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THE PRESENT VALUE OF A PORTFOLIO OF ASSETS WITH PRESENT VALUES DETERMINED BY TRAPEZOIDAL ORDERED FUZZY NUMBERS*

We consider the obvious thesis that the present value of a portfolio is equal to the sum of the present values of its components. The main goal of this paper is the implementation of this thesis in the case when present values are determined by trapezoidal ordered fuzzy numbers. We apply the revised sum of ordered fuzzy numbers. The associativity of such a revised sum is investigated here. In addition, we show that the multiple revised sum of a finite sequence of trapezoidal ordered fuzzy numbers depends on the ordering of its summands. Without any obstacles, the results obtained can be generalized to the case of any ordered fuzzy numbers.

Keywords: *present value, ordered fuzzy number, portfolio, revised sum*

1. Introduction

In [19] the present value (*PV*) of a cash flow to be obtained in the present or future was defined to be the monetary value to be received now that we would deem to be equivalent. This equivalent may be imprecisely estimated. Thus it is commonly accepted that the *PV* of a future cash flow can be imprecise. The natural consequence of this approach to estimate *PV* by fuzzy numbers. Ward [27] defined a fuzzy *PV* as a discounted fuzzy forecast of the value of a future cash flow. Sheen [25] expanded this

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definition to the case of a fuzzy future cash flow. A more general definition of a fuzzy *PV* was proposed by Tsao [26], who assumed that a future cash flow is a fuzzy probabilistic set. All of these authors depicted *PV* as a discounted, imprecisely estimated future cash flow. A different approach was given in [18] and [23], where the behavioural present value (*BPV*) was defined as such an approximation of the current market price which is imprecisely estimated under the impact of behavioural factors.

In [12], the information described by *BPV* was supplemented with a subjective forecast of the trend in the market price. This forecast was implemented in the model of *BPV* as the orientation of a fuzzy number. In this way, the *BPV* was replaced by an oriented *BPV* described by an ordered fuzzy number [9]. The positive orientation of a fuzzy number indicates a subjective prediction of a rise in the market price. The negative orientation of fuzzy number indicates a subjective prediction of a fall in the market price.

In general, a portfolio's *PV* is equal to the sum of its components' *PV*s. In [22], a two-asset portfolio was considered in the case where its components are assessed by oriented *BPV*s. The considered portfolio is assessed by an oriented *BPV* determined in the same way as its components' *BPV*s. Thus adding the components' *PV*s is not necessary.

The results obtained in the articles [1, 4, 10] and [11] have justified the applicability of triangular or trapezoidal fuzzy numbers as a tool in financial arithmetic. Therefore, the main goal of this paper is to calculate a portfolio's *PV* in the case where its components' *PV*s are given as trapezoidal ordered fuzzy numbers.

The paper is organized as follows. Section 2 outlines ordered fuzzy numbers and their basic properties. The content of this section is the theoretical background for our later considerations. The next three sections contain our original contribution. In Section 3, we introduce a simple formula for the revised sum of trapezoidal fuzzy numbers. Moreover, we find some previously unknown properties of revised sums. These conclusions are used in Section 4, where the notion of a portfolio's oriented fuzzy *PV* is introduced. In addition, we present an original method for determining a portfolio's oriented fuzzy *PV*. Section 5 contains a case study based on real data. This case study sufficiently explains the proposed method for evaluating a portfolio. On the other hand, the considered case study is an original proposition of applying Japanese Candlesticks to portfolio analysis. Finally, Section 6 concludes the article, summarizes the main findings of this research and proposes some future research directions.

2. The basic notions

An imprecise number is a family of values in which each considered value belongs to the imprecise number to a varying degree. A commonly accepted model of an imprecise number is a fuzzy number (*FN*), defined as a fuzzy subset of the real line *R*. The most general definition of *FN* is given as follows [3]:

Definition 1. A fuzzy number is a fuzzy subset $S \in F(R)$ represented by an upper semi-continuous membership function $\mu_S \in [0, 1]^R$ satisfying the conditions:

$$\exists_{x \in R} \ \mu_S(x) = 1 \tag{1}$$

$$\forall_{(x,y,z) \in R^3} \ x \le y \le z \Rightarrow \mu_S(y) \ge \min\{\mu_S(x), \ \mu_S(z)\} \square \tag{2}$$

We denote the set of all *FN* by the symbol $\overline{\overline{F}}$. Dubois and Prade [2] first introduced arithmetic operations on *FN*. These arithmetic operations are coherent with the Zadeh extension principle [28–30].

The concept of ordered fuzzy numbers (*OFN*) was introduced by Kosiński et al. in a series of papers [6–9] as an extension of the concept of fuzzy numbers. Thus, any *OFN* should be determined as a fuzzy subset of the real line R. On the other hand, Kosiński defined *OFN* as a ordered pair of functions from the unit interval $[0,1]$ into *R*. This kind of pair is not a fuzzy subset in *R*. Thus we cannot accept Kosiński's original terminology. However, Kosiński's intuitive approach to the notion of *OFN* is very useful. This usefulness follows from the fact that Kosiński's definition describes the orientation of an *OFN* which is understood as a linear order on the real line *R*. This orientation may be negative or positive. A negative orientation means a linear order on *R* from bigger numbers to smaller ones. A negatively oriented number is interpreted as such a number which may decrease. A positive orientation means a linear order on *R* from smaller numbers to bigger ones. A positively oriented number is interpreted as such a number which may increase. We will denote any orientation from $a \in R$ to $b \in R$ by the symbol $[a \mapsto b]$. According to Kosiński's interpretation, an *OFN* should not be considered as information about an *FN* and its orientation.

On the other hand, any *OFN* should be considered as information about an *FN* and its orientation. For these reasons, a revised general definition of *OFN* was presented in [21], where an *OFN* was defined as a pair of an *FN* and its orientation. This definition fully corresponds to the intuitive definition by Kosiński. The space of all *OFN* is denoted by the symbol *K*. The space *K* may be described as the following union:

$$K = K^+ \cup K^- \cup R \tag{3}$$

where: K^+ is the space of all positively oriented *OFN*, K^- – the space of all negatively oriented *OFN*.

In this paper, we will limit our considerations to the following kind of *OFN*.

Definition 2. For any monotonic sequence $\{a,b,c,d\} \subset R$ the trapezoidal ordered fuzzy number (*TrOFN*) $\overleftrightarrow{Tr}(a,b,c,d)$ is defined as the pair of *FN*s determined by the membership function $\mu_{\overleftrightarrow{Tr}}(\cdot \mid a,b,c,d) \in [0,1]^R$ given by the identity[3]

$$\mu_{\overleftrightarrow{Tr}}(x \mid a,b,c,d) = \begin{cases} 0 & x \notin [a,d] = [d,a] \\ \dfrac{x-a}{b-a} & x \in [a,b[=]b,a] \\ 1 & x \in [b,c[=]c,b] \\ \dfrac{x-d}{c-d} & x \in]c,d] = [d,c[\end{cases} \tag{4}$$

and orientation $[a \mapsto d]$. □

Fulfilment of the condition $a < d$ corresponds to the positive orientation $[a \mapsto d]$ of *TrOFN* $\overleftrightarrow{Tr}(a,b,c,d)$. Fulfilment of the condition $a > d$ corresponds to the negative orientation $[a \mapsto d]$ of *TrOFN* $\overleftrightarrow{Tr}(a,b,c,d)$. In the case $a = d$, *TrOFN* $\overleftrightarrow{Tr}(a,a,a,a)$ represents the crisp number $a \in R$, which is not oriented.

For the case of *OFN* as defined by Kosiński, the arithmetic operators of summation $\oplus$ and the dot product $\otimes$ are defined in [9]. In this paper, the sum determined by the operator $\oplus$ will be called the *K*-sum. Without any obstacles, these two operators can be equivalently implemented as arithmetic operators on the space *K*. The *K*-sum is associative and commutative. If two *OFN*s have identical orientations, then their *K*-sum is identical to the sum obtained by means of the arithmetic introduced by Dubois and Prade [2]. However, Kosiński [9] showed that if two *OFN*s have different orientations, then their *K*-sum may be different to the result obtained by the arithmetic introduced by Dubois and Prade [2].

The *K*-sum of any two *TrOFN*s $\overleftrightarrow{Tr}(a_1, b_1, c_1, d_1)$ and $\overleftrightarrow{Tr}(a_2, b_2, c_2, d_2)$ can be calculated as follows:

$$\overline{\overline{K}}(a_1+a_2, b_1+b_2, c_1+c_2, d_1+d_2) = \overleftrightarrow{Tr}(a_1,b_1,c_1,d_1) \oplus \overleftrightarrow{Tr}(a_2,b_2,c_2,d_2) \tag{5}$$

where the membership relation $\mu_{Tr}(\cdot \mid a,b,c,d)$ of the sum $\overline{\overline{K}}(a,b,c,d)$ is given by (4).

[3]Let us note that this identity describes an extended concept of numerical intervals used in this article.

The dot product of any real number $\beta \in R$ and any *TrOFN* $\overleftrightarrow{Tr}\left(a_1, b_1, c_1, d_1\right)$ can be calculated as follows:

$$\overleftrightarrow{Tr}\left(\beta \cdot a_1, \beta \cdot b_1, \beta \cdot c_1, \beta \cdot d_1\right) = \beta \otimes \overleftrightarrow{Tr}\left(a_1, b_1, c_1, d_1\right) \tag{6}$$

3. Revised sum

In [21], it was shown that there exist pairs of *TrOFN* such that their *K*-sum is not an *OFN*. Therefore, we ought to modify the operation of adding *OFN*s in a way that the sum of two *OFN*s is always *OFN*. In this paper, we define the revised sum of two *TrOFN*s as follows:

$$\overleftrightarrow{Tr}\left(a,b,c,d\right)\ [+]\ \overleftrightarrow{Tr}\left(p-a, q-b, r-c, s-d\right)$$
$$= \begin{cases} \overleftrightarrow{Tr}\left(\min\{p,q\},\ q, r,\ \max\{r,s\}\right) & \left(q<r\right) \vee \left(q=r \wedge p \le s\right) \\ \overleftrightarrow{Tr}\left(\max\{p,q\},\ q, r,\ \min\{r,s\}\right) & \left(q>r\right) \vee \left(q=r \wedge p > s\right) \end{cases} \tag{7}$$

For any pair of *TrOFN*s, their modified sum $[+]$ is equal to the *TrOFN* determined by such a membership function that its graph is nearest to the graph of the relation determined by (4) and (5). This implies that

$$\forall_{\overleftrightarrow{A}, \overleftrightarrow{B} \in K}\ \overleftrightarrow{A} \oplus \overleftrightarrow{B} \in K \Rightarrow \overleftrightarrow{A}[+]\ \ \overleftrightarrow{B} = \overleftrightarrow{A} \oplus \overleftrightarrow{B} \tag{8}$$

If we find two different *TrOFN*s nearest to the considered *K*-sum $\oplus$, then we choose the positively oriented one, of which there exists exactly one. Moreover, for any pair of *TrOFN*s $\overleftrightarrow{A}$, $\overleftrightarrow{B}$ we have

$$\forall_{\overleftrightarrow{A}, \overleftrightarrow{B} \in K^+}\ \overleftrightarrow{A}\,[+]\,\overleftrightarrow{B} = \overleftrightarrow{A} \oplus \overleftrightarrow{B} \in K \tag{9}$$

$$\forall_{\overleftrightarrow{A}, \overleftrightarrow{B} \in K^-}\ \overleftrightarrow{A}\,[+]\,\overleftrightarrow{B} = \overleftrightarrow{A} \oplus \overleftrightarrow{B} \in K \tag{10}$$

Example 1. Some cases of the modified sum $[+]$ are presented below. Let us observe that in all of these cases, the *K*-sum $\oplus$ does not exist.

$$\overleftrightarrow{Tr}(1, 2, 4, 6)\ [+]\ \overleftrightarrow{Tr}(5, 3, 2, 1) = \overleftrightarrow{Tr}(5,\ 5,\ 6,\ 7)$$

$$\overleftrightarrow{Tr}(6,\ 4,\ 2,\ 1)\ [+]\ \overleftrightarrow{Tr}(1,\ 2,\ 3,\ 5) = \overleftrightarrow{Tr}(7,\ 6,\ 5,\ 5)$$

$$\overleftrightarrow{Tr}(1,\ 2,\ 4,\ 4)\ [+]\ \overleftrightarrow{Tr}(5,\ 3,\ 2,\ 1) = \overleftrightarrow{Tr}(5,\ 5,\ 6,\ 6)$$

$$\overleftrightarrow{Tr}(4,\ 4,\ 2,\ 1)\ [+]\ \overleftrightarrow{Tr}(1,\ 2,\ 3,\ 5) = \overleftrightarrow{Tr}(6,\ 6,\ 5,\ 5)$$

$$\overleftrightarrow{Tr}(1,\ 2,\ 3,\ 4)\ [+]\ \overleftrightarrow{Tr}(6,\ 3,\ 2,\ 2) = \overleftrightarrow{Tr}(7,\ 5,\ 5,\ 5)\ \square$$

Now we will study the basic properties of this revised sum. For a more detailed view of further considerations, we define the following special kind of *TrOFN*.

Definition 3. For any monotonic sequence $\{a, b, c\} \subset R$, a triangular ordered fuzzy number (*TOFN*) $\overleftrightarrow{T}(a, b, c)$ is defined by the identity.

$$\overleftrightarrow{T}(a, b, c) = \overleftrightarrow{Tr}(a, b, b, c)\ \square \tag{11}$$

Many mathematical applications require that a finite multiple sum is independent of the ordering of the summands. Any associative and commutative sum satisfies this property. The sum of real numbers is associative and commutative. The sum of fuzzy numbers [2] and the *K*-sum $\oplus$ of *OFN* are also associative and commutative. It is very easy to check that the revised sum $[+]$ is commutative. Now we investigate the associativity of the revised sum $[+]$.

Counterexample 1. Consider the following four *TOFN*:

$$A = \overleftrightarrow{T}(10, 40, 70),\ B = \overleftrightarrow{T}(110, 100, 60),\ C = \overleftrightarrow{T}(50, 65, 105),\ D = \overleftrightarrow{T}(120, 90, 67)$$

The number of different ways of associating the three applications of the addition operator $[+]$ is equal to the Catalan number $C_3 = 5$. Therefore, we have the following five different associations of four summands [8]:

$$(A\,[+]\,B)[+](C\,[+]\,D),\ \ ((A\,[+]\,B)[+]\,C)[+]D,\ \ (A\,[+](B\,[+]\,C))[+]\,D$$

$$(A\,[+](B\,[+]\,C))[+]\,D,\ \ A\,[+](B\,[+]\,(C\,[+]\,D))$$

In [14], it is shown that in the case considered here

$$(A\,[+]\,B)[+](C\,[+]\,D) = A\,[+]\big(B\,[+](C\,[+]\,D)\big) = \overleftrightarrow{T}(275, 295, 302)$$

$$\big(A\,[+](B\,[+]\,C)\big)[+]\,D = A\,[+]\big((B\,[+]\,C)[+]\,D\big) = \overleftrightarrow{T}(290, 295, 302)$$

$$\big((A\,[+]\,B)\,[+]\,C\big)\,[+]\,D = \overleftrightarrow{T}(290, 295, 312) \;\square$$

The results of the above counterexample prove that the revised sum $[+]$ of *TrOFN* is not associative.

Counterexample 2. We determine the multiple sum for all the permutations of the four *TOFN*s $A, B, C, D \in K$ described in Counterexample 1. In [14], it is shown that in the case considered here

$$C\,[+]\,D\,[+]\,A\,[+]\,B = C\,[+]\,D\,[+]\,B\,[+]\,A = D\,[+]\,A\,[+]\,B\,[+]\,C$$
$$= D\,[+]\,C\,[+]\,A\,[+]\,B = D\,[+]\,C\,[+]\,B\,[+]\,A = \overleftrightarrow{T}(275, 295, 302)$$
$$A\,[+]\,C\,[+]\,B\,[+]\,D = A\,[+]\,C\,[+]\,D\,[+]\,B = A\,[+]\,D\,[+]\,B\,[+]\,C$$
$$= A\,[+]\,D\,[+]\,C\,[+]\,B = B\,[+]\,C\,[+]\,D\,[+]\,A = B\,[+]\,D\,[+]\,A\,[+]\,C$$
$$= B\,[+]\,D\,[+]\,C\,[+]\,A = C\,[+]\,A\,[+]\,B\,[+]\,D = C\,[+]\,A\,[+]\,D\,[+]\,B$$
$$= C\,[+]\,B\,[+]\,A\,[+]\,D = C\,[+]\,B\,[+]\,D\,[+]A = D\,[+]\,A\,[+]\,B\,[+]\,C$$
$$= D\,[+]\,A\,[+]\,C\,[+]\,B = D\,[+]\,B\,[+]\,A\,[+]\,C = D\,[+]\,B\,[+]\,C\,[+]\,A$$
$$= \overleftrightarrow{T}(290,\; 295,\; 302)$$
$$A\,[+]\,B\,[+]\,C\,[+]\,D = A\,[+]\,B\,[+]\,D\,[+]\,C = B\,[+]\,A\,[+]\,C\,[+]\,D$$
$$= B\,[+]\,A\,[+]\,D\,[+]\,C = B\,[+]\,C\,[+]\,A\,[+]\,D = \overleftrightarrow{T}(290, 295, 312) \;\square$$

The results of the above counterexample prove that the multiple revised sum $[+]$ of a finite sequence of *TrOFN*s depends on the ordering of its summands. The ordering of summands should be clearly defined for each practical application of the multiple revised sum $[+]$ of a finite sequence of *TrOFN*s. Such an ordering of summands must be sufficiently justified in the appropriate field of application.

All of the results obtained in this section may be generalized to the case of any *OFN*.

4. Present value of a portfolio

By a financial portfolio we will understand an arbitrary, finite set of financial assets. We consider a multi-asset portfolio π, consisting of the assets Y_i $(i = 1, 2, ..., n)$. Each of these assets $Y_i \in \pi$ is characterized by a non-decreasing sequence $\left\{V_{\min}^{(i)}, V_{\subseteq}^{(i)}, V_{\supseteq}^{(i)}, V_{\max}^{(i)}\right\}$ of values given as follows:

- $V_{\min}^{(i)}$ is the infimum of the values perceptibly equal to *PV*,
- $V_{\max}^{(i)}$ is the supremum of the values perceptibly equal to *PV*,
- $V_{\subseteq}^{(i)}$ is the supremum of the values perceptibly less than *PV*,
- $V_{\supseteq}^{(i)}$ is the infimum of the values perceptibly greater than *PV*.

Let us assume that for each asset $Y_i \in \pi$ we have a subjective prediction of future trends in its market price. Then we estimate the *PV* of any asset $Y_i \in \pi$ by the *TrOFN* $\overset{\leftrightarrow}{PV}_i$ determined in the following way:

If we predict a rise in the market price of the asset Y_i, then its *PV* is determined as the positively oriented *TrOFN*

$$\overset{\leftrightarrow}{PV}_i = \overset{\leftrightarrow}{Tr}\left(V_{\min}^{(i)}, V_{\subseteq}^{(i)}, V_{\supseteq}^{(i)}, V_{\max}^{(i)}\right) \tag{12}$$

if we predict a fall in the market price of the asset Y_i, then its *PV* is determined as the negatively oriented *TrOFN*

$$\overset{\leftrightarrow}{PV}_i = \overset{\leftrightarrow}{Tr}\left(V_{\max}^{(i)}, V_{\supseteq}^{(i)}, V_{\subseteq}^{(i)}, V_{\min}^{(i)}\right) \tag{13}$$

In this way, each asset $Y_i \in \pi$ is evaluated by an oriented fuzzy *PV* given by the identity

$$\overset{\leftrightarrow}{PV}_i = \overset{\leftrightarrow}{Tr}\left(V_b^{(i)}, V_f^{(i)}, V_l^{(i)}, V_e^{(i)}\right) \tag{14}$$

where the individual parameters are reinterpreted as follows:

- $V_b^{(i)}$ is the beginning *PV*,
- $V_f^{(i)}$ is the first *PV*,
- $V_l^{(i)}$ is the last *PV*,
- $V_e^{(i)}$ is the end *PV*.

In this way, we distinguish a portfolio of rising securities $\pi^{+} \subset \pi$ and a portfolio of falling securities $\pi^{-} \subset \pi$ as follows:

$$\pi^{+} = \left\{ Y_i \in \pi : \overset{\leftrightarrow}{PV}_i \in K^{+} \right\} \tag{15}$$

$$\pi^{-} = \left\{ Y_i \in \pi : \overset{\leftrightarrow}{PV}_i \in K^{-} \right\} \tag{16}$$

A portfolio's *PV* is always equal to the sum of its components' *PV*s. In the case where the components' *PV*s are estimated by *TrOFN*s, addition should be modelled by the revised sum $[+]$. In the previous section, it was shown that, in this case, the result of multiple additions depends on the order of the summands. This implies that a portfolio's *PV*, given as any multiple revised sum of its components' *PV*s, is not explicitly determined. Therefore, in the considered case, any method of calculating the portfolio's *PV* should be supplemented with a reasonable method for determining the ordering of the portfolio's components. Below, we propose such a method of ordering the assets, which seems to us to be reasonable.

First, we propose to calculate the *PV* of a portfolio of rising securities π^{+}, denoted by the symbol $\overset{\leftrightarrow}{PV}^{(\pi+)}$ and the *PV* of a portfolio of falling securities π^{-} denoted by the symbol $\overset{\leftrightarrow}{PV}^{(\pi-)}$. From (9), (10) and (5), we obtain

$$\begin{aligned}
\overset{\leftrightarrow}{PV}^{(\pi+)} &= \overset{\leftrightarrow}{Tr}\left(\sum_{Y_i \in \pi+} V_{\min}^{(i)},\ \sum_{Y_i \in \pi+} V_{\subseteq}^{(i)},\ \sum_{Y_i \in \pi+} V_{\supseteq}^{(i)},\ \sum_{Y_i \in \pi+} V_{\max}^{(i)} \right) \\
&= \overset{\leftrightarrow}{Tr}\left(V_{\min}^{(\pi+)},\ V_{\subseteq}^{(\pi+)},\ V_{\supseteq}^{(\pi+)},\ V_{\max}^{(\pi+)} \right) \\
&= \overset{\leftrightarrow}{Tr}\left(V_{b}^{(\pi+)},\ V_{f}^{(\pi+)},\ V_{l}^{(\pi+)},\ V_{e}^{(\pi+)} \right)
\end{aligned} \tag{17}$$

$$\begin{aligned}
\overset{\leftrightarrow}{PV}^{(\pi-)} &= \overset{\leftrightarrow}{Tr}\left(\sum_{Y_i \in \pi-} V_{\max}^{(i)},\ \sum_{Y_i \in \pi-} V_{\supseteq}^{(i)},\ \sum_{Y_i \in \pi-} V_{\subseteq}^{(i)},\ \sum_{Y_i \in \pi-} V_{\min}^{(i)} \right) \\
&= \overset{\leftrightarrow}{Tr}\left(V_{\max}^{(\pi-)},\ V_{\supseteq}^{(\pi-)},\ V_{\subseteq}^{(\pi-)},\ V_{\min}^{(\pi-)} \right) \\
&= \overset{\leftrightarrow}{Tr}\left(V_{b}^{(\pi-)},\ V_{f}^{(\pi-)},\ V_{l}^{(\pi-)},\ V_{e}^{(\pi-)} \right)
\end{aligned} \tag{18}$$

Both of the above *PV*s are determined explicitly. Therefore, in the next step, from (15), we can determine the *PV*, $\overset{\leftrightarrow}{PV}^{(\pi)}$ of the portfolio in an explicit manner as the sum

$$\begin{aligned}\overset{\leftrightarrow}{PV}^{(\pi)} &= \overset{\leftrightarrow}{\mathrm{PV}}^{(\pi)} = \overset{\leftrightarrow}{\mathrm{PV}}^{(\pi+)}\ [+]\ \overset{\leftrightarrow}{\mathrm{PV}}^{(\pi-)} \\ &= \overset{\leftrightarrow}{Tr}\left(V_b^{(\pi+)},\ V_f^{(\pi+)},\ V_l^{(\pi+)},\ V_e^{(\pi+)}\right)\ [+]\ \overset{\leftrightarrow}{Tr}\left(V_b^{(\pi-)},\ V_f^{(\pi-)},\ V_l^{(\pi-)},\ V_e^{(\pi-)}\right) \\ &= \begin{cases} \overset{\leftrightarrow}{Tr}\left(\min\{b,\ f\},\ f,\ l,\ \max\{l,\ e\}\right) & (f < l) \vee (f = l \wedge b \le e) \\ \overset{\leftrightarrow}{Tr}\left(\max\{b,\ f\},\ f,\ r,\ \min\{l,\ e\}\right) & (f > l) \vee (f = l \wedge b > e) \end{cases}\end{aligned} \tag{19}$$

where

$$b = V_b^{(\pi+)} + V_b^{(\pi-)} \tag{20}$$

$$f = V_f^{(\pi+)} + V_f^{(\pi-)} \tag{21}$$

$$l = V_l^{(\pi+)} + V_l^{(\pi-)},\ \ \pi^- = \left\{Y_i \in \pi : \overset{\leftrightarrow}{PV}_i \in K^-\right\} \tag{22}$$

$$e = V_e^{(\pi+)} + V_e^{(\pi-)} \tag{23}$$

Moreover, the *PV*s of the portfolios π^+ and π^- may be determined using the results obtained in [23]. This is sufficient to manage portfolio risk, because only rising securities can get BUY or ACCUMULATE recommendations and only falling securities can get SELL or REDUCE recommendations. The complex form of the relationships (19)–(23) allows us to use them only for evaluating an already constructed portfolio. Such evaluation may be carried out using the analytical tools described in [15] and [20].

5. Case study

Japanese candlesticks [17] are a very useful tool supporting investors' decisions on the exchange market. The concept of Japanese candles can be interpreted as estimation of an ambiguous *PV*. In [16], it is shown that any Japanese candlestick can be represented by a *TrOFN* $\overset{\leftrightarrow}{Tr}\left(Pb, Po, Pc, Pf\right)$, where the individual parameters are given as follows: Pb is the back price, Po is the opening price, Pc is the closing price, and Pf is the face price.

The back price, Pb, and the face price, Pf, are determined using the extreme prices: the minimal price, Pl[4], and the maximal price, Ph[5]. All Japanese candles can be divided into three groups as follows:

- white candle ($Po < Pc$), where $Pb = Pl$, and $Pf = Ph$,
- black candle ($Po > Pc$), where $Pb = Ph$, and $Pf = Pl$,
- Doji ($Po < Pc$) described by a *TrOFN* with orientation determined by the direction from the earlier extreme price to the later extreme price.

Based on the closing of the session on the Warsaw Stock Exchange on January 15, 2018, we evaluate the portfolio π composed of:

- a block B_1 of 10 shares in Assecopol (ACP),
- a block B_2 of 30 shares in ENERGA (ENG),
- a block B_3 of 5 shares in JSW (JSW),
- a block B_4 of 5 shares in KGHM (KGH),
- a block B_5 of 10 shares in LOTOS (LTS),
- a block B_6 of 100 shares in ORANGEPL (OPL),
- a block B_7 of 10 shares in PKOBP (PKO).

The stock quotes of these shares observed on January 15, 2018 are presented in Table 1.

Table 1. Selected quotes on the Warsaw Stock Exchange on January 15, 2018

Stock company	Opening price	Minimal price	Maximal price	Closing price
ACP	46.96	46.04	46.96	46.58
ENG	12.46	12.24	12.51	12.42
JSW	107.55	106.45	108.90	108.15
KGH	115.50	114.00	115.95	115.00
LTS	56.10	56.00[a]	56.18[b]	56.10
OPL	6.02	5.98	6.11	6.11
PKO	45.35	44.92	45.70	45.50

[a]Earlier extreme price.
[b]Later extreme price.
Source: [31].

[4]In the original terminology of Japanese candlesticks, the minimal price is called the low price.
[5]In the original terminology of Japanese candlesticks, the maximal price is called the high price.

Let us evaluate the components of the portfolio using their oriented fuzzy *PV*s as determined by Japanese candlesticks. If these Japanese candlesticks are represented by *TrOFN*, then we have:

$$\overset{\leftrightarrow}{PV}_{ACP} = \overset{\leftrightarrow}{Tr}\left(46.96,\ 46.96,\ 46.58,\ 46.04\right)$$

$$\overset{\leftrightarrow}{PV}_{ENG} = \overset{\leftrightarrow}{Tr}\left(12.51,\ 12.46,\ 12.42,\ 12.24\right)$$

$$\overset{\leftrightarrow}{PV}_{JSW} = \overset{\leftrightarrow}{Tr}\left(106.45,\ 107.55,\ 108.15,\ 108.90\right)$$

$$\overset{\leftrightarrow}{PV}_{KGH} = \overset{\leftrightarrow}{Tr}\left(115.95,\ 115.50,\ 115.00,\ 114.00\right)$$

$$\overset{\leftrightarrow}{PV}_{LTS} = \overset{\leftrightarrow}{Tr}\left(56.00,\ 56.10,\ 56.10,\ 56.18\right)$$

$$\overset{\leftrightarrow}{PV}_{OPL} = \overset{\leftrightarrow}{Tr}\left(5.98,\ 6.02,\ 6.11,\ 6.11\right)$$

$$\overset{\leftrightarrow}{PV}_{PKO} = \overset{\leftrightarrow}{Tr}\left(44.92,\ 45.35,\ 45.50,\ 45.70\right)$$

We notice that:

- the stocks of the companies JSW, OPL and PKO are evaluated by white candles, which predict a rise in the market price,
- the stocks of the companies ACP, ENG and KGH are evaluated by black candles, which predict a fall in the market price,
- the stocks of the company LTS are evaluated by a positively oriented Doji, which predicts a rise in the market price.

In this way, we distinguish a portfolio of rising shares π^{+} from a portfolio of falling shares π^{-} as follows:

$$\pi^{+} = \left\{JSW,\ OPL,\ LTS,\ PKO\right\}$$

$$\pi^{-} = \left\{ACP,\ ENG,\ KGH\right\}$$

Using Equation (6) for each considered block, B_i $\left(i = 1,\ 2,\ ...,\ 7\right)$, of shares, we calculate the corresponding oriented fuzzy $\overset{\leftrightarrow}{PV}_i$ as follows:

$$\overleftrightarrow{PV}_1 = 10 \otimes \overleftrightarrow{PV}_{ACP} = \overleftrightarrow{Tr}\left(469.60,\ 469.60,\ 465.80,\ 460.40\right)$$

$$\overleftrightarrow{PV}_2 = 30 \otimes \overleftrightarrow{PV}_{ENG} = \overleftrightarrow{Tr}\left(375.30,\ 373.80,\ 372.60,\ 367.20\right)$$

$$\overleftrightarrow{PV}_3 = 5 \otimes \overleftrightarrow{PV}_{JSW} = \overleftrightarrow{Tr}\left(532.25,\ 537.75,\ 540.75,\ 544.50\right)$$

$$\overleftrightarrow{PV}_4 = 5 \otimes \overleftrightarrow{PV}_{KGH} = \overleftrightarrow{Tr}\left(579.75,\ 577.50,\ 575.00,\ 570.00\right)$$

$$\overleftrightarrow{PV}_5 = 10 \otimes \overleftrightarrow{PV}_{LTS} = \overleftrightarrow{Tr}\left(560.00,\ 561.00,\ 561.00,\ 561.80\right)$$

$$\overleftrightarrow{PV}_6 = 100 \otimes \overleftrightarrow{PV}_{OPL} = \overleftrightarrow{Tr}\left(598.00,\ 602.00,\ 611.00,\ 611.00\right)$$

$$\overleftrightarrow{PV}_7 = 10 \otimes \overleftrightarrow{PV}_{PKO} = \overleftrightarrow{Tr}\left(449.20,\ 453.50,\ 455.00,\ 457.40\right)$$

First, we calculate the oriented fuzzy *PV*s of the portfolio of rising shares π^{+} denoted by the symbol $\overleftrightarrow{PV}^{(\pi+)}$, and of the portfolio of the falling shares π^{-} denoted by the symbol $\overleftrightarrow{PV}^{(\pi-)}$. From (17) and (18), we obtain

$$\overleftrightarrow{PV}^{(\pi+)} = \overleftrightarrow{PV}_3 \oplus \overleftrightarrow{PV}_5 \oplus \overleftrightarrow{PV}_6 \oplus \overleftrightarrow{PV}_7 = \overleftrightarrow{Tr}\left(2139.45,\ 2154.25,\ 2167.75,\ 2174.30\right)$$

$$\overleftrightarrow{PV}^{(\pi-)} = \overleftrightarrow{PV}_1 \oplus \overleftrightarrow{PV}_2 \oplus \overleftrightarrow{PV}_4 = \overleftrightarrow{Tr}\left(1424.65,\ 1420.90,\ 1413.40,\ 1397.60\right)$$

In the last step, from (19) we determine the oriented fuzzy *PV* of the portfolio π as the revised sum

$$\overleftrightarrow{PV}^{(\pi)} = \overleftrightarrow{PV}^{(\pi+)}\ [+]\ \overleftrightarrow{PV}^{(\pi-)} = \overleftrightarrow{Tr}\left(3564.10,\ 3575.15,\ 3581.15,\ 3581.15\right)$$

We notice that the portfolio π corresponds to a white candle, which predicts a rise in the market value of the portfolio.

Counterexample 3. Let us assume that the oriented fuzzy *PV* of the considered portfolio π may be determined as the *K*-sum

$$\overline{\overline{PV}}^{(\pi)} = \overleftrightarrow{PV}^{(\pi+)} \oplus \overleftrightarrow{PV}^{(\pi-)}$$

Then, using equation (5), we obtain

$$\overline{\overline{PV}}^{(\pi)} = \overline{\overline{K}}\left(3564.10,\ 3575.15,\ 3581.15,\ 3571.90\right)$$

which means that the maximal price $Ph = 3571.90$ is lower than the closing price $Pc = 3581.15$. This is a contradiction! □

The above counterexample proves the need to use revised sums for portfolio analysis.

6. Summary

In this paper, the *PV* of a portfolio is determined explicitly for any portfolio consisting of assets with oriented fuzzy *PV*s estimated by *TrOFN*s. The authors are convinced that the proposed method of determining the *PV* is well-reasoned. On the other hand, this paper has shown that particular methods of determining the *PV* of a portfolio can give significantly different results according to the order of summation of the oriented fuzzy *PV*s of the portfolio's components. Therefore, it is obvious to us that there may be proposals for other methods of determining the *PV* of such portfolios. All of these proposals should be carefully compared.

The above results well justify the need for further research into two-asset portfolios consisting of one rising asset and one falling asset.

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