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ON SOME TESTS OF VARIANCE COMPONENTS FOR LINEAR MIXED MODELS

Introduction

Special cases of the general or the generalized mixed linear models are widely used in different areas including for example economics (e.g. Fay and Herriot, 1979), genetics (e.g. Bernardo, 1996) or statistical image analysis (e.g. Demidenko, 2004, chapter 12). Assumptions of the linear mixed model are as follows:

$$\begin{cases} \mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{v} + \mathbf{e} \\ D^2(\mathbf{v}) = \mathbf{G} \\ D^2(\mathbf{e}) = \mathbf{R} \\ Cov(\mathbf{v}, \mathbf{e}) = \mathbf{0} \end{cases}, \tag{1}$$

where $\mathbf{Y}$ and $\mathbf{e}$ are random vectors of sizes $n \times 1$, $\mathbf{X}$ and $\mathbf{Z}$ are known matrices of sizes $n \times p$ and $n \times q$, respectively, the random vector $\mathbf{v}$ is of size $q \times 1$ and the vector of parameters $\boldsymbol{\beta}$ is of size $p \times 1$. Hence, the variance-covariance matrix of $\mathbf{Y}$ is given by:

$$D^2(\mathbf{Y}) = \mathbf{V} = \mathbf{V}(\boldsymbol{\delta}) = \mathbf{Z}\mathbf{G}\mathbf{Z}^{\mathrm{T}} + \mathbf{R}, \tag{2}$$

where $\boldsymbol{\delta}$ is the vector of unknown in practice variance parameters.

Moreover, if in the model (1) we additionally assume that elements of random vectors $\mathbf{e}$ and $\mathbf{v}$ are independent with zero expected values and variances σ_e^2 and σ_v^2, respectively, then $D^2(\mathbf{e}) = \mathbf{R} = \sigma_e^2 \mathbf{I}_{n\times n}$ and $D^2(\mathbf{v}) = \mathbf{G} = \sigma_v^2 \mathbf{I}_{q\times q}$, where $\mathbf{I}$ is the identity matrix. In this case, the variance-covariance matrix (2) simplifies to the following formula:

$$D^2(\mathbf{Y}) = \mathbf{V} = \mathbf{V}(\boldsymbol{\delta}) = \sigma_v^2 \mathbf{Z}\mathbf{Z}^{\mathbf{T}} + \sigma_e^2 \mathbf{I}_{n \times n}, \tag{3}$$

where $\boldsymbol{\delta} = \left[\sigma_e^2 \ \sigma_v^2 \right]^T$.

If we estimate $\boldsymbol{\delta}$ using REML − Restricted Maximum Likelihood Method (Jiang, 1996), the vector of estimators will be denoted by $\hat{\boldsymbol{\delta}}_{REML}$. When ML – Maximum Likelihood is used, the vector of estimators will be denoted by $\hat{\boldsymbol{\delta}}_{ML}$.

In the paper the classic and permutation tests of variance components are studied including the situation when the assumption of normality of random effects and random components is not met.

1. Tests of variance components

In the section we will introduce classic tests: log-likelihood ratio test and Wald test. In both tests we consider hypotheses: $H_0 : \sigma_v^2 = 0$, $H_1 : \sigma_v^2 > 0$ (e.g. Biecek, 2012, p. 161). If the null hypothesis is true, it means that we should not consider a model which belongs to the class of the General Linear Mixed Model, but the model without the random effect. Because the random effect implies some non-zero covariances in the variance-covariance matrix, it implies the lack of some correlations assumed in the mixed model. For example, if the random effect is subpopulation-specific it means that we assume some correlations between the random variables within the subpopulation.

When we consider null hypothesis $H_0 : \sigma_v^2 \in \Theta_{\sigma,0}$, where $\Theta_{\sigma,0}$ is a subspace of Θ_σ which is parameter space of the variance components, the test statistic for the log-likelihood ratio test (LRT) has the following form (e.g. Verbecke, Molenbergs, 2000, pp. 65-66):

$$-2 \ln \lambda_N = -2 \ln \left[\frac{L_{ML}\left(\sigma^2_{v(ML,0)}\right)}{L_{ML}\left(\sigma^2_{v(ML)}\right)} \right], \tag{4}$$

where $\sigma^2_{v(ML,0)}$ and $\sigma^2_{v(ML)}$ are maximum likelihood estimates obtained from maximizing log-likelihood function over space $\Theta_{\sigma,0}$ and Θ_σ, respectively. According to the classic likelihood theory (under some regularity conditions), statistic (4) has χ^2 distribution with $s - p$ degrees of freedom which is difference of dimensions of Θ_σ and $\Theta_{\sigma,0}$. This test is conservative (Stram and Lee 1994,

p. 1176). Self and Liang (1994), likewise Stram and Lee (1994) explain that p-value based on χ^2_{s-p} obtained in this test is higher than it should be.

Second classic test is Wald test. In this case the test statistic is a ratio of estimator $\hat{\sigma}_v^2$ and estimator of its asymptotic standard error, what can be written as follows:

$$\frac{\hat{\sigma}_v^2}{\hat{D}(\hat{\sigma}_v^2)}, \tag{5}$$

where $\hat{D}^2(\hat{\sigma}_v^2)$ can be obtained as appropriate element of the diagonal of the estimated inverse of Fisher information matrix.

Based on the classic likelihood theory and when some regularity conditions are fulfilled, distribution of ML and REML estimators of σ_v^2 can be approximated by normal distribution with mean vector σ_v^2 and covariance matrix given by inverse Fisher information matrix (Verbecke, Molenbergs, 2000, p. 64).

Both classic test presented in section allow testing the same hypotheses and have similar properties (McCulloch, Searle, 2001, p. 148; Bishop et al., 1975, sec. 14.9).

2. Permutation tests

In this section we present three permutation test: permution test of variance components based on log-likelihood, studied i.a. by Biecek (2012), and permutation versions of two classic tests showed in previous section.

Permutation procedure for each of these tests we can divide into four steps. In the first stage of permutation test based on likelihood we should calculate L_0 – test statistic (log-likelihood) for original data. Next we generate $\pi^{*,b}$ which is permutation of vector [1 2 … n], where n is sample size and b is the iteration number. In the third phase, the permutation of grouping variable is made and log-likelihood is calculated for data with this permutation ($L_0^{*,b}$). This and the previous step are repeated B times (e.g. Biecek, 2012, p. 177). In the last phase, p-value is calculated as the fraction of the cases where log-likelihood for model with permutation is larger than for the model under original data:

$$p = \frac{1 + \#\left\{b : L_0^{*,b} > L_0\right\}}{1 + B}. \tag{6}$$

If the permutation version of the log-likelihood ratio (LRT) test is considered, at beginning we have to compute LRT_0 – value of the LRT statistic (4) for original data. In the second stage, we generate $\pi^{*,b}$ permutation of the vector [1 2 … n]. In the next phase we calculate permutation of grouping variable and we compute test statistic for the data with the permutation (denoted by $LRT_0^{*,b}$). The second and the third phase are repeated B times, as in the previous test. In the last phase we calculate p-value:

$$p = \frac{1 + \#\left\{b : LRT_0^{*,b} > LRT_0\right\}}{1 + B} \tag{7}$$

as the fraction of cases where LRT statistic for the model under the original data is smaller than for the model under the data with the permutation.

The last permutation test is the permutation version of the classic Wald test. Similarly, like in other permutation tests, firstly we compute test statistic (5) for original data which is denoted by W_0. In the second stage we generate $\pi^{*,b}$ as a permutation of vector [1 2 … n]. In the third phase we compute permutation of the grouping variable and the value of the test statistic for data with permutation ($W_0^{*,b}$). The second and the third phase are repeated B times. In the last phase we calculate p-value:

$$p = \frac{1 + \#\left\{b : W_0^{*,b} > W_0\right\}}{1 + B}. \tag{8}$$

It is fraction of cases when the Wald test statistic for the model under data with the permutation is larger than for the model without the permutation.

3. Simulation study

In the section we present results Monte Carlo simulation study which widely used to assess properties of different methods (e.g. Białek, 2014; Gamrot, 2013; Krzciuk, Mierzwa i Wywiał, 2003). It is prepared using R software (R Development Core Team, 2013). Similarly to Kończak (2010, 2012) we will compare properties of classic and permutation tests. Using data on revenues from municipal taxation in 284 Swedish municipalities, presented in Särndal, Swensson, Wretman (1992), we analyze properties of all test presented in this paper. This data set is available in R program in package *sampling*. In the paper we study five variables from this data set:

– RMT85 – revenues from municipal taxation in 1985 (millions of kronor),
– P75 – population in 1985 in municipalities in thousands,

– REV84 – real estate values in 1984 (millions of kronor),
– CL – clusters indicator,
– REG – geographic region indicator.

In this data set country is divided into eight regions and fifty clusters.

In this study we consider five types of models:

$$Y_{id} = \beta_0 + \beta_1 X_{1id} + \beta_2 X_{2id} + v_d + e_{id}\ , \tag{9}$$

$$Y_{id} = \beta_0 + \left(\beta_1 + v_d\right) X_{1id} + \beta_2 X_{2id} + e_{id}\ , \tag{10}$$

$$Y_{id} = \beta_0 + \beta_1 X_{1id} + \left(\beta_2 + v_d\right) X_{2id} + e_{id}\ , \tag{11}$$

$$Y_{id} = \left(\beta_1 + v_d\right) X_{1id} + \beta_2 X_{2id} + e_{id}\ , \tag{12}$$

$$Y_{id} = \beta_1 X_{1id} + \left(\beta_2 + v_d\right) X_{2id} + e_{id}\ . \tag{13}$$

where i = 1, 2, … , n (where n is the sample size), d = 1, 2, … , D (the population is divided into D subpopulations). Response variable in all of the models is RMT85 and auxiliary variables – P75 and REV84. We take into account classifications by two grouping variables – CL and REG, what means that we analyze ten models.

Model selection is based on the special cases of the General Information Criterion – Akkaike Information Cirterion (AIC) and Bayesian Information Criterion (BIC). General information Criterion has a form:

$$GIC\left(M\right) = -2\ln L\left(M\right) + h\left|M\right|, \tag{14}$$

where L is likelihood for analyzed model M and $\left|M\right|$ is number of parameters of model M. In the Akkaike criterion parameter h is 2 and in the Bayesian criterion – ln(n), where n it is the number of observations. We obtained smallest values of both information criteria for the model (10), where grouping variable is the indicator of cluster (CL). For the model, based on the results obtained by Żądło (2006), we have

$$D(\hat{\sigma}_v^2) = \sqrt{2b^{-1}\sum_{d=1}^{D} b_d^{-2}\left(\sum_{i\in s_d} x_{1id}^2\right)^2}\ , \tag{15}$$

where

$$b_d = \sigma_e^2 + \sigma_v^2 \sum_{i\in s_d} x_{1id}^2\ ,$$

$$b=\left(\sum_{d=1}^{D}\left((n_d-1)(\sigma_e^2)^{-2}+b_d^{-2}\right)\right)\left(\sum_{d=1}^{D}b_d^{-2}\left(\sum_{i\in s_d}x_{1id}^2\right)^2\right)-\left(\sum_{d=1}^{D}b_d^{-2}\left(\sum_{i\in s_d}x_{1id}^2\right)\right)^2,$$

which may be used in the formula (5) to obtain the value of statistics of Wald test.

In the simulation study we perform two experiments. In the first experiment we generate data based on the model:

$$Y_{id}=\beta_0+\beta_1X_{1id}+\beta_2X_{2id}+e_{id} \tag{16}$$

– linear models with two auxiliary variables where values of β_0, β_1, β_2, and σ_e^2 are obtained based on the real data under assumption of (16). The model does not contain the random effect. Hence, we analyze type I error – fraction of cases when we reject true null hypothesis.

In the second experiment we generate data based on the model (10), where values of β_0, β_1, β_2, σ_v^2 and σ_e^2 are obtained based on the real data. In this part we consider type II error – fraction of cases when we cannot reject H_0. We consider two variants of both experiments. In the first one random components are generated from normal distribution (variant "a"), in the second (variant "b") from shifted exponential distribution, in both cases expected values are 0, variances – σ_v^2 and σ_e^2.

In the simulation study the number of iterations is 1000 and the number of permutations within each iteration is 500. Firstly, we analyze type I error. Results for variant "a" are presented in the Table 1 and for variant "b" in the Table 2. Although, in the case of variant "b" (random effects and random components are generated from shifted exponential distribution) and classic test should not be used, their properties are acceptable. In both variants, classic tests were conservative (values of type I errors are smaller than assumed level o significance) and all presented permutation test – anticonservative (values of type I errors are larger than assumed level of significance).

Table 1

Summary of experiment 1(a) − type I error

Test	Assumed level of significance (α)		
	α = 0,01	α = 0,05	α = 0,1
LRT	0	0	0
Wald test	0	0,001	0,003
Permutation lnL test	0,014	0,069	0,127
Permutation LRT	0,016	0,063	0,125
Permutation Wald test	0,345	0,517	0,588

Table 2

Summary of experiment 1(b) − type I error

Test	Assumed level of significance (α)		
	α = 0,01	α = 0,05	α = 0,1
LRT	0,002	0,002	0,003
Wald test	0,004	0,017	0,032
Permutation lnL test	0,012	0,062	0,118
Permutation LRT	0,011	0,064	0,117
Permutation Wald test	0,412	0,556	0,631

In the first experiment we obtained very high values of type I errors for the Wald test. It may result from the fact that using the Wald test we assume normality of the test statistic while the possible values of the parameter (σ_v^2) are non-negative. It is a serious problem especially if the true unknown value of the parameter is very close to the border of the parameter space. Therefore, the properties of the Wald test will not be studied in the second simulation study.

Table 3

Summary of experiment 2(a) − type II error

Test	Assumed level of significance (α)		
	α = 0,01	α = 0,05	α = 0,1
LRT	0	0	0
Wald test	0	0	0
Permutation lnL test	0	0	0
Permutation LRT	0	0	0

Table 4

Summary of experiment 2(b) − type II error

Test	Assumed level of significance (α)		
	α = 0,01	α = 0,05	α = 0,1
LRT	0	0	0
Wald test	0	0	0
Permutation lnL test	0	0	0
Permutation LRT	0	0	0

In the second experiment we consider type II error. Results for variant “a” are presented in the Table 3 and for variant “b” in the Table 4. Although, in the case of variant “b” (random effects and random components are generated from shifted exponential distribution) and classic test should not be used, values of type II error in the simulation (using value of σ_v^2 obtained based on real data) are equal zero. For classic and permutation tests values of type II error are for all of the considered cases are equal zero or very close to zero. Therefore, values of the power of the tests are 1 or very close to 1.

Conclusions

In the paper three permuation test of variance components are studied – one based on the log-likelihood, other two are permutation versions of classic tests. Their properites are considered in the Monte Carlo simulation study based on the real data on revenues of municipal taxations in Swedish municipalities. The results may be very useful for practitioners who can use permutations test without testing assumptions which must be met in the case of classic tests.

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Summary

In the paper three permutation tests of significance of variance components in the linear mixed model are presented. Two of them are permutation versions of classic tests. The third one is based on log-likelihood. In the Monte Carlo simulation studies properties of the permutation tests are compared with properties of the classic likelihood ratio test and Wald test.