \phi_{k'}} + \theta S_{\tau_{i'} \phi_{k'(2)}}\right]_{q_2 \times q_6}$,

$\boldsymbol{\Delta_{34}} = E(\mathbf{d_z d_\omega^t}) = \left[\lambda_3 S_{z_j \omega_{j'}} + \theta S_{z_j \omega_{j'(2)}}\right]_{q_3 \times q_4}$, $\boldsymbol{\Delta_{44}} = E(\mathbf{d_\omega d_\omega^t}) = \left[\lambda_3 S_{\omega_{j'} \omega_{j'}} + \theta S_{\omega_{j'} \omega_{j'(2)}}\right]_{q_4 \times q_4}$.

Expression given in (2.6) can be written as

$$\begin{aligned}
Bias(t_m) = {} & (\eta - 1)\bar{Y} + cd\Big[a\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{13}\mathbf{Z}\boldsymbol{\alpha}_3 + b\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{23}\mathbf{Z}\boldsymbol{\alpha}_3 + 2^{-1}\eta\bar{Y}d\boldsymbol{\alpha}_3^{2^{\mathbf{t}}}\mathbf{Z}^2\left(\lambda_4\mathbf{S}^2_{\mathbf{z}}\right. \\
& \left. + \theta\mathbf{S}^2_{(2)\mathbf{z}}\right) + 2^{-1}\bar{Y}\eta\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}^2\left(\lambda_3\mathbf{S}^2_{\mathbf{z}} + \theta\mathbf{S}^2_{(2)\mathbf{z}}\right) - \eta\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\boldsymbol{\delta}_3\Big] \\
& + ch\Big[a\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{14}\boldsymbol{\Psi}\boldsymbol{\alpha}_4 + b\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{24}\boldsymbol{\Psi}\boldsymbol{\alpha}_4 + 2^{-1}\bar{Y}\eta d\boldsymbol{\alpha}_4^{2^{\mathbf{t}}}\boldsymbol{\Psi}^2(\lambda_4\mathbf{S}^2_{\boldsymbol{\omega}} \\
& + \theta\mathbf{S}^2_{(2)\boldsymbol{\omega}} + 2^{-1}\bar{Y}\eta\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}^2\left(\lambda_3\mathbf{S}^2_{\boldsymbol{\omega}} + \theta\mathbf{S}^2_{(2)\boldsymbol{\omega}}\right) - \eta\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\boldsymbol{\delta}_4 \\
& + \bar{Y}\eta d\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\boldsymbol{\Delta}_{34}\boldsymbol{\Psi}\Big] + ef\Big[-2^{-1}a\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{15}\mathbf{W}\boldsymbol{\alpha}_5 - 2^{-1}b\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{25}\mathbf{W}\boldsymbol{\alpha}_5 \\
& - 4^{-1}\bar{Y}\eta\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}^2(\lambda_3\mathbf{S}^2_{\mathbf{w}} + \theta\mathbf{S}^2_{(2)\mathbf{w}}) + 2^{-1}\eta\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}\boldsymbol{\delta}_5\Big] + el\Big[-2^{-1}a\boldsymbol{\alpha}_1^{\mathbf{t}}\boldsymbol{\Delta}_{16}\mathbf{E}\boldsymbol{\alpha}_6 \\
& - 2^{-1}b\boldsymbol{\alpha}_2^{\mathbf{t}}\boldsymbol{\Delta}_{26}\mathbf{E}\boldsymbol{\alpha}_6 - 4^{-1}\bar{Y}\eta\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}^2\left(\lambda_3\mathbf{S}^2_\phi + \theta\mathbf{S}^2_{(2)\phi}\right) + 2^{-1}\eta\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\boldsymbol{\delta}_6\Big].
\end{aligned} \tag{2.7}$$

Bias(t_{mix}) = Bias due to regression – cum – ratio (quantitative) + Bias due to regression – cum – ratio (qualitative) + Bias due to regression – cum – exponential (quantitative) + Bias due to regression – cum – exponential (qualitative)

For obtaining the mean square error and optimum value of generalized class, ignoring the second and higher order terms after multiplication from (2.5), we have

$$\begin{aligned}
t_{mix} - \bar{Y} = {} & \eta\bar{e}_{y_{(2)}} + (\eta - 1)\bar{Y} - a\boldsymbol{\alpha}_1^{\mathbf{t}}\mathbf{d_x} - b\boldsymbol{\alpha}_2^{\mathbf{t}}\mathbf{d_\tau} - \bar{Y}\eta cd\boldsymbol{\alpha}_3^{\mathbf{t}}\mathbf{Z}\mathbf{d_z} \\
& - \bar{Y}\eta ch\boldsymbol{\alpha}_4^{\mathbf{t}}\boldsymbol{\Psi}\mathbf{d}_{\boldsymbol{\omega}} + 2^{-1}\bar{Y}\eta ef\boldsymbol{\alpha}_5^{\mathbf{t}}\mathbf{W}\mathbf{d_w} + 2^{-1}\bar{Y}\eta el\boldsymbol{\alpha}_6^{\mathbf{t}}\mathbf{E}\mathbf{d}_\phi,
\end{aligned}$$

or

$$t_{mix} - \bar{Y} = \eta\bar{e}_{y_{(2)}} + (\eta - 1)\bar{Y} - \mathbf{h^t H},$$

where $\mathbf{h^t} = \left[\boldsymbol{\alpha}_1^{\mathbf{t}} \quad \boldsymbol{\alpha}_2^{\mathbf{t}} \quad \boldsymbol{\alpha}_3^{\mathbf{t}} \quad \boldsymbol{\alpha}_4^{\mathbf{t}} \quad \boldsymbol{\alpha}_5^{\mathbf{t}} \quad \boldsymbol{\alpha}_6^{\mathbf{t}}\right]_{1 \times m}$

and $\mathbf{H^t} = \left[a\mathbf{d_x} \;\; b\mathbf{d_\tau} \;\; \bar{Y}\eta cd\mathbf{Z}\mathbf{d_z} \;\; \bar{Y}\eta ch\mathbf{\Psi}\mathbf{d_\omega} \;\; -2^{-1}\bar{Y}\eta ef\mathbf{W}\mathbf{d_w} \;\; -2^{-1}\bar{Y}\eta el\mathbf{E}\mathbf{d_\phi} \right]_{1\times m}$.

Squaring and taking the expectation, we have

$$E(t_{mix} - \bar{Y})^2 = E(\eta \bar{e}_{y_{(2)}} + (\eta-1)\bar{Y} - \mathbf{h^t H})^2 . \tag{2.8}$$

To find the optimum value of the unknown vector of row vectors h for which mean square error will be the minimum, differentiating (4.8) with respect to h and equating to zero, we get

$$\eta E(\mathbf{H}\bar{e}_{y_{(2)}}) + (\eta-1)\bar{Y}E(\mathbf{H}) - E(\mathbf{HH^t})\mathbf{h} = \mathbf{0}$$

or

$$\eta\mathbf{\Omega} - \mathbf{\Lambda h} = \mathbf{0} , \tag{2.9}$$

where

$E(\mathbf{H}\bar{e}_{y_2}) = \mathbf{\Omega}$, with , $\mathbf{\Omega^t} = \left[a\boldsymbol{\delta_1} \;\; b\boldsymbol{\delta_2} \;\; \bar{Y}cd\mathbf{Z}\boldsymbol{\delta_3} \;\; \bar{Y}ch\mathbf{\Psi}\boldsymbol{\delta_4} \;\; -2^{-1}\bar{Y}ef\mathbf{W}\boldsymbol{\delta_5} \;\; -2^{-1}\bar{Y}el\mathbf{E}\boldsymbol{\delta_6} \right]$,

$$E(\mathbf{H}) = \mathbf{0} \text{ and } E(\mathbf{HH^t}) = \mathbf{\Lambda} = \left[\Lambda_{ij}\right]_{m\times m} . \tag{2.10}$$

The elements in $\left[\Lambda_{ij}\right]$ are

$\mathbf{\Lambda_{11}} = a^2E(\mathbf{d_x d_x^t}) = a^2\mathbf{\Delta_{11}}$, $\quad \mathbf{\Lambda_{12}} = aE(\mathbf{d_x d_\tau^t})b = ab\mathbf{\Delta_{12}}$,

$\mathbf{\Lambda_{13}} = \eta acd\bar{Y}E(\mathbf{d_x d_z^t})\mathbf{Z} = \eta acd\bar{Y}\mathbf{\Delta_{13}Z}$, $\quad \mathbf{\Lambda_{14}} = \bar{Y}\eta achE(\mathbf{d_x d_\omega^t})\mathbf{\Psi} = \bar{Y}\eta ach\mathbf{\Delta_{14}\Psi}$,

$\mathbf{\Lambda_{15}} = -2^{-1}\bar{Y}\eta aefE(\mathbf{d_x d_w^t})\mathbf{W} = -2^{-1}\bar{Y}\eta aef\mathbf{\Delta_{15}W}$,

$\mathbf{\Lambda_{16}} = -2^{-1}\bar{Y}\eta ael(\mathbf{d_x d_\phi^t})\mathbf{E} = -2^{-1}\bar{Y}\eta ael\mathbf{\Delta}_{16}\mathbf{E}$, $\mathbf{\Lambda_{22}} = bE(\mathbf{d_\tau d_\tau^t})b^t = b^2\mathbf{\Delta_{22}}$,

$\mathbf{\Lambda_{23}} = \eta bcd\bar{Y}E(\mathbf{d_\tau d_z^t})\mathbf{Z} = \eta bcd\bar{Y}\mathbf{\Delta_{23}Z}$, $\mathbf{\Lambda_{24}} = \bar{Y}\eta bchE(\mathbf{d_\tau d_\omega^t})\mathbf{\Psi} = \bar{Y}\eta bch\mathbf{\Delta_{24}\Psi}$,

$\mathbf{\Lambda_{25}} = -2^{-1}\bar{Y}\eta befE(\mathbf{d_\tau d_w^t})\mathbf{W} = -2^{-1}\bar{Y}\eta bef\mathbf{\Delta_{25}W}$,

$\mathbf{\Lambda_{26}} = -2^{-1}\bar{Y}\eta belE(\mathbf{d_\tau d_\phi^t})\mathbf{E} = -2^{-1}\bar{Y}\eta bel\mathbf{\Delta_{26}E}$,

$\mathbf{\Lambda_{33}} = (\eta cd\bar{Y})^2E(\mathbf{d_z d_z^t})\mathbf{Z^2} = (\eta cd\bar{Y})^2\mathbf{\Delta_{33}Z^2}$,

$\mathbf{\Lambda_{34}} = (\bar{Y}\eta c)^2dh\mathbf{Z}E(\mathbf{d_z d_\omega^t})\mathbf{\Psi} = (\bar{Y}\eta c)^2dh\mathbf{Z\Delta_{34}\Psi}$,

$\mathbf{\Lambda_{35}} = -2^{-1}\eta^2\bar{Y}^2cdef\mathbf{Z}E(\mathbf{d_z d_w^t})\mathbf{W} = -2^{-1}\eta^2\bar{Y}^2cdef\mathbf{Z\Delta_{35}W}$,

$\mathbf{\Lambda_{36}} = -2^{-1}\eta^2\bar{Y}^2cdel\,\mathbf{Z}E(\mathbf{d_z d_\phi^t})\mathbf{E} = -2^{-1}\eta^2\bar{Y}^2cdel\,\mathbf{Z\Delta_{36}E}$,

$\mathbf{\Lambda_{44}} = (\bar{Y}\eta ch)^2E(\mathbf{d_\omega d_\omega^t})\mathbf{\Psi^2} = (\bar{Y}\eta ch)^2\mathbf{\Delta_{44}\Psi^2}$,

$\mathbf{\Lambda_{45}} = -2^{-1}\eta^2\bar{Y}^2chef\;\;\mathbf{\Psi}E(\mathbf{d_\omega d_w^t})\mathbf{W} = -2^{-1}\eta^2\bar{Y}^2chef\mathbf{\Psi\Delta_{45}W}$,

$\mathbf{\Lambda_{46}} = -2^{-1}\bar{Y}^2\eta^2ehcl\mathbf{\Psi}E(\mathbf{d_\omega d_\phi^t})\mathrm{E} = -2^{-1}\bar{Y}^2\eta^2ehcl\mathbf{\Psi\Delta_{46}E}$,

$\mathbf{\Lambda_{55}} = \left(2^{-1}\bar{Y}\eta ef\right)^2E(\mathbf{d_w d_w^t})\mathbf{W^2} = \left(2^{-1}\bar{Y}ef\right)^2\mathbf{\Delta_{55}W^2}$,

$\mathbf{\Lambda_{56}} = 4^{-1}\bar{Y}^2\eta^2e^2fl\mathbf{W}E(\mathbf{d_w d_\phi^t})\mathbf{E} = 4^{-1}\bar{Y}^2e^2fl\mathbf{W\Delta_{56}E}$,

$\mathbf{\Lambda_{66}} = \left(2^{-1}\bar{Y}\eta efl\right)^2E(\mathbf{d_\phi d_\phi^t})\mathbf{E^2} = \left(2^{-1}\bar{Y}\eta efl\right)^2\mathbf{\Delta_{66}E^2}$, Now, (2.9) can be written as

$$\mathbf{h} = \eta \mathbf{\Lambda}^{-1}\mathbf{\Omega} . \tag{2.11}$$

From (2.8)

$$MSE(t_{mix}) = E\left\{\eta \bar{e}_{y_{(2)}} + (\eta-1)\bar{Y} - \mathbf{h}^{\mathbf{t}}\mathbf{H}\right\}\left\{\eta \bar{e}_{y_{(2)}} + (\eta-1)\bar{Y} - \mathbf{h}^{\mathbf{t}}\mathbf{H}\right\}$$

or

$$MSE(t_{mix}) = E\left\{\eta \bar{e}_{y(2)} + (\eta-1)\bar{Y}\right\}\left\{\eta \bar{e}_{y(2)} + (\eta-1)\bar{Y} - \mathbf{h}^{\mathbf{t}}\mathbf{H}\right\}$$

or

$$MSE(t_{mix}) = \eta^2 E\left(\bar{e}_{y(2)}\right)^2 + \left\{(\eta-1)\bar{Y}\right\}^2 - \eta \mathbf{h}^{\mathbf{t}} E\left(\mathbf{H}\bar{e}_{y(2)}\right)$$

or

$$MSE(t_{mix}) = \eta^2 \lambda_2 S_y^2 + (\eta-1)^2 \bar{Y}^2 - \eta \mathbf{h}^{\mathbf{t}}\mathbf{\Omega} .$$

By using (2.11), we have

$$MSE(t_{mix}) = \bar{Y}^2\left(\eta^2 - 2\eta + 1\right) + \eta^2\left\{\lambda_2 S_y^2 - \mathbf{\Omega}^{\mathbf{t}}\mathbf{\Lambda}^{-1}\mathbf{\Omega}\right\} \tag{2.12}$$

or

$$MSE(t_{mix}) = \left(\eta^2 - 2\eta + 1\right)\bar{Y}^2 + \eta^2 \Gamma ,$$

where $\Gamma = \lambda_2 S_y^2 - \mathbf{\Omega}^{\mathbf{t}}\mathbf{\Lambda}^{-1}\mathbf{\Omega}$.

Differentiating MSE w.r.t η and equating to zero

$$2\eta\bar{Y}^2 - 2\bar{Y}^2 + 2\eta\Gamma = 0 ,$$

where $\eta\left(\bar{Y}^2 + \Gamma\right) = \bar{Y}^2$.

$$\eta_{opt} = \bar{Y}^2\left(\bar{Y}^2 + \Gamma\right)^{-1} = \left(1 + \Gamma\bar{Y}^{-2}\right)^{-1} .$$

Then, the minimum *MSE* of the general class is

$$MSE(t_{mix}) = \left[\left\{\left(1 + \bar{Y}^{-2}\Gamma\right)^{-1} - 1\right\}\bar{Y}\right]^2 + \left(1 + \bar{Y}^{-2}\Gamma\right)^{-2}\Gamma . \tag{2.13}$$

Remark 1. The general class of Ahmad et al. (2012) is a member of our proposed class after substituting $b = h = l = 0$ and $\eta = 1$ in (2.1).

As the proposed class is general in nature, special cases of the proposed class (2.1) may be deduced under the assumption $c + e = 1$ using different values of generalizing constants. The special cases with their expressions of bias and MSE's are given in the Remarks 2 and 3. Further, special cases for two and three components are given in Tables 1 and 2 respectively.

Remark 2. *(using two components of generalized class)*

We can obtain a regression (qualitative)-cum-ratio (quantitative) estimator by substituting $\eta = b = c = d = 1, \quad a = e = f = h = l = 0$ in (2.1), i.e.

$$t_{mix(23)} = \left[\bar{y}_2 + \sum_{i'=1}^{q_2} \alpha_{2i'} \left(\tau^*_{(1)i'} - \tau_{(2)i'} \right) \right] \left[\prod_{j=1}^{q_3} \left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}} \right)^{\alpha_{3j}} \right] \tag{2.14}$$

The bias of (2.14) can be obtained by substituting $\eta = b = c = d = 1$, $a = e = f = h = l = 0$ in (2.7) as

$$Bias(t_{mix(23)}) = \boldsymbol{\alpha}_2^t \boldsymbol{\Delta}_{23} \mathbf{Z} \boldsymbol{\alpha}_3 + 2^{-1} \bar{Y} \boldsymbol{\alpha}_3^{2^t} \mathbf{Z}^2 \left(\lambda_4 \mathbf{S}_z^2 + \theta \mathbf{S}_{(2)z}^2 \right)$$
$$+ 2^{-1} \bar{Y} \boldsymbol{\alpha}_3^t \mathbf{Z}^2 \left(\lambda_3 \mathbf{S}_z^2 + \theta \mathbf{S}_{(2)z}^2 \right) - \boldsymbol{\alpha}_3^t \mathbf{Z} \boldsymbol{\delta}_3 .$$

The optimum values are

$$\boldsymbol{\alpha}_2 = \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 + \bar{Y}^2 Z^2 \left(\boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\Delta}_{23} \mathbf{M}^{-1} \boldsymbol{\Delta}_{32} \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 - \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\Delta}_{23} \mathbf{M}^{-1} \boldsymbol{\delta}_3 \right) \text{ and}$$
$$\boldsymbol{\alpha}_3 = -\bar{Y} \left(\mathbf{M}^{-1} \mathbf{Z} \boldsymbol{\Delta}_{32} \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 - \mathbf{M}^{-1} \mathbf{Z} \boldsymbol{\delta}_3 \right).$$

Substituting $\eta = b = c = d = 1$, $a = e = f = h = l = 0$ in (2.12) and using (2.10) and Result-1 of the Appendix, the mean square error of (2.14) can be obtained as

$$MSE(t_{mix(23)}) = \lambda_2 S_y^2 - \left\{ \boldsymbol{\delta}_2^t \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 + \bar{Y}^2 \left(\boldsymbol{\delta}_2^t \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\Delta}_{23} \mathbf{Z}^2 \mathbf{M}^{-1} \boldsymbol{\Delta}_{32} \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 - \boldsymbol{\delta}_3^t \mathbf{Z}^2 \mathbf{M}^{-1} \boldsymbol{\Delta}_{32} \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\delta}_2 \right) \right\}$$
$$+ \bar{Y}^2 \left(\boldsymbol{\delta}_2^t \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\Delta}_{23} \mathbf{M}^{-1} \mathbf{Z} - \boldsymbol{\delta}_3^t \mathbf{M}^{-1} \mathbf{Z} \right) \mathbf{Z} \boldsymbol{\delta}_3 ,$$

where $\mathbf{M}^{-1} = \bar{Y}^{-1} \mathbf{R}^{-1}$ and $\mathbf{R}^{-1} = \bar{Y}^{-2} \left(\boldsymbol{\Delta}_{33} \mathbf{Z}^2 - \boldsymbol{\Delta}_{32} \boldsymbol{\Delta}_{22}^{-1} \boldsymbol{\Delta}_{23} \mathbf{Z}^2 \right)^{-1}$.

Remark 3. *(using three components of generalized class)*

A regression-cum-ratio estimator using a mixture of auxiliary variables can be obtained by substituting $\eta = a = b = c = d = 1$, $h = e = f = l = 0$ in generalized class (2.1) and we get

$$t_{mix(123)} = \left[\bar{y}_2 + \sum_{i=1}^{q_1} \alpha_{1i} \left(\bar{x}^*_{(1)i} - \bar{x}_{(2)i} \right) + \sum_{i'=1}^{q_2} \alpha_{2i'} \left(\tau^*_{(1)i'} - \tau_{(2)i'} \right) \right] \left[\prod_{j=1}^{q_3} \left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}} \right)^{\alpha_{3j}} \right]. \tag{2.15}$$

The bias of (2.15) can be obtained by substituting $\eta = a = b = c = d = 1$ and $h = e = f = l = 0$ in (2.7) as

$$Bias(t_{mix(123)}) = \boldsymbol{\alpha}_1^t \boldsymbol{\Delta}_{13} \mathbf{Z} \boldsymbol{\alpha}_3 + \boldsymbol{\alpha}_2^t \boldsymbol{\Delta}_{23} \mathbf{Z} \boldsymbol{\alpha}_3 + 2^{-1} \bar{Y} \boldsymbol{\alpha}_3^{2^t} \mathbf{Z}^2 \left(\lambda_4 \mathbf{S}_z^2 + \theta \mathbf{S}_{(2)z}^2 \right)$$
$$+ 2^{-1} \bar{Y} \boldsymbol{\alpha}_3^t \mathbf{Z}^2 \left(\lambda_3 \mathbf{S}_z^2 + \theta \mathbf{S}_{(2)z}^2 \right) - \boldsymbol{\alpha}_3^t \mathbf{Z} \boldsymbol{\delta}_3 .$$

where $\boldsymbol{\alpha}_1^t$, $\boldsymbol{\alpha}_2^t$ and $\boldsymbol{\alpha}_3^t$ are vectors of unknown constants of the vector $\mathbf{h}^t$.

Let $\mathbf{h}^t = \begin{bmatrix} \boldsymbol{\alpha}_1^t & \boldsymbol{\alpha}_2^t & \boldsymbol{\alpha}_3^t \end{bmatrix}_{1 \times m_1}$, where $m_1 = q_1 + q_2 + q_3$.

The optimum value of $\mathbf{h}^t$ for which MSE of $t_{mix(123)}$ will be the minimum can be written directly from (2.11) as:

$$\mathbf{h}_{\mathbf{m_1} \times 1} = \eta \boldsymbol{\Lambda}^{-1}{}_{\mathbf{m_1} \times \mathbf{m_1}} \boldsymbol{\Omega}_{\mathbf{m_1} \times 1} .$$

The mean square error of (2.15) can be obtained by substituting $\eta = a = b = c = d = 1$ and $h = e = f = l = 0$ in (2.13) as:

$$MSE(t_{mix(m_1)}) = \left[\left\{\left(1+\bar{Y}^{-2}\Gamma_{m_1}\right)^{-1} - 1\right\}^2 \bar{Y}^2\right] + \left(1+\bar{Y}^{-2}\Gamma_{m_1}\right)^{-2}\Gamma_{m_1},$$

where $\Gamma_{m_1} = \lambda_2 S_y^2 - \mathbf{\Omega}_{1\times m_1}\mathbf{\Lambda}^{-1}{}_{m_1\times m_1}\mathbf{\Omega}_{m_1\times 1}$;

$$\mathbf{\Lambda}_{m_1\times m_1} = \begin{bmatrix} \mathbf{\Delta}_{11} & \mathbf{\Delta}_{12} & \bar{Y}\mathbf{\Delta}_{13}\mathbf{Z} \\ \mathbf{\Delta}_{21} & \mathbf{\Delta}_{22} & \bar{Y}\mathbf{\Delta}_{23}\mathbf{Z} \\ \bar{Y}\mathbf{Z}\mathbf{\Delta}_{31} & \bar{Y}\mathbf{Z}\mathbf{\Delta}_{32} & \bar{Y}^2\mathbf{\Delta}_{33}\mathbf{Z}^2 \end{bmatrix} \text{ and } \mathbf{\Omega}_{m_1\times 1} = \begin{bmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \\ \bar{Y}\mathbf{Z}\boldsymbol{\delta}_3 \end{bmatrix}.$$

The inverse of $\mathbf{\Lambda}_{m_1\times m_1}$ i.e. $\mathbf{\Lambda}^{-1}{}_{m_1\times m_1}$ can be obtained using the Result-1 given in the Appendix.

The proposed general class comprises six components, three pairs are based on regression, ratio and exponential forms and each form utilizes categorical and continuous auxiliary variables separately. Moreover, it is assumed that $c + e = 1$. Following the Remarks 2 and 3, special cases consist of four and five components and even the single case based on all the components can be deduced using suitable values of generalizing constants. The special cases in which either c or e are involved, need no additional work, but the cases that involve both c and e need one extra step in finding the optimum value of either c or e. After finding this additional optimum value, the bias, existing optimum values and means square errors will be changed accordingly for these particular special cases. We have not included these cases in the article due to the limitation of length of the article. The special cases for two and three are given in the following tables.

Table1. Special cases of generalized class using two components

Estimator	(a,b,c,d,e,f,h,l,η)
$t_{mix(23)} = \left[\bar{y}_2 + \sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'} - \tau_{(2)i'}\right)\right]\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\right]$	(0,1,1,1,0,0,0,0,1)
$t_{mix(16)} = \left[\bar{y}_2 + \sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i} - \bar{x}_{(2)i}\right)\right]\left[\exp\left\{\sum_{k'=1}^{q_6}\alpha_{6k'}\left(\frac{\phi_{(2)k'} - \phi^*_{(1)k'}}{\phi_{(2)k'} + \phi^*_{(1)k'}}\right)\right\}\right]$	(1,0,0,0,1,0,0,1,1)
$t_{mix(25)} = \left[\bar{y}_2 + \sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'} - \tau_{(2)i'}\right)\right]\left[\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k} - \bar{w}^*_{(1)k}}{\bar{w}_{(2)k} + \bar{w}^*_{(1)k}}\right)\right\}\right]$	(0,1,0,0,1,1,0,0,1)
$t_{mix(15)} = \left[\bar{y}_2 + \sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i} - \bar{x}_{(2)i}\right)\right]\left[\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k} - \bar{w}^*_{(1)k}}{\bar{w}_{(2)k} + \bar{w}^*_{(1)k}}\right)\right\}\right]$	(1,0,0,0,1,1,0,0,1)

Table1. Special cases of generalized class using two components (cont.)

Estimator	(a,b,c,d,e,f,h,l,η)
$t_{mix(26)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\exp\left\{\sum_{k'=1}^{q_6}\alpha_{6k'}\left(\frac{\phi_{(2)k'}-\phi^*_{(1)k'}}{\phi_{(2)k'}+\phi^*_{(1)k'}}\right)\right\}\right]$	(0,1,0,0,1,0,0,1,1)
$t_{mix(14)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i}-\bar{x}_{(2)i}\right)\right]\left[\prod_{j'=1}^{q_4}\left(\frac{\omega^*_{(1)j'}}{\omega_{(2)j'}}\right)^{\alpha_{4j'}}\right]$	(1,0,1,0,0,0,1,0,1)
$t_{mix(12)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'}-\tau_{(2)i'}\right)\right]$	(1,1,1,0,0,0,0,0,1)
$t_{mix(34)}=\bar{y}_2\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\prod_{j'=1}^{q_4}\left(\frac{\omega^*_{(1)j'}}{\omega_{(2)j'}}\right)^{\alpha_{4j'}}\right]$	(0,0,1,1,0,0,1,0,1)
$t_{mix(24)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\prod_{j'=1}^{q_4}\left(\frac{\omega^*_{(1)j'}}{\omega_{(2)j'}}\right)^{\alpha_{4j'}}\right]$	(0,1,1,0,0,0,1,0,1)
$t_{mix(13)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i}-\bar{x}_{(2)i}\right)\right]\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\right]$	(1,0,1,1,0,0,0,0,1)
$t_{mix(56)}=\bar{y}_2\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k}-\bar{w}^*_{(1)k}}{\bar{w}_{(2)k}+\bar{w}^*_{(1)k}}\right)+\sum_{k'=1}^{q_6}\alpha_{6k'}\left(\frac{\phi_{(2)k'}-\phi^*_{(1)k'}}{\phi_{(2)k'}+\phi^*_{(1)k'}}\right)\right\}$	(0,0,1,0,1,1,0,1,1)

Table 2. Special cases of generalized class using three components

Estimator	(a,b,c,d,e,f,h,l,η)
$t_{mix(123)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^*_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\right]$	(1,1,1,1,0,0,0,0,1)
$t_{mix(134)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^*_{(1)i}-\bar{x}_{(2)i}\right)\right]\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^*_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\prod_{j'=1}^{q_4}\left(\frac{\omega^*_{(1)j'}}{\omega_{(2)j'}}\right)^{\alpha_{4j'}}\right]$	(1,0,1,1,0,0,1,0,1)

Table 2. Special cases of generalized class using three components (cont.)

Estimator	(a,b,c,d,e,f,h,l,η)
$t_{mix(234)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\prod_{j=1}^{q_3}\left(\frac{\bar{z}^{*}_{(1)j}}{\bar{z}_{(2)j}}\right)^{\alpha_{3j}}\prod_{j'=1}^{q_4}\left(\frac{\omega^{*}_{(1)j'}}{\omega_{(2)j'}}\right)^{\alpha_{4j'}}\right]$	(0,1,1,1,0,0,1,0,1)
$t_{mix(125)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k}-\bar{w}^{*}_{(1)k}}{\bar{w}_{(2)k}+\bar{w}^{*}_{(1)k}}\right)\right\}\right]$	(1,1,0,0,1,1,0,0,1)
$t_{mix(126)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\exp\left\{\sum_{k'=1}^{q_6}l\alpha_{6k'}\left(\frac{\phi_{(2)k'}-\phi^{*}_{(1)k'}}{\phi_{(2)k'}+\phi^{*}_{(1)k'}}\right)\right\}\right]$	(1,1,0,0,0,1,0,1,1)
$t_{mix(256)}=\left[\bar{y}_2+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k}-\bar{w}^{*}_{(1)k}}{\bar{w}_{(2)k}+\bar{w}^{*}_{(1)k}}\right)+\sum_{k'=1}^{q_6}\alpha_{6k'}\left(\frac{\phi_{(2)k'}-\phi^{*}_{(1)k'}}{\phi_{(2)k'}+\phi^{*}_{(1)k'}}\right)\right\}\right]$	(0,1,0,0,1,1,0,1,1)
$t_{mix(156)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^{*}_{(1)i}-\bar{x}_{(2)i}\right)\right]\left[\exp\left\{\sum_{k=1}^{q_5}\alpha_{5k}\left(\frac{\bar{w}_{(2)k}-\bar{w}^{*}_{(1)k}}{\bar{w}_{(2)k}+\bar{w}^{*}_{(1)k}}\right)+\sum_{k'=1}^{q_6}\alpha_{6k'}\left(\frac{\phi_{(2)k'}-\phi^{*}_{(1)k'}}{\phi_{(2)k'}+\phi^{*}_{(1)k'}}\right)\right\}\right]$	(1,0,0,0,1,1,0,1,1)
$t_{mix(124)}=\left[\bar{y}_2+\sum_{i=1}^{q_1}\alpha_{1i}\left(\bar{x}^{*}_{(1)i}-\bar{x}_{(2)i}\right)+\sum_{i'=1}^{q_2}\alpha_{2i'}\left(\tau^{*}_{(1)i'}-\tau_{(2)i'}\right)\right]\left[\prod_{j'=1}^{q_4}\left(\frac{\omega^{*}_{(1)j'}}{\omega_{(2)j'}}\right)^{h\alpha_{4j'}}\right]$	(1,1,1,0,0,0,1,0,1)

3. Empirical study

In this section we have empirically compared special cases that are discussed in sections 4.1 and 4.2. For this comparison, the data of Census report of district Jhang (1998), Pakistan (see Ahmad et al. (2009)) is used. The population size is 368 (N). From N , 276(N_1) are considered as respondent group of population and the remaining 92(N_2) are non-respondent group. A sample of 160(n_1) is selected at the first phase sample and from first phase, a sample of 90(n_2) is selected as the second phase sample. From the second phase sample, a sub-sample of 10(r) is selected as the re-contacted sample and it is assumed that there is full response from this sample.

The data set, which is considered for the empirical study, consists of three quantitative and three qualitative auxiliary variables along with one response variable. The variables description is given in Table 3. The variances and co-variances and the mean of all variables for both complete and non-respondent populations are given in Table 4 and Table 5 respectively. The bias and MSE's of all special cases given in Table 1 and Table 2 are given in Table 6 and Table 7 respectively.

Table 3. Description of variables *(each variable is taken from rural locality)*

Description of Variables	
Y	Literacy ratio
x	Household size
z	Population of both sexes
w	Household characteristics
τ	Male above and below average education
ω	Female above and below average education
ϕ	Persons below and above average age

Table 4. Variance co-variances and mean of complete population N = 368

		Variance Co-variance Matrix						
Variable	**Mean**	y	x	z	w	τ	ω	ϕ
y	29.831	62.860	0.548	5780.0000	593.123	1.936	0.5630	1.440
x	6.372	0.548	0.266	138.7860	-7.139	0.015	0.0020	0.0110
z	5901.46	5780.0	138.786	26660000.0	1262000.0	711.523	1191.0	1173.0
w	897.71	593.123	-7.139	1262000.0	211500.0	113.702	162.718	157.043
τ	0.35	1.936	0.015	711.5230	113.702	0.227	0.1060	0.1560
ω	0.42	0.563	0.002	1191.0	162.718	0.106	0.2440	0.1740
ϕ	0.43	1.440	0.011	1173.0	157.043	0.156	0.1740	0.2450

Table 5. Variances co-variances and mean of non-respondent population $N_2 = 92$

	Variance Co-variance Matrix							
Variable	y	x	z	w	τ	ω	ϕ	**Mean**
y	40.0700	0.1680	7508.0	1176.00	1.5780	1.2480	1.5370	25.1467
x	0.1680	0.1270	-17.55	-21.4730	0.0030	-0.0190	-0.0080	6.2130
z	7508.00	-17.550	6239000.0	999300.0	668.870	1001.0	991.744	6164.5761
w	1176.00	-21.4730	999300.0	163700.0	105.675	163.453	160.103	994.3152
τ	1.5780	0.0030	668.87	105.6750	0.1950	0.0870	0.1360	0.2609
ω	1.2480	-0.0190	1001.0	163.4530	0.0870	0.2510	0.1750	0.5435
ϕ	1.5370	-0.0080	991.7440	160.1030	0.1360	0.1750	0.2430	0.4022

From Table 6, biases of estimators show that some estimators are overestimating and some are underestimating the population mean of study variable except the regression estimator that is unbiased and similar information is described from Table 7 for estimators based on biases.

From Table 6, the ranked absolute biases show that the estimator $t_{mix(13)}$ has larger bias as compared to others whereas $t_{mix(26)}$ has smaller bias. The estimators at rank 2, 3, 4 and 5 have a very small amount of bias whereas the remaining ones have large amount of bias. Considering the ranks of MSE, the estimator $t_{mix(25)}$ is more efficient then all others whereas $t_{mix(34)}$ is the least efficient. However, the differences in MSE's for all estimators are very small but there is a lot of variation in biases. Considering the trade-off between biases and MSE's, the sum of ranks of bias and MSE suggests that the biased estimator $t_{mix(26)}$ is useful for practical situations where only qualitative variables are considered, $t_{mix(25)}$ is suitable when there is a mixture of auxiliary variables and $t_{mix(15)}$ can be used for only quantitative auxiliary variables.

Table 6. Bias and MSE of members of generalized class for two factors

Estimators	**Bias**	**Absolute Bias**	**Ranked Absolute Bias**	**MSE**	**Ranked MSE**	**Sum of Ranks**	**Ranks of Sum**
$t_{mix(23)}$	$-8.31E+03$	8310	09	15.3249	5	14	7
$t_{mix(25)}$	$-1.30E+03$	1300	6	15.3217	1	7	3
$t_{mix(15)}$	1.41E+03	1410	7	15.3601	8	15	8.5
$t_{mix(26)}$	1.74E-04	0.000174	2	15.3248	4	6	2
$t_{mix(14)}$	5.39E-04	0.000539	3	15.3620	10	13	6
$t_{mix(12)}$	0	0	1	15.3222	2	3	1
$t_{mix(34)}$	6.64E+04	66400	10	15.3693	11	21	11
$t_{mix(24)}$	1.35E-03	0.00135	5	15.3246	3	8	4
$t_{mix(13)}$	8.31E+04	83100	11	15.3608	9	20	10
$t_{mix(56)}$	$-2.17E+03$	2170	8	15.3446	7	15	8.5

In the case of estimators based on three components, from Table 7, the ranked absolute biases show that $t_{mix(124)}$ has smaller bias whereas $t_{mix(134)}$ is highly biased. $t_{mix(126)}$ has also a very small amount of bias. Based on ranks of MSE, $t_{mix(256)}$ is more efficient than all others whereas $t_{mix(134)}$ is the least efficient. Considering the trade-off between bias and MSE's, the sum of ranks of bias and MSE suggests that $t_{mix(124)}$ is useful for practical situations when there is a mixture of auxiliary variables.

Table 7. Bias and MSE of members of generalized class using three components

Estimators	Bias	Absolute Bias	Ranked Absolute Bias	MSE	Ranked MSE	Sum of Ranks	Ranks of Sum
$t_{mix(123)}$	-8.31E+03	8310	6	15.3222	5	11	5
$t_{mix(134)}$	1.34E+08	134000000	8	15.4720	8	16	8
$t_{mix(234)}$	-2.82E+04	28200	7	15.3243	6	13	7
$t_{mix(125)}$	-1.17E+03	1170	3	15.3196	2	5	2.5
$t_{mix(126)}$	3.01E-04	0.000301	2	15.3220	4	6	4
$t_{mix(256)}$	-2.33E+03	2330	4	15.3181	1	5	2.5
$t_{mix(156)}$	-5.72E+03	5720	5	15.3416	7	12	6
$t_{mix(124)}$	1.81E-04	0.000181	1	15.3218	3	4	1

As the suggested class consists of regression, ratio and exponential components, it is obvious that the regression component contributes in terms of reduction of MSE and ratio and exponential components will increase bias and decrease MSE. This statement can be verified from Table 6 and 7. For example, from Table 6 $t_{mix(15)}$ is a regression-cum-exponential estimator with one exponential component having bias 1410 and MSE 15.3601, and by adding another ratio component in this estimator we obtain the regression-cum-exponential $t_{mix(156)}$ (given in Table 7) with bias 5720 and MSE 15.3416 as the result. This type of change can be observed for other estimators with this property.

What is specific to this empirical study is that the qualitative auxiliary variables are performing better than the continuous auxiliary variables. For example, $t_{mix(125)}$ [regression-cum-exponential (quantitative)] has bias 1170 and MSE 15.3196 whereas $t_{mix(126)}$ [regression-cum-exponential (qualitative)] has bias 0.000301 and MSE 15.3220. Similar information can be observed considering other such pairs of estimators.

Summarizing the discussion on both tables, the three-component estimator $t_{mix(124)}$ is better than all others while considering bias and MSE simultaneously. This estimator comprises two regression components of quantitative and qualitative auxiliary variables and one ratio component of qualitative auxiliary variables.

4. Conclusions

In this paper, a general class of regression-cum-ratio-exponential estimators is developed for two-phase sampling in the presence of non-responses at the first phase. Both quantitative and qualitative auxiliary variables are used in the construction of the class to increase the efficiency of the class as well as its members. The general expression of bias and mean square error is also derived. As the proposed class is general in nature, some suitable special cases are deduced along with their bias and mean square errors. On the basis of the empirical study it is concluded that both types of auxiliary variables can play a role in reducing the bias and the mean square error of an estimator. The bias and mean square error can be reduced by increasing the number of auxiliary variables. An increase in ratio or exponential components increases the bias. Our findings show that an estimator based on three components performs better then all others. This estimator comprises two regression components of quantitative and qualitative auxiliary variables and one ratio component of qualitative auxiliary variables.

This paper also fills the gap in the literature as it attempts to estimate the finite population mean using both qualitative and quantitative multi-auxiliary variables in the presence of non-response at the first phase under two-phase sampling. It can also provide an opportunity to the applied survey statisticians if they consider estimation of finite population mean using several qualitative and quantitative auxiliary variables.

REFERENCES

AHMAD, Z., HANIF, M., (2010). Generalized Multi-Phase Multivariate Regression Estimators for Partial Information Case using Multi-Auxiliary Variables. World Applied Sciences Journal, 10(3): 370–379.

AHMAD, Z., HANIF, M., AHMAD, M., (2009a). Generalized Multivariate Ratio Estimators for Multi-Phase Sampling using Multi-Auxiliary Variables. Pakistan Journal of Statistics, 25(4), 615–629.

AHMAD, Z., HANIF, M., AHMAD, M., (2009b). Generalized Regression-Cum-Ratio Estimators for Two-Phase Sampling using Multi-Auxiliary Variables, Pakistan Journal of Statistics, 25(2), 1–14.

AHMAD, Z., HANIF, M., AHMAD, M., (2010). Generalized Multi-Phase Multivariate Ratio Estimators for Partial Information Case using Multi-Auxiliary Variables. Communications of the Korean Statistical Society, 17(5), 625–637.

AHMAD, Z., MAQSOOD, I., HANIF, M., (2013). Generalized estimator of population mean for two phase sampling using multi-auxiliary variables in the presence of non-response at first phase for no information case. Pakistan Journal of Statistics, 29(2), 155–180.

AHMAD, Z., ZAFAR, I., HANIF, M., (2013a). Generalized estimator of population mean for two phase sampling using multi-auxiliary variables in the presence of non-response at first phase for no information case, Journal of Mathematical Sciences and Applied E-Notes. 1(2), 238–256.

AHMAD, Z., ZAFAR, I., BANO, Z., (2013b). Generalized Class of Mean Estimators for Two-Phase Sampling in The Presence of Nonresponse. J. Japan Statist. Soc., 43(2), 163–185.

BOWLEY, A. L., (1926). Measurements of precision attained in sampling. Bull. Inst. Inte. Statist., 22, 1–62.

HANSEN, M. H., HURWITZ, W. N., (1946). The problem of no response in sample surveys. American Statistical Association, 41, 517–529.

HANIF, M., HAQ, I., SHAHBAZ, M. Q., (2010). Ratio Estimators using Multiple Auxiliary Attributes, World Applied Sciences Journal, 8(1), 133–136.

HIDIROGLOU, M. A., SARNDAL, C. E., (1998). Use of auxiliary information for two-phase sampling. Survey Methodology, 24(1), 11–20.

JHAJJ, H. S., SHARMA, M. K., GROVER, L. K., (2006). Dual of ratio estimators of finite population mean obtained on using linear transformation to auxiliary variables. Journal of Japan Statistical Society, 36(1), 107–119.

KHARE, B. B., SRIVASTAVA, S., (1993). Estimation of Population Mean using Auxiliary Character in the Presence of Non Response. National. Acad. Sci. Letters. (India), 16(3), 111–114.

KHARE, B. B., SRIVASTAVA, S., (1995). Study of Conventional and Alternative Two Phase Sampling Ratio, Product and Regression Estimators in the Presence of Non-Response. Proc. Nat. Acad. Sci. (India), 65(A), 195–203.

MOHANTY, S., (1967). Combination of regression and ratio estimate, J. Ind. Statist. Assoc., 5, 16–19.

NEYMAN, J., (1934). On the two different aspects of representative method: The method of stratified sampling and the method of purposive selection. J. Roy. Statist. Soc., 97, 558–606.

NEYMAN, J. (1938). Contribution to the theory of sampling human populations. J. Amer. Statist. Assoc., 33, 101–116.

NAIK, V. D., GUPTA, P. C., (1996). A note on estimation of mean with known population of an auxiliary character. J. Ind. Soc. Agri. Statist., 48(2), 151–158.

RAO, J. N. K., (1973). On double sampling for stratification and analytic surveys. Biometrika, 6, 125–133.

ROY, D. C., (2003). A regression type estimates in two-phase sampling using two auxiliary variables. Pak. J. Statist., 19(3), 281–290.

SAMIUDDIN, M., HANIF, M., (2007). Estimation of Population Means in Single and Two-Phase Sampling with or without Additional Information. Pakistan Journal of Statistics, 23 (2), 99–118.

SHABBIR, J., GUPTA, S., (2007). On estimating the finite population mean with known population proportion of an auxiliary variable. Pakistan Journal of Statistics, 23(1), 1–9.

SINGH, H. P., KUMAR, S., (2008). Estimation of Mean in Presence of Non Response using Two-Phase Sampling Scheme. Statist. Pap. DOI 10.1007/s00362-008-040-5.

SINGH, R., CHAUHAN, P., SAWAN, N., (2010). Ratio-Product Type Exponential Estimator for Estimating Finite Population Mean Using Information on Auxiliary Attribute. Unpublished manuscript.

TABASUM, R., KHAN, I. A., (2004). Double Sampling for Ratio Estimation with Non-Response. J. Ind. Soc. Agril. Statist. 58(3), 300–306.

WU, C., LUAN, Y., (2003). Optimal Calibration Estimators under Two-Phase Sampling. Journal of Official Statistics, 19(2), 119–131.

APPENDIX

Result 1. Inverse of matrix of matrices:

Let $\mathbf{T}$ be a matrix of matrices of order 4×4,

$$\mathbf{T}=\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} & \mathbf{T}_{14} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} & \mathbf{T}_{24} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} & \mathbf{T}_{34} \\ \mathbf{T}_{41} & \mathbf{T}_{42} & \mathbf{T}_{43} & \mathbf{T}_{44} \end{bmatrix}.$$

The inverse of $\mathbf{T}$ is

$$\mathbf{T}^{-1}=\begin{bmatrix} \mathbf{B}_{33} & \mathbf{B}_{31} \\ \mathbf{B}_{13} & \mathbf{B}_{11} \end{bmatrix}^{-1}=\begin{bmatrix} \mathbf{B}_{33}^{-1}+\left(\mathbf{B}_{33}^{-1}\mathbf{B}_{31}\right)\mathbf{G}_{11}^{-1}\left(\mathbf{B}_{13}\mathbf{B}_{33}^{-1}\right) & -\left(\mathbf{B}_{33}^{-1}\mathbf{B}_{31}\right)\mathbf{G}_{11}^{-1} \\ -\mathbf{G}_{11}^{-1}\left(\mathbf{B}_{13}\mathbf{B}_{33}^{-1}\right) & \mathbf{G}_{11}^{-1} \end{bmatrix},$$

where $\mathbf{G}_{11}^{-1}=\left(\mathbf{B}_{11}-\mathbf{B}_{13}\mathbf{B}_{33}^{-1}\mathbf{B}_{31}\right)^{-1}$,

$$\mathbf{B}_{33}=\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix},\ \mathbf{B}_{31}=\begin{bmatrix} \mathbf{T}_{14} \\ \mathbf{T}_{24} \\ \mathbf{T}_{34} \end{bmatrix},\ \mathbf{B}_{13}=\begin{bmatrix} \mathbf{T}_{41} & \mathbf{T}_{42} & \mathbf{T}_{43} \end{bmatrix} \text{ and } \mathbf{B}_{11}=\mathbf{T}_{44}.$$

and

$$\mathbf{B}_{33}^{-1}=\begin{bmatrix} \mathbf{A}_{22} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{11} \end{bmatrix}^{-1}=\begin{bmatrix} \mathbf{A}_{22}^{-1}+\left(\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)\mathbf{H}_{11}^{-1}\left(\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\right) & -\left(\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)\mathbf{H}_{11}^{-1} \\ -\mathbf{H}_{11}^{-1}\left(\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\right) & \mathbf{H}_{11}^{-1} \end{bmatrix},$$

where $\mathbf{H}_{11}^{-1}=\left(\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21}\right)^{-1}$,

$$\mathbf{A}_{22}=\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix},\ \mathbf{A}_{21}=\begin{bmatrix} \mathbf{T}_{13} \\ \mathbf{T}_{23} \end{bmatrix},\ \mathbf{A}_{12}=\begin{bmatrix} \mathbf{T}_{31} & \mathbf{T}_{32} \end{bmatrix} \text{ and } \mathbf{A}_{11}=\mathbf{T}_{33}.$$

and

$$\mathbf{A}_{22}^{-1}=\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix}^{-1}=\begin{bmatrix} \mathbf{T}_{11}^{-1}+\left(\mathbf{T}_{11}^{-1}\mathbf{T}_{12}\right)\mathbf{R}_{22}^{-1}\left(\mathbf{T}_{21}\mathbf{T}_{11}^{-1}\right) & -\left(\mathbf{T}_{11}^{-1}\mathbf{T}_{12}\right)\mathbf{R}_{22}^{-1} \\ -\mathbf{R}_{22}^{-1}\left(\mathbf{T}_{21}\mathbf{T}_{11}^{-1}\right) & \mathbf{R}_{22}^{-1} \end{bmatrix},$$

where $\mathbf{R}_{22}^{-1}=\left(\mathbf{T}_{22}-\mathbf{T}_{21}\mathbf{T}_{11}^{-1}\mathbf{T}_{12}\right)^{-1}$.