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INCOMPLETE PAIRWISE COMPARISON MATRIX AND ITS APPLICATION TO RANKING OF ALTERNATIVES

Abstract

A fuzzy preference matrix is the result of pairwise comparison - a powerful method in multi-criteria optimization. When comparing two elements, the decision maker assigns a value between 0 and 1 to any pair of alternatives representing the element of the fuzzy preference matrix. Here, we investigate relations between transitivity and consistency of fuzzy preference matrices and multiplicative preference ones. The results obtained are applied to decision situations where some elements of the fuzzy preference matrix are missing. We propose a new method for completing the fuzzy preference matrix with missing elements called the extension of the fuzzy preference matrix and investigate an important particular case of the fuzzy preference matrix with missing elements. Next, using the eigenvector of the transformed matrix we obtain the corresponding priority vector. Illustrative numerical examples are supplied.

Keywords: pairwise comparison matrix, fuzzy preference matrix, reciprocity, consistency, transitivity, fuzzy preference matrix with missing elements.

1 Introduction

In various fields of evaluation, selection, and prioritization processes the decision makers (DM) try to find the best alternative(s) from a feasible set of alternatives. In many cases, the comparison of different alternatives according to their desirability in decision problems cannot be done by one person or using only a single criterion. In many DM problems, procedures have been established to combine opinions about alternatives related to different points of view. These procedures are often based on pairwise comparisons, in the sense that processes

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are linked to some degree of preference of one alternative over another. According to the nature of the information expressed by every DM, for every pair of alternatives different representation formats can be used to express preferences, e.g. multiplicative preference relations (see Alonso et al., 2008; Saaty, 1980; Saaty, 1991), fuzzy preference relations (see Fodor, Roubens, 1994; Herrera-Viedma et al., 2004; Ramík, 2011), interval-valued preference relations and also linguistic preference relations (see Alonso et al., 2008).

Usually, experts are characterized by their own personal background and knowledge of the problem to be solved. Expert opinions may differ substantially, some of them cannot efficiently express a preference degree between two or more of the available options. This may be true due to an imprecise or insufficient level of knowledge of the problem on the part of an expert, or because the expert is unable to determine the degree to which some options are better than others. In such situations the expert will provide an incomplete fuzzy preference relation (see Alonso et al., 2008; Herrera-Viedma et al., 2007).

In this paper, we present a general method to estimate the missing information in the form of incomplete fuzzy preference relations- multiplicative or fuzzy. Our proposal is different to the approach described in Herrera-Viedma, et al., (2007) and Ma et al., (2006), where special averages of expert evaluations and consistency/transitivity properties are applied. In the literature (Xu and Da, 2005), the problem is solved by the least deviation method to obtain a priority vector of a fuzzy preference relation. Here, we propose the classical result of Perron-Frobenius theory to obtain the priority vector for a transformed fuzzy preference matrix. Moreover, our approach enables us to obtain a priority vector for additive-transitive preference relations and also for additive-consistent ones, i.e. additive-reciprocal and multiplicative-transitive ones. It also allows for completing a pairwise comparison matrix with missing elements and for finding out the closest consistent/transitive matrix to the inconsistent/intransitive one, i.e. by repairing the inconsistency of fuzzy preference relations.

2 Multiplicative and additive preferences

The DM problem can be formulated as follows. Let $X=\{x_1, x_2,...,x_n\}$ be a finite set of alternatives. These alternatives have to be ordered from best to worst, using the information given by a DM in the form of pairwise comparison matrix.

The preferences over the set of alternatives, X, may be represented in two ways: multiplicative and additive (also called fuzzy preference relations). Let us assume that the preferences on X are described by a preference relation on X given by a positive $n\times n$ matrix $\boldsymbol{A}=\{a_{ij}\}$, where $a_{ij} > 0$ for all i,j indicates the preference intensity for the alternative x_i to that of x_j. The elements of $\boldsymbol{A}=\{a_{ij}\}$ satisfy the following reciprocity condition.

A positive n×n matrix $\mathbf{A}=\{a_{ij}\}$ is multiplicative-reciprocal (m-reciprocal), if:

$$a_{ij}.\ a_{ji} =1 \text{ for all } i,j\in\{1,2,\ldots,n\}. \quad (1)$$

A positive n×n matrix $\mathbf{A}=\{a_{ij}\}$ is multiplicative-consistent (or, m-consistent), if:

$$a_{ij} = a_{ik}.\, a_{kj} \quad \text{for all } i,j,k\in\{1,2,\ldots,n\}. \tag{2}$$

Note that $a_{ii} = 1$ for all i, and that an m-consistent matrix is m-reciprocal (however, not vice versa). Here, $a_{ij} > 0$ and m-consistency is not restricted to the Saaty scale. In particular, we extend this scale to the closed interval $[1/\sigma; \sigma]$, where $\sigma>1$.

Sometimes it is more natural, when comparing x_i to x_j, that the decision maker (DM) assigns nonnegative values b_{ij} to x_i and b_{ji} to x_j, such that $b_{ij} + b_{ji} = 1$. With this interpretation, the preferences on X can be understood as a fuzzy preference relation, with membership function $\mu_R : X\times X \to [0;1]$, where $\mu_R(x_i, x_j) = b_{ij}$ denotes the preference of the alternative x_i over x_j. The most important properties of the above mentioned matrix $\boldsymbol{B} =\{b_{ij}\}$, called here the *fuzzy preference matrix*, can be summarized as follows.

An $n\times n$ matrix $\boldsymbol{B} =\{b_{ij}\}$ with $0 \leq b_{ij} \leq 1$ for all i and j is *additive-reciprocal* (*a-reciprocal*), if:

$$b_{ij} + b_{ji} = 1 \text{ for all } i,j\in\{1,2,\ldots,n\}. \tag{3}$$

Evidently, if (3) holds, then $b_{ii} = 0.5$ for all $i\in\{1,2,\ldots,n\}$.

To make a coherent choice of evaluations b_{ij} (when assuming fuzzy preference matrix $\boldsymbol{B} =\{b_{ij}\}$), a set of properties to be satisfied by such relations has been suggested in the literature, the terminology of properties of relations is, however, not established yet, compare e.g. Alonso et al. (2008); Fodor, Roubens (1994); Tanino (1984). Here, we use the usual terminology which is as close as possible to the one used in the literature.

Transitivity is one of the most important properties of preferences, and it represents the idea that the preference intensity obtained by comparing two alternatives directly should be equal to or greater than the preference intensity between those two alternatives obtained using an indirect chain of alternatives.

Let $\boldsymbol{B} =\{b_{ij}\}$ be an $n\times n$ a-reciprocal matrix with $0 \leq b_{ij} \leq 1$ for all i and j.

We say that $\mathbf{B} =\{b_{ij}\}$ is multiplicative-transitive (m-transitive), if:

$$\frac{b_{ij}}{b_{ji}} = \frac{b_{ik}}{b_{ki}} \frac{b_{kj}}{b_{jk}} \qquad \text{for all } i,j,k\in\{1,2,\ldots,n\}. \tag{4}$$

Note that if $\boldsymbol{B}$ is m-consistent then $\boldsymbol{B}$ is m-transitive. Moreover, if $\boldsymbol{B}$ is m-reciprocal, then $\boldsymbol{B}$ is m-transitive iff $\boldsymbol{B}$ is m-consistent.

We say that $\mathbf{B} =\{b_{ij}\}$ is additive-transitive (a-transitive), if:

$$(b_{ij} - 0{,}5) = (b_{ik} - 0{,}5) + (b_{kj} - 0{,}5) \text{ for all } i,j,k\in\{1,2,\ldots,n\}. \tag{5}$$

This property is also called additive consistency; here, we reserve, however, this name for a different notion, see below.

Now, we shall investigate some relationships between a-reciprocal and m-reciprocal pairwise comparison matrices. We start with an extension of the result published in Herrera-Viedma et al. (2004). For this purpose, given $\sigma > 1$, we define the following function φ_σ as:

$$\varphi_\sigma(t) = \sigma^{2t-1} \text{ for } t \in [0;1]. \tag{6}$$

We obtain the following result, characterizing a-transitive and m-consistent matrices; for the proof see Ramik, Vlach (2013).

Proposition 1. Let $\sigma > 1$, $\boldsymbol{B} = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all i, $j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is a-transitive then $\boldsymbol{A} = \{\varphi_\sigma(b_{ij})\}$ is m-consistent.

Now, let us define the function Φ as follows:

$$\Phi(t) = \frac{t}{1-t} \text{ for } 0 < t < 1. \tag{7}$$

We obtain the following result (see Ramik, Vlach, 2013).

Proposition 2. Let $\boldsymbol{B} = \{b_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all $i,j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is m-transitive then $\boldsymbol{A} = \{a_{ij}\} = \{\Phi(b_{ij})\}$ is m- consistent.

From Proposition 2 it is clear that the notion of m-transitivity plays a similar role for a-reciprocal fuzzy preference matrices as the notion of m-consistency does for m-reciprocal matrices. That is why it is reasonable to introduce the following *definition*:

Any a-reciprocal m-transitive $n \times n$ matrix $\boldsymbol{B} = \{b_{ij}\}$ is called *additively consistent* (*a-consistent*).

According to this definition Proposition 2 can be reformulated as follows:

Proposition 2*. Let $\boldsymbol{B} = \{b_{ij}\}$ be an $n \times n$ matrix with $0 \le b_{ij} \le 1$ for all $i,j \in \{1,2,\ldots,n\}$. If $\boldsymbol{B}$ is a-consistent then $\boldsymbol{A} = \{a_{ij}\} = \{\Phi(b_{ij})\}$ is m-consistent.

By Proposition 1, resp. Proposition 2* we can transform a-transitive, resp. a-consistent matrices into m-consistent ones by an appropriate transformation functions φ_σ, resp. Φ.

In practice, perfect consistency/transitivity is difficult to obtain, particularly when measuring preferences on a set with a large number of alternatives.

3 Inconsistency of pairwise comparison matrices, priority vectors

If for some positive $n \times n$ matrix $\boldsymbol{A} = \{a_{ij}\}$ and for some $i,j,k \in \{1,2,...,n\}$, the multiplicative consistency condition (2) does not hold, then $\boldsymbol{A}$ is said to be *multiplicative-inconsistent* (or *m-inconsistent*). If for some $n \times n$ fuzzy preference matrix $\boldsymbol{B} = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some indices $i,j,k \in \{1,2,...,n\}$, (4) does not hold, then $\boldsymbol{B}$ is said to be *additive-inconsistent* (or, *a-inconsistent*). Finally, if for some $n \times n$ fuzzy matrix $\boldsymbol{B} = \{b_{ij}\}$ with $0 \le b_{ij} \le 1$ for all i and j, and for some indices $i,j,k \in \{1,2,...,n\}$, (5) does not hold, then $\boldsymbol{B}$ is said to be *additive-intransitive* (*a-intransitive*). In order to measure the degree of inconsistency/intransitivity of a given matrix several measurement methods have been proposed in the literature (see e.g. Alonso et al., 2008). In AHP, multiplicative reciprocal matrices have been considered (Saaty, 1991).

As far as additive-reciprocal matrices are concerned, some methods for measuring a-inconsistency/a-intransitivity are proposed here. We start, however, with measuring the inconsistency of positive matrices which is based on Perron-Frobenius theory (see e.g. Fiedler, Nedoma, Ramík, Rohn, 2006). Later on, we shall deal with measuring a-inconsistent and a-intransitive matrices.

The Perron-Frobenius theorem describes some of the remarkable properties enjoyed by the eigenvalues and eigenvectors of irreducible nonnegative matrices (e.g. positive matrices).

Theorem (Perron-Frobenius). Let $\boldsymbol{A}$ be an irreducible nonnegative $n\times n$ matrix. Then the spectral radius, $\rho(\boldsymbol{A})$, is a positive (real) eigenvalue, with a positive (real) eigenvector $\boldsymbol{w}$ such that $\boldsymbol{Aw}= \rho(\boldsymbol{A})\boldsymbol{w}$.

In the decision making context the above mentioned eigenvalue $\rho(\boldsymbol{A})$ is called the *principal eigenvalue of* $\boldsymbol{A}$. It is a simple eigenvalue (i.e. it is not a multiple root of the characteristic equation), and its eigenvector, called the *priority vector*, is unique up to a multiplicative constant.

Now, let $\boldsymbol{A}$ be a nonnegative m-reciprocal $n\times n$ matrix. The m-consistency of $\boldsymbol{A}$ is characterized by the *m-consistency index* $I_{mc}(\boldsymbol{A})$ defined in (Saaty, 1980) as:

$$I_{mc}(\boldsymbol{A}) = \frac{\rho(\boldsymbol{A}) - n}{n-1}, \tag{8}$$

where $\rho(\boldsymbol{A})$ is the spectral radius of $\boldsymbol{A}$ (in particular, the principal eigenvalue of $\boldsymbol{A}$).

Moreover, we suppose that $\boldsymbol{A}=\{a_{ij}\}$ is a pairwise comparison matrix with elements a_{ij} based on evaluation of alternatives x_i and x_j, for all i and j. For the purpose of decision making, the rank of the alternatives in $X=\{x_1, x_2,...,x_n\}$ is determined by the vector of weights $\boldsymbol{w} = (w_1,w_2,...,w_n)$, where $w_i > 0$, for all $i\in\{1,2,...,n\}$, such that $\sum_{i=1}^{n} w_i = 1$, satisfying the characteristic equation $\boldsymbol{Aw} = \rho(\boldsymbol{A})\boldsymbol{w}$. This vector $\boldsymbol{w}$ is the (normalized) priority vector of $\boldsymbol{A}$. Since the element of the priority vector w_i is interpreted as the relative importance of the alternative x_i, the alternatives $x_1, x_2,...,x_n$ in X are ranked by their relative importance. The following result has been derived in Saaty (1980).

Proposition 3. If $\boldsymbol{A} =\{a_{ij}\}$ is an $n\times n$ positive m-reciprocal matrix, then $I_{mc}(\boldsymbol{A}) \geq 0$. Moreover, $\boldsymbol{A}$ is m-consistent if and only if $I_{mc}(\boldsymbol{A}) = 0$.

To provide a consistency measure independently of the dimension n of the matrix $\boldsymbol{A}$, T. Saaty (1980) proposed the consistency ratio. To distinguish it here from the other consistency measures, we shall call it *m-consistency ratio*. This is obtained by taking the ratio of I_{mc} to its mean value R_{mc}, estimated by an arithmetic average over a large number of positive m-reciprocal matrices of dimension n, whose entries are randomly and uniformly generated (see Saaty, 1980), i.e.:

$$CR_{mc} = I_{mc}/R_{mc}. \tag{9}$$

It was proposed that a pairwise comparison matrix could be accepted (in a DM process) if its m-consistency ratio does not exceed 0.1 (see Saaty, T.L., 1980). The m-consistency index I_{mc} has been defined for m-reciprocal matrices; here, we investigate the inconsistency/intransitivity of a-reciprocal matrices. For this purpose we use relations between m-consistent and a-transitive/a-consistent matrices derived in Propositions 1 and 2*.

Let $\boldsymbol{B} = \{b_{ij}\}$ be an m-reciprocal matrix with $0 \le b_{ij} \le 1$ for all i and j. We define the a-consistency index $I_{ac}(\boldsymbol{B})$ of $\boldsymbol{B} = \{b_{ij}\}$ as:

$$I_{ac}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}), \text{ where } \boldsymbol{A} = \{\, \Phi(b_{ij}) \,\}. \tag{10}$$

From (10) we easily obtain the following result, which is parallel to Proposition 3.

Proposition 4. If $\boldsymbol{B} = \{b_{ij}\}$ is an a-reciprocal $n\times n$ fuzzy matrix with $0 \le b_{ij} \le 1$ for all i and j, then $I_{ac}(\boldsymbol{B}) \ge 0$. Moreover, $\boldsymbol{B}$ is a-consistent if and only if $I_{ac}(\boldsymbol{B}) = 0$.

Now, we shall deal with measuring a-intransitivity of a-reciprocal matrices. Let $\sigma > 1$ be a given value characterizing the scale. Let $\boldsymbol{B} = \{b_{ij}\}$ be an a-reciprocal $n\times n$ fuzzy matrix with $0 \le b_{ij} \le 1$ for all i and j. We define the *a-transitivity index* $I^{\sigma}_{at}(\boldsymbol{B})$ of $\boldsymbol{B} = \{b_{ij}\}$ as:

$$I^{\sigma}_{at}(\boldsymbol{B}) = I_{mc}(\boldsymbol{A}^{\sigma}), \tag{11}$$

where:

$$\boldsymbol{A}^{\sigma} = \{\, \varphi_{\sigma}(b_{ij}) \,\}. \tag{12}$$

By applying (8) and (12) we obtain the following results corresponding to Propositions 3 and 4.

Proposition 5. If $\boldsymbol{B} = \{b_{ij}\}$ is an a-reciprocal $n\times n$ matrix with $0 \le b_{ij} \le 1$ for all i and j, then $I^{\sigma}_{at}(\boldsymbol{B}) \ge 0$. Moreover, $\boldsymbol{B}$ is a-transitive if and only if $I^{\sigma}_{at}(\boldsymbol{B}) = 0$.

Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n\times n$ matrix. In (12), the m-consistency ratio of $\boldsymbol{A}$ denoted by $CR_{mc}(\boldsymbol{A})$ is obtained by taking the ratio of $I_{mc}(\boldsymbol{A})$ to its mean value $R_{mc}(n)$, i.e.:

$CR_{mc}(\boldsymbol{A}) = I_{mc}(\boldsymbol{A})/R_{mc}(n)$.

The values of $R_{mc}(n)$ for n=3,4,…, can be found in Saaty (1980). Similarly, we define the a-consistency ratio $CR_{ac}(\boldsymbol{A})$ and the a-transitivity ratio $CR^{\sigma}_{at}(\boldsymbol{A})$.

Denote $\Phi(B) = \{\Phi(b_{ij})\}$, then the corresponding priority vector $\boldsymbol{w}^{ac}$ is given by the characteristic equation $\Phi(B)\,\boldsymbol{w}^{ac} = \rho(\Phi(B))\boldsymbol{w}^{ac}$.

Given $\sigma > 1$, let us denote $\varphi_{\sigma}(B) = \{\varphi_{\sigma}(b_{ij})\}$, then the priority vector $\boldsymbol{w}^{at}$ is defined by the characteristic equation $\varphi_{\sigma}(B)\,\boldsymbol{w}^{at} = \rho(\varphi_{\sigma}(B))\boldsymbol{w}^{at}$.

In practice, a-inconsistency of a positive a-reciprocal fuzzy priority matrix $\boldsymbol{B}$ is “acceptable” if $CR_{ac}(\boldsymbol{B}) < 0.1$. Also, a-intransitivity of a positive a-reciprocal pairwise comparison matrix $\boldsymbol{B}$ is "acceptable" if $CR_{at}^{\sigma}(\boldsymbol{B}) < 0.1$. The final ranking of alternatives is given by the corresponding priority vector, see Example 1.

The following two results give a characterization of a m-consistent matrix as well as an a-consistent matrix by vectors of weights, i.e. positive vectors with the sum of their elements equal to one. The proofs are straightforward and can be found in Ramik, Vlach (2013).

Proposition 6. Let $\boldsymbol{A} = \{a_{ij}\}$ be a positive $n \times n$ matrix. $\boldsymbol{A}$ is m-consistent if and only if there exists a vector $\boldsymbol{w} = (w_1, w_2, ..., w_n)$ with $w_i > 0$ for all $i \in \{1,2,...,n\}$, and $\sum_{i=1}^{n} w_i = 1$ such that:

$$a_{ij} = \frac{w_i}{w_j} \quad \text{for all i,j} \in \{1,2,...,n\}. \tag{13}$$

Proposition 7. Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < a_{ij} < 1$ for all $i,j \in \{1,2,...,n\}$. $\boldsymbol{A}$ is a-consistent if and only if there exists a vector $\boldsymbol{v} = (v_1, v_2, ..., v_n)$ with $v_i > 0$ for all $i \in \{1,2,...,n\}$, and $\sum_{i=1}^{n} v_i = 1$ such that:

$$a_{ij} = \frac{v_i}{v_i + v_j} \quad \text{for all i,j} \in \{1,2,...,n\}. \tag{14}$$

An associated result can be derived also for a-transitive matrices.

Proposition 8. Let $\boldsymbol{A} = \{a_{ij}\}$ be an a-reciprocal $n \times n$ matrix with $0 < a_{ij} < 1$ for all $i,j \in \{1,2,...,n\}$. $\boldsymbol{A}$ is a-transitive if and only if there exists a vector $\boldsymbol{u} = (u_1, u_2, ..., u_n)$ with $u_i > 0$ for all $i \in \{1,2,...,n\}$, and $\sum_{i=1}^{n} u_i = 1$ such that:

$$a_{ij} = \tfrac{1}{2}(1 + nu_i - nu_j) \quad \text{for all } i,j \in \{1,2,...,n\}. \tag{15}$$

The proof of this proposition is based on the observation that for a-transitive matrix $\boldsymbol{A} = \{a_{ij}\}$ we have:

Setting for all $i \in \{1,2,...,n\}$, we obtain the required result.

Example 1

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of 4 alternatives. The preferences on X are described by a positive matrix $\boldsymbol{B} = \{b_{ij}\}$:

$$\boldsymbol{B} = \begin{pmatrix} 0.5 & 0.6 & 0.6 & 0.9 \\ 0.4 & 0.5 & 0.6 & 0.7 \\ 0.4 & 0.4 & 0.5 & 0.5 \\ 0.1 & 0.3 & 0.5 & 0.5 \end{pmatrix}. \tag{16}$$

Here, $\boldsymbol{B}$ is a-reciprocal and a-inconsistent, which may be directly verified by (7), e.g. $b_{12}.b_{23}.b_{31} \neq b_{21}.b_{32}.b_{13}$. At the same time, $\boldsymbol{B}$ is a-intransitive as $b_{12}+b_{23}+b_{31} = 1.9 \neq 1.5$. We consider $\sigma = 9$ and calculate:

$$\boldsymbol{E} = \Phi(B) = \begin{pmatrix} 1 & 1.50 & 1.50 & 9.0 \\ 0.67 & 1 & 1.50 & 2.33 \\ 0.67 & 0.67 & 1 & 1 \\ 0.11 & 0.43 & 1 & 1 \end{pmatrix},$$

$$\boldsymbol{F} = \varphi_{\sigma}(B) = \begin{pmatrix} 1 & 1.55 & 1.55 & 5.80 \\ 0.64 & 1 & 1.55 & 2.41 \\ 0.64 & 0.64 & 1 & 1 \\ 0.17 & 0.42 & 1 & 1 \end{pmatrix}.$$

We calculate $\rho(\boldsymbol{E}) = 4.29$, $\rho(\boldsymbol{F}) = 4.15$, then we obtain $CR_{ac}(\boldsymbol{B}) = 0.11 > 0.1$ with the priority vector $\boldsymbol{w}^{ac} = (0.47; 0.25; 0.18; 0.10)$, which gives the ranking of alternatives: $x_1 > x_2 > x_3 > x_4$. Similarly, $CR_{at}^{9}(\boldsymbol{B}) = 0.056 < 0.1$ with the priority vector $\boldsymbol{w}^{at} = (0.44;0.27;0.18;0.12)$, with the same ranking of alternatives: $x_1 > x_2 > x_3 > x_4$.

As it is evident, a-consistency ratio $CR_{ac}(\boldsymbol{B})$ is too high for the matrix $\boldsymbol{B}$ to be considered a-consistent. On the other hand, a-transitivity ratio $CR_{at}^{9}(\boldsymbol{B})$ is sufficiently low for the matrix $\boldsymbol{B}$ to be considered a-transitive. The ranking of the alternatives given by both methods remains, however, the same.

In this example we can see that the values of consistency ratio and transitivity ratio can be different for an a-reciprocal matrix. In order to investigate a possible relationship between the inconsistency/in-transitivity indices, we performed a simulation experiment with randomly generated 1000 a-reciprocal matrices, (n=4 and n=15). Then we calculated the corresponding consistency and transitivity indexes. Numerical experiments show that there is no strong relationship between a-consistency and a-transitivity.

4 Fuzzy preference matrix with missing elements

In many decision-making procedures we assume that experts are capable of providing preference degrees between any pair of possible alternatives. However, this may not always be true, which creates a missing information problem. A missing value in a fuzzy preference matrix is not equivalent to a lack of preference of one alternative over another. A missing value can be the result of the incapacity of an expert to quantify the degree of preference of one alternative over another. In this case he/she may decide not to guess the preference degree between some pairs of alternatives. It must be clear that when an expert is not able to express a particular value a_{ij}, because he/she does not have a clear idea of how the alternative x_i is better than the alternative x_j, this does not mean that he/she prefers both options with the same intensity. The DM could be also bored by evaluating too many pairs of alternatives. To model these situations, in the following we introduce the incomplete preference relation matrix. Here, we use a different approach and different notation than Alonso (2008).

Let $n > 2$, $I = \{1,2,...,n\}$ be a set of indices, $I^2 = I \times I$ the Cartesian product of I, i.e. $I^2 = \{(i,j) | i,j \in I\}$. Here, we assume that the reciprocity condition is satisfied. Therefore, we shall consider only a-reciprocal fuzzy preference matrices.

Let $L \subseteq I^2$, $L = \{(i_1,j_1),(i_2,j_2),...,(i_q,j_q)\}$ be the set of pairs (i,j) of indices such that there exists a pairwise comparison value a_{ij}, $0 \le a_{ij} \le 1$. By L' we denote the symmetric subset to L, i.e. $L' = \{(j_1,i_1),(j_2,i_2),...,(j_q,i_q)\}$. By reciprocity, each subset $K \subseteq I^2$ of the given elements can be expressed as follows

$$K = L \cup L' \cup D, \tag{17}$$

where L is a set of pairs of indices (i,j) of the evaluated elements a_{ij} and D is the diagonal of the fuzzy preference matrix, $D = \{(1,1),(2,2),...,(n,n)\}$, here a_{ii} = 0.5 for all i. The elements a_{ij} with $(i,j) \in I^2 - K$ are called *missing elements*.

Now we define the fuzzy preference matrix $\mathbf{B}(\mathrm{K}) = \{\mathrm{b}_{\mathrm{ij}}\}_{\mathrm{K}}$ with missing elements by:

$$b_{ij} = \begin{cases} a_{ij} & \text{if} \quad (i,j) \in K, \\ - & \text{if} \quad (i,j) \notin K. \end{cases}$$

Here, the missing elements of the matrix $\boldsymbol{B}(K)$ are denoted by a dash "-". On the other hand, the elements evaluated by the experts are denoted by a_{ij} where $(i,j) \in K$. By a-reciprocity, it is sufficient that the expert quantifies only elements a_{ij} where $(i,j) \in L$, such that $K = L \cup L' \cup D$; the other elements are calculated automatically by (3). In what follows we shall investigate a particular important case of L, namely, $L = \{(1,2);(2,3);...,(n-1,n)\}$.

5 Extension of fuzzy preference matrix with missing elements and its consistency/transitivity

In this section we shall deal with the problem of finding the values of missing elements of a given fuzzy preference matrix so that the extended matrix is as much a-consistent/a-transitive as possible. In the ideal case the extended matrix will become a-consistent/a-transitive. We start with the a-consistency property.

Let $K \subseteq I^2$, let $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. The matrix $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ called an *ac-extension of* $\boldsymbol{B}(K)$ is defined as follows:

$$b_{ij}^{ac} = \begin{cases} b_{ij} & \text{if} \quad (i,j) \in K, \\ \dfrac{v_i^*}{v_i^* + v_j^*} & \text{if} \quad (i,j) \notin K. \end{cases} \tag{18}$$

Here, $v^* = (v_1^*, v_2^*, ..., v_n^*)$, called the *ac-priority vector with respect to K*, is the optimal solution of the following optimization problem:

$$(\mathbf{P_{ac}}) \qquad d_{ac}(\boldsymbol{v},K) = \sum_{(i,j)\in K} (b_{ij} - \frac{v_i}{v_i + v_j})^2 \to \min; \tag{19}$$

subject to:

$\sum_{j=1}^{n} v_j = 1$, $v_i \geq \varepsilon > 0$ for all i=1,2,...,n

($\varepsilon > 0$ is a given sufficiently small number).

Note that the a-consistency index of the matrix $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ is defined by (15) as $I_{ac}(\boldsymbol{B}^{ac}(K))$. The following proposition follows directly from Proposition 7.

Proposition 9. $\boldsymbol{B}^{ac}(K) = \{ b_{ij}^{ac} \}_K$ is a-consistent, i.e. $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$ if and only if $d_{ac}(\boldsymbol{v},K) = 0$.

Now, we look for the values of missing elements of a given fuzzy preference matrix so that the extended matrix is as much a-transitive as possible. In the ideal case the extended matrix will become a-transitive.

Again, let $K \subseteq I^2$, $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements.

The matrix $\mathbf{B}^{\mathbf{at}}(\mathrm{K}) = \{ b_{ij}^{at} \}_{\mathrm{K}}$ called an at-extension of $\mathbf{B}(\mathrm{K})$ is defined as follows:

$$b_{ij}^{at} = \begin{cases} b_{ij} & \text{if} \quad (i,j) \in K, \\ \max\{0, \min\{1, \frac{1}{2}(1+nu_i^*-nu_j^*)\}\} & \text{if} \quad (i,j) \notin K. \end{cases} \tag{20}$$

Here, $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ called the *at-priority vector with respect to K* is the optimal solution of the following optimization problem:

$$(\mathbf{P_{at}}) \qquad d_{at}(\boldsymbol{u},K) = \sum_{(i,j)\in K} (b_{ij} - \tfrac{1}{2}(1 + nu_i - nu_j))^2 \to \min; \tag{21}$$

subject to:

$\sum_{j=1}^{n} u_j = 1$, $u_i \geq \varepsilon > 0$ for all i=1,2,...,n

($\varepsilon > 0$ is a given sufficiently small number).

In general, the optimal solution $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of (P_{at}) does not satisfy the following condition:

$$0 \le \tfrac{1}{2}(1 + nu_i^* - nu_j^*) \le 1, \tag{22}$$

i.e. $\boldsymbol{B} = \{b_{ij}\} = \{\tfrac{1}{2}(1 + nu_i - nu_j)\}$ fails to be a fuzzy preference matrix. That is why in the definition of the at-extension of $\boldsymbol{B}(K)$ we use formula (20) ensuring that all the elements b_{ij} belong to the unit interval [0;1]. In the next section we shall derive the necessary and sufficient conditions for (22) to be satisfied.

Note that the a-transitivity index (for a given $\sigma > 1$) of the matrix $\boldsymbol{B}^{at}(K) = \{ b_{ij}^{at} \}_K$ is defined by (11) as $I_{at}^{\sigma}(\boldsymbol{B}^{at}(K))$. The next proposition follows directly from Proposition 8.

Proposition 10. Let $\sigma > 1$. If $\boldsymbol{B}^{at}(K) = \{ b_{ij}^{at} \}_K$ is a-transitive, i.e. $I_{at}^{\sigma}(\boldsymbol{B}^{ac}(K)) = 0$, then $d_{at}(\boldsymbol{v},K) = 0$.

6 A particular case of fuzzy preference matrix with missing elements

For a complete definition of a reciprocal fuzzy preference $n \times n$ matrix we need $N = \dfrac{n(n-1)}{2}$ pairs of elements to be evaluated by an expert. For example, if n=10, then N=45, which is a considerable number of pairwise comparisons. In practice we ask that the expert evaluates only around n pairwise comparisons of alternatives which seems a reasonable number. In this section we shall deal with an important particular case of fuzzy preference matrix with missing elements where the expert should evaluate only n-1 pairwise comparisons of elements.

Let $K \subseteq I^2$ be a set of indexes given by an expert, $\boldsymbol{B}(K) = \{b_{ij}\}_K$ be a fuzzy preference matrix with missing elements. Moreover, let $K = L \cup L' \cup D$. In fact, it is sufficient that the expert evaluates matrix elements only from L.

Here, we assume that the expert evaluates the following n-1 elements of the fuzzy preference matrix $\boldsymbol{B}(K)$: $b_{12}, b_{23}, \ldots, b_{n-1,n}$.

First, we investigate the ac-extension of $\boldsymbol{B}(K)$. We obtain the following result.

Proposition 11. Let $L=\{(1,2);(2,3);\ldots,(n-1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$, and $L' = \{(2,1);(3,2);\ldots,(n, n-1)\}$, $D = \{(1,1),\ldots,(n,n)\}$. Then the ac-priority vector $v^* = (v_1^*, v_2^*, \ldots, v_n^*)$ with respect to K is given as:

$$v_1^* = \frac{1}{S}, \tag{23}$$

$$v_{i+1}^* = a_{i,i+1} v_i^*, \text{ for } i=1,2,..,n-1, \tag{24}$$

where:

$$a_{ij} = \frac{1-b_{ij}}{b_{ij}} \text{ for all } (i,j) \in L \text{ and:} \tag{25}$$

$$S = 1 + \sum_{i=1}^{n-1} a_{i,i+1} a_{i+1,i+2} \cdot \ldots \cdot a_{n-1,n} \,. \tag{26}$$

Remark. The proof of Proposition 11 is straightforward by using (25), (26) and the optimal solution of (19). By Proposition 7 it follows that the ac-extension of $\boldsymbol{B}(K)$, i.e. the matrix $\boldsymbol{B}^{ac}(K)=\{ b_{ij}^{ac} \}_K$ is a-consistent.

Now, we investigate the at-extension $\boldsymbol{B}^{at}(K)$ of $\boldsymbol{B}(K)$. We obtain the following result.

Proposition 12. Let $L=\{(1,2);(2,3);\ldots,(n-1,n)\}$, $0 < b_{ij} < 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Then the at-priority vector $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ with respect to K is given as:

$$u_i^* = \frac{2}{n^2} \sum_{j=1}^{n-1} \alpha_j - \frac{2}{n} \alpha_{i-1} - \frac{n-i-1}{n} \quad \text{for } i=1,2,\ldots,n, \tag{27}$$

where:

$$\alpha_0 = 0,\ \alpha_j = \sum_{i=1}^{j} b_{i,i+1} \quad \text{for } j=1,2,\ldots,n-1. \tag{28}$$

Remark. The proof of Proposition 12 is straightforward by using (27), (28) and the optimal solution of (21). In general, the optimal solution $\boldsymbol{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of (P_{at}) does not satisfy the condition:

$$0 \le \tfrac{1}{2}(1 + nu_i^* - nu_j^*) \le 1, \text{ for all } i,j=1,2,\ldots,n, \tag{29}$$

i.e. $\boldsymbol{B} = \{b_{ij}\} = \{\tfrac{1}{2}(1 + nu_i - nu_j)\}$ is not a fuzzy preference matrix. We can easily prove the necessary and sufficient condition for satisfying (29) based on evaluations $b_{i,i+1}$.

Proposition 13. Let $L=\{(1,2);(2,3);\ldots,(n-1,n)\}$, $0 \le b_{ij} \le 1$ with $b_{ij} + b_{ji} = 1$ for all $(i,j) \in L$, let $K = L \cup L' \cup D$. Then the at-extension $\boldsymbol{B}^{at}(K)=\{ b_{ij}^{at} \}_K$ is a-transitive if and only if:

$$\left| \sum_{k=i}^{j-1} b_{k,k+1} - \frac{j-i}{2} \right| \le \frac{1}{2} \text{ for } i=1,2,\ldots,n-1,\ j=i+1,\ldots,n. \tag{30}$$

The proof of Proposition 13 follows directly from Proposition 8 and Proposition 12.

Example 2

Let L={(1,2);(2,3);(3,4)}, let expert evaluations be b_{12}=0.9, b_{23}=0.8, b_{34}=0.6, with $b_{ij}+b_{ji}$=1 for all $(i,j) \in L$, $K = L \cup L' \cup D$. Then $\boldsymbol{B}(K) = \{b_{ij}\}_K$ is a fuzzy preference matrix with missing elements as follows:

$$\boldsymbol{B}(K) = \begin{pmatrix} 0.5 & 0.9 & - & - \\ 0.1 & 0.5 & 0.8 & - \\ - & 0.2 & 0.5 & 0.6 \\ - & - & 0.4 & 0.5 \end{pmatrix}.$$

Solving (P_{ac}) we obtain the ac-priority vector $\boldsymbol{v}$* with respect to K, in particular, $v^* = (0{,}864; 0{,}096; 0{,}024; 0{,}016)$. By (20) we obtain $\boldsymbol{B}^{ac}(K)$ - the ac-extension of $\boldsymbol{B}(K)$ as follows:

$$\boldsymbol{B}^{ac}(K) = \begin{pmatrix} 0.5 & 0.9 & 0.97 & 0.98 \\ 0.1 & 0.5 & 0.8 & 0.86 \\ 0.03 & 0.2 & 0.5 & 0.6 \\ 0.02 & 0.14 & 0.4 & 0.5 \end{pmatrix}.$$

By Proposition 9, $\boldsymbol{B}^{ac}(K)$ is a-consistent, hence $I_{ac}(\boldsymbol{B}^{ac}(K)) = 0$. Solving ($P_{at}$) we obtain the at-priority vector $\boldsymbol{u}$* with respect to K as follows:

$u^* = (0{,}487; 0{,}287; 0{,}137; 0{,}088)$. By (27) we obtain $\boldsymbol{B}^{at}(K)$ – the at-extension of $\boldsymbol{B}(K)$ as follows:

$$\boldsymbol{B}^{at}(K) = \begin{pmatrix} 0.5 & 0.9 & 1.0 & 1.0 \\ 0.1 & 0.5 & 0.8 & 0.9 \\ 0.0 & 0.2 & 0.5 & 0.6 \\ 0.0 & 0.1 & 0.4 & 0.5 \end{pmatrix},$$

where, by Proposition 10, $\boldsymbol{B}^{at}(K)$ is a-intransitive, as $d_{ac}(\boldsymbol{v},K) > 0$. In particular, $I^9_{at}(\boldsymbol{B}^{ac}(K))$= 0.057.

7 Conclusions

In this paper we have dealt with some properties of fuzzy preference relations, in particular with reciprocity, consistency and transitivity of relations given in the form of square nonnegative matrices. We have shown how to measure the degree of consistency and/or transitivity, and also how to extend crisp comparisons to fuzzy ones, i.e. how to evaluate pairs of elements by fuzzy values. Also, we have proposed a new method for measuring inconsistency based on Saaty´s principal eigenvector method. Moreover, we have dealt with the problem of the incomplete fuzzy preference matrix, where some elements of pairwise comparison are missing. We have proposed a special method for dealing with that case. Some illustrating examples have been presented to clarify the theory proposed.

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