FUNCTIONAL REGRESSION IN SHORT-TERM PREDICTION OF ECONOMIC TIME SERIES

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ABSTRACT

We compare four methods of forecasting functional time series including fully functional regression, functional autoregression FAR(1) model, Hyndman & Shang principal component scores forecasting using one-dimensional time series method, and moving functional median. Our comparison methods involve simulation studies as well as analysis of empirical dataset concerning the Internet users behaviours for two Internet services in 2013. Our studies reveal that Hyndman & Shao predicting method outperforms other methods in the case of stationary functional time series without outliers, and the moving functional median induced by Frainman & Muniz depth for functional data outperforms other methods in the case of smooth departures from stationarity of the time series as well as in the case of functional time series containing outliers.

Key words: functional data analysis, functional time series, prediction.

1. Introduction

A variety of economic phenomena directly leads to functional data: yield curves, income densities, development trajectories, price trajectories, life of a product, and electricity or water consumption within a day (see Kosiorowski et al. 2014). The Functional Data Analysis (FDA) over the last two decades proved its usefulness in the context of decomposition of income densities or yield curves, analyses of huge, sparse economic datasets or analyses of ultra-high frequency financial time series. The FDA enables an effective statistical analysis when the number of variables exceeds the number of observations. Using FDA we can effectively analyse economic data streams, i.e., for example, perform an analysis of non-equally spaced observed time series, which cannot be predicted using, e.g. common moving average or ARIMA framework, by analysing or predicting a whole future trajectory of a stream rather than iteratively predict single observations.

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Using a functional regression where both the predictor as well as the response are functions, we can express relations between complex economic phenomena without dividing them into parts. Recently proposed models for functional time series give us a hope for overcoming the so-called *curse of dimensionality* related to nonparametric analysis of huge economic data sets (see Horvath and Kokoszka, 2012). From other perspective, functional medians defined within the *data depth concept* for functional objects may have useful applications in the context of robust time series analysis – in the case of existence paths of outliers in the data.

The analysis of functional time series (FTS) was considered, among others, in the literature in the contexts of: breast cancer mortality rate modelling and forecasting, call volume forecasting, climate forecasting, demographical modelling and forecasting, electricity demand forecasting, credit card transaction and Eurodollar futures (see Ferraty, 2011 for an overview), yield curves and the Internet users behaviours forecasting (Kosiorowski et al. 2014b), extraction of information from huge economic databases (Kosiorowski et al. 2014a).

The FTS undoubtedly brings up conceptually new areas of economic research and provides new methodology for applications. It is not clear, however, which approaches proposed in the FTS literature up to now are the most promising in the context of FTS prediction. The main aim of this paper is to compare main approaches for FTS prediction using real data set related to day and night Internet users behaviours in 2013. Our paper refers to similar simulation studies of the selected FTS prediction methods presented in Didieriksen et al. (2011) and Besse et al. (2000). Additionally, we considered Hyndeman and Shang (2010) nonparametric FTS prediction and moving Frainman & Muniz functional median forecasting methods.

The rest of the paper is organized as follows. In Section 2 we briefly describe selected approaches for FTS prediction. In Section 3 we compare the approaches using empirical examples. We conclude with Section 4 which discusses advantages and disadvantages of the approaches presented in Section 2.

2. Functional time series prediction

2.1. Preliminaries – functional time series

Functions considered within the FDA are usually elements of a certain separable Hilbert space H with certain inner product $\langle \cdot, \cdot \rangle$ which generates a norm $\|\cdot\|$. A typical example is a space $L^2 = L^2\left([t_0, t_L]\right)$ - a set of measurable real-valued functions x defined on $[t_0, t_L]$ satisfying $\int_{t_0}^{t_L} x^2(t)dt < \infty$. The space L^2 is a separable Hilbert space with an inner product $\langle x, y \rangle = \int x(t)y(t)dt$. We

usually treat the random curve $X = \{X(t), t \in [t_0, t_L]\}$ as a random element of L^2 equipped with the Borel σ algebra. Recently, within a nonparametric FDA, authors have successfully used certain wider functional spaces, i.e. for example, Sobolev spaces (Ferraty and Vieu, 2006).

In order to apply FDA into the economic researches, first we have to transform discrete observations into functional objects using smoothing, kernel methods or orthogonal systems representations. Then we can calculate and interpret functional analogues of basic descriptive measures such as mean, variance and covariance (for details see Ramsay and Silvermann, 2005; Górecki and Krzyśko, 2012).

For the iid observations $X_1, X_2, ..., X_N$ in L^2 with the same distribution as X, which is assumed to be square integrable we can define the following **descriptive characteristics**:

$$\mu(t) = E[X(t)], \text{ mean function,} \tag{1}$$

$$c(t,s) = E\big[(X(t) - \mu(t))(X(s) - \mu(s))\big], \text{ covariancefunction,} \tag{2}$$

$$C = E\big[\langle X - \mu, \cdot \rangle (X - \mu)\big], \text{ covariance operator} \tag{3}$$

and correspondingly their **sample estimators**

$$\hat{\mu}(t) = N^{-1} \sum_{i=1}^{N} X_i(t), \tag{4}$$

$$\hat{c}(t,s) = N^{-1} \sum_{i=1}^{N} (X_i(t) - \hat{\mu}(t))(X_i(s) - \hat{\mu}(t)), \tag{5}$$

$$\hat{C}(x) = N^{-1} \sum_{i=1}^{N} \langle x_i - \hat{\mu}, x \rangle (x_i - \hat{\mu}), \; x \in L^2, \tag{6}$$

It is worth noting that $\hat{C}$ maps L^2 into a finite dimensional subspace spanned by $X_1, X_2, ..., X_N$.

A **functional analogue** of the **principal component analysis** plays a central role in the FTS. For a covariance operator C, the *eigenfunctions* v_j and the *eigenvalues* λ_j are defined by $Cv_j = \lambda_j v_j$, so if v_j is an eigenfunction, then so is av_j – for any nonzero scalar a. The v_j are typically normalized so that $\|v_j\| = 1$.

In a sample case we define the estimated eigenfunctions $\hat{v}_j$ and eigenvalues by

$$\int \hat{c}(t,s)\hat{v}_j(s)ds = \hat{\lambda}_j \hat{v}_j(t), \; j = 1, 2, ..., N, \tag{7}$$

where $\hat{c}(t,s)$ denotes estimated covariance function (see Górecki and Krzyśko, 2012).

Let $y_t(x)$ denote a function, such as monthly income for the continuous age variable x in year t. We assume that there is an underlying smooth function $f_t(x)$ which is observed with an error at discretized grid points of x. **A special case of functional time series $\{y_t(x)\}_{t\in\square}$ is when the continuous variable x is also a time variable**. For example, let $\{Z_w, w \in [1, N]\}$ be a *seasonal time series* which has been observed at N equispaced time points. We divide the observed time series into n trajectories, and then consider each trajectory of length p as a curve rather than p distinct data points. The functional time series is then given by

$$y_t(x) = \{Z_w, w \in (p(t-1), pt]\} , \ t = 1, 2, ..., n . \tag{8}$$

The problem of interest is to forecast $y_{n+h}(x)$, where h denotes forecast horizon.

In the context of FTS prediction, several methods have been considered in the literature up to now. Ramsay and Silverman (2005) and Kokoszka (2007) studied several functional linear models. Theoretical background related to the prediction using functional autoregressive processes can be found in Bosq (2000). Functional kernel prediction was considered in Ferraty and Vieu (2006), Ferraty (2011). An application of a functional principal component regression to FTS prediction can be found in Shang and Hyndeman (2011).

For evaluating prediction quality of main approaches for FTS prediction in the case of our empirical data set related to the Internet users of certain services analysis, we refer to frameworks presented in two finite sample studies: Besse et al. (2000) and Didericksen et al. (2011). Within simulation studies, these authors have studied predictions at time n errors E_n and R_n, $1 < n < N$, defined in the following way:

$$E_n = \sqrt{\int_{t_0}^{t_L} \left(X_n(t) - \hat{X}_n(t) \right)^2 dt} , \tag{9}$$

$$R_n = \int_{t_0}^{t_L} \left| X_n(t) - \hat{X}_n(t) \right| dt , \tag{10}$$

for several N=50, 100, 200, several processes models and innovation processes.

2.2. Prediction using fully functional model

In the simple linear regression we consider observations from the following point of view

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i , i = 1, 2, ..., N , \tag{11}$$

where all random variables Y_i as well the regressors x_i are scalars.

In the case of a *functional linear model*, predictors, responses as well as analogues of the coefficients β_0 and β_1 may be curves and have to be appropriately defined.

The **fully functional** model is defined as

$$Y_i(t) = \int \psi(t,s) X_i(s) ds + \varepsilon_i(t), \ i = 1, 2, ..., N , \tag{12}$$

where responses Y_i are curves and so are regressors X_i.

The fully functional model can alternatively be written as

$$Y(t) = \int X(s) \beta(s,t) ds + \varepsilon(t), \tag{13}$$

where $\beta(s,t) = \psi(t,s)$, $Y(t) = \left[Y_1(y), ..., Y_N(t) \right]^T$, $X(s) = \left[X_1(s), ..., X_N(s) \right]^T$, and $\varepsilon(t) = \left[\varepsilon_1(t), ..., \varepsilon_N(t) \right]^T$.

Suppose $\{ \eta_k, k \geq 1 \}$ and $\{ \theta_l, l \geq 1 \}$ are some bases which need not be orthonormal. Assume that the functions η_k are suitable for expanding the functions X_i and θ_i for expanding the Y_i. For estimating the kernel $\beta(\cdot, \cdot)$, let us consider estimates of the form

$$\beta^*(s,t) = \sum_{k=1}^{K} \sum_{l=1}^{L} b_{kl} \eta_k(s) \theta_l(t), \tag{14}$$

in which K and L are relatively small numbers which are used as smoothing parameters.

We obtain a **least squares estimator** by finding b_{kl} which minimizes the residual sum

$$\sum_{i=1}^{N} \left\| Y_i - \int X_i(s) \beta^*(s, \cdot) \right\|^2 . \tag{15}$$

Derivation of normal equations can be found in Horvath and Kokoszka (2012). Alternative estimators for (14) can be found in Ramsay and Silverman (2005), where authors used large K and L but introduced a roughness penalty on the estimates.

Effective application of the model (12) relates to fulfilling an assumption that the conditional expectation $E[Y(t) | X]$ is a linear function of X. It is worth noting that within the functional regression setup it is possible to perform an analogue of regression diagnostics using **functional residuals** defined as

$$\hat{\varepsilon}_i(t) = Y_i(t) - \int \hat{\psi}(t,s) X_i(s) ds, \ i = 1, 2, ..., N , \tag{16}$$

and calculate an analogue of the **coefficient of determination**

$$R^2(t) = \frac{Var\left[E\left[Y(t)\,|\,X\right]\right]}{Var\left[Y(t)\right]}, \tag{17}$$

note that since $Var\left[E\left[Y(t)\,|\,X\right]\right] \le Var\left[Y(t)\right]$, $0 \le R^2(t) \le 1$. The coefficient $R^2(t)$ quantifies the degree to which the functional linear model explains the variability of the response curves at a fixed point t. For the global measure we can integrate $R^2(t)$.

2.3. Hyndman & Shang FPC regression

Let $\mathbf{f}(\mathbf{x}) = \left[f_1(x), f_2(x), ..., f_n(x)\right]^T$ denote a sample of functional data. Note that at a population level, a stochastic process denoted by f can be decomposed into the mean function and the products of orthogonal functional principal components and uncorrelated principal component scores. It can be expressed as

$$f = \mu + \sum_{k=1}^{\infty} \beta_k \phi_k, \tag{18}$$

where μ is the unobservable population mean function, β_k is the kth principal component score. Assume that we observe n realizations of f evaluated on a compact interval $x \in [t_0, t_L]$, denoted by $f_t(x)$, for $t = 1, 2, ..., n$. At a sample level, the functional **principal component decomposition** can be written as

$$f_t(x) = \overline{f}(x) + \sum_{k=1}^{K} \hat{\beta}_{t,k} \hat{\phi}_k(x) + \hat{\varepsilon}_t(x), \tag{19}$$

where $\overline{f}(x) = n^{-1} \sum_{t=1}^{n} f_t(x)$ is the estimated mean function, $\hat{\phi}_k(x)$ is the kth estimated orthonormal eigenfunction of the empirical covariance operator

$$\hat{C}(x) = n^{-1} \sum_{t=1}^{n} [f_t(x) - \overline{f}(x)][f_t(x) - \overline{f}(x)]. \tag{20}$$

The coefficient $\hat{\beta}_{t,k}$ is the kth principal component score for year t. It is given by the projection of $f_t(x) - \overline{f}(x)$ in the direction of kth eigenfunction $\hat{\phi}_k(x)$, that is,

$$\hat{\beta}_{t,k} = \left\langle f_t(x) - \overline{f}(x), \hat{\phi}_k(x) \right\rangle = \int_x [f_t(x) - \overline{f}(x)] \hat{\phi}_k(x) dx, \tag{21}$$

where $\hat{\varepsilon}_t(x)$ is the residual, and K is the optimal number of components, which can be chosen for example by cross validation.

By conditioning on the set of smoothed functions $\mathbf{f}(\mathbf{x}) = \left[f_1(x), f_2(x), ..., f_n(x) \right]^T$ and the fixed functional principal components $B = \left[\hat{\phi}_1(x), \hat{\phi}_2(x), ..., \hat{\phi}_K(x) \right]^T$, **the Hyndman and Shangh-step-ahead forecast** of $y_{n+h}(x)$ can be obtained as

$$\hat{y}_{n+h|n}(x) = E[y_{n+h}(x) \,|\, \mathbf{f}(\mathbf{x}), \mathbf{B}] = \overline{f}(x) + \sum_{k=1}^{K} \hat{\beta}_{n+h|n,k} \hat{\phi}_k(k), \qquad (22)$$

where $\hat{\beta}_{n+h|n,k}$ denotes **the h-step-ahead forecast of** $\beta_{n+h,k}$ using **univariate time series forecasting** methods (i.e., for example, ARIMA, linear exponential smoothing).

Note: because of orthogonality, the forecast variance can be approximated by the sum of component variances.

2.4. Moving functional median

For one dimensional sample $X^N = \{X_1, X_2, ..., X_N\}$ and empirical cumulative density function (ecdf) $F_N(x) = N^{-1} \sum_{n=1}^{N} I\{X_i \le x\}$ we can define the **halfspace depth** of X_i as

$$HD_N(x_i) = \min\{F_N(x_i), 1 - F_N(x_i)\}. \qquad (23)$$

We can obtain another one-dimensional depth using the following formula

$$D_N(x_i) = 1 - \left|1/2 - F_N(x_i)\right|. \qquad (24)$$

For N functions $\{X_i(t), t \in [t_0, t_L]\}$ and $F_{N,t}(x) = N^{-1} \sum_{n=1}^{N} I\{X_i(t) \le x\}$ we can define a **functional depth** by integrating one of the univariate depth (see Zuo and Serfling, 2000 or Kosiorowski, 2012 for a detailed introduction to the data depth concept).

Frainman and Muniz (2001) proposed to calculate the depth of the curve as

$$FD_N(X_i \,|\, X^n) = \int_{t_0}^{t_L} \left[1 - \left|1/2 - F_{N,t}(X_i(t))\right|\right] dt . \qquad (25)$$

Frainman and Muniz median is defined as

$$MED_{FM}(X^n) = \arg\max_i FD(X_i \,|\, X^n). \qquad (26)$$

We can **predict next observations** by means of the following formula

$$\hat{X}_{n+1}(t) = MED_{FM}(W_{n,k}), \tag{27}$$

where $W_{n,k}$ denotes a **moving window** of length k ending at moment n, i.e., $W_{n,k} = \{X_{n-k+1}(t), ..., X_n(t)\}$.

3. Empirical example

In order to check properties of the selected method of forecasting FTS we considered an empirical example related to behaviours of the Internet users of two services in 2012 and 2013. The services were considered with respect to the number of unique users and number of page views during an hour. Fig. 1 presents raw data for the year 2013. Fig. 2 presents the main idea of obtaining functional time series on the basis of a periodic one-dimensional time series (in the considered series the period equals 24 hours). Fig. 3 – 6 present obtained functional observations for the corresponding number of users in the first service, the number of users in the second service, the numbers of page views in the first service and the number of page views in the second service. Additionally, we added corresponding functional means and Frainman & Muniz functional medians to the Fig. 3 – 6.

We considered a fully functional model, Hyndman and Shang principal component scores forecasting method, Ferraty and View (2006) functional kernel regression, functional autoregressive FAR(1) model described by Horvath and Kokoszka (2012) and estimated by their improved *estimated kernel* method and using moving Frainman and Muniz median. All calculations were conducted using **fda** (Ramsay et al., 2009), **ftsa** (Shang, 2013), **fda.usc** (Febrero-Bande and Oviedo de la Fuente, 2012) and **DepthProc** (Kosiorowski and Zawadzki, 2014). Below we present selected outputs for the methods which performed best within our empirical analysis. In all the situations we used 7–9 spline basis systems for transforming discrete data to the functional objects.
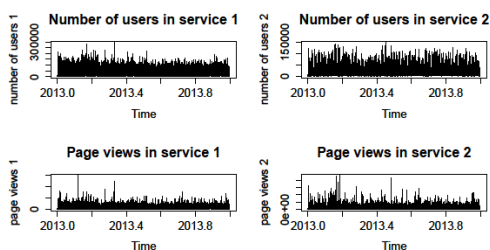

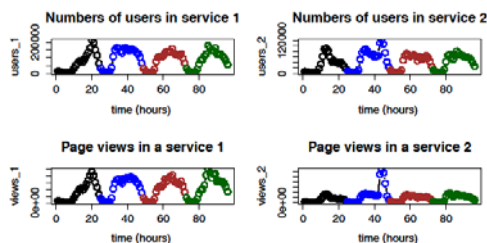**Figure 1.** The behaviour of Internet users of two services in 2013

Figure 2. An idea of transformation of the data from univariate to functional time series
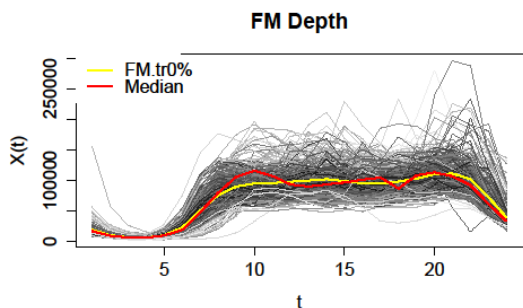
Figure 3. Functional data – number of unique users during 24 hours in service 1
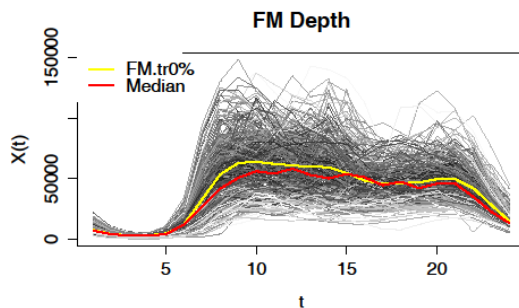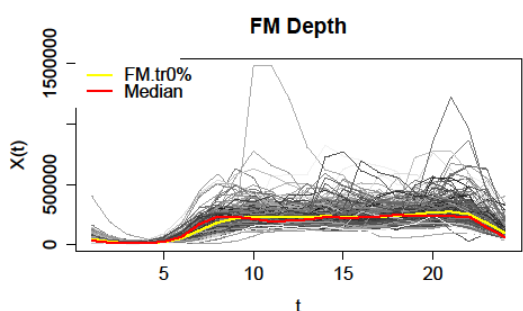

Figure 4. Functional data – number of unique users during 24 hours in service 2

Fig. 7 presents the results of a functional principal component analysis for functional data related to the number of users in the first considered service. We can see there the first two principal component functions and biplots for the observations. It is easy to propose an interpretation according to which the first component relates to using the service at work whereas the second component relates to using the Internet at home. Fig. 7 – 11 present the functional regression method proposed by Hyndman and Shang applied to the corresponding **number of users in the first service, the number of users in the second service, the numbers of page views in the first service** and **the number of page views in the second service**. Each time we used three basis functions (upper panel) and calculated principal component scores (down panel).

Figure 5. Functional data – number of page views during 24 hours in service 1
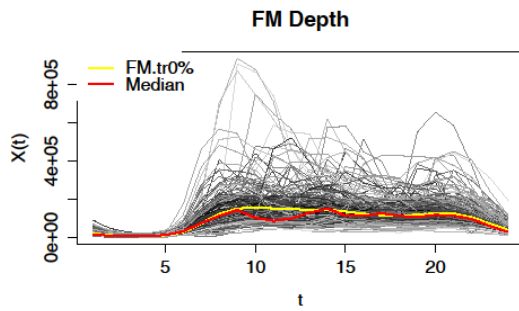
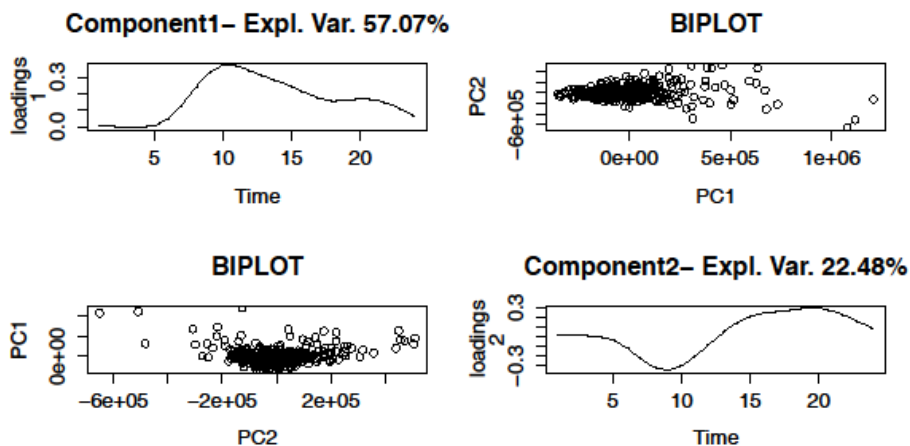**Figure 6.** Functional data – number of page views during 24 hours in service 2

Figure 7. Functional principal components for number of unique users in service 1 in 2013
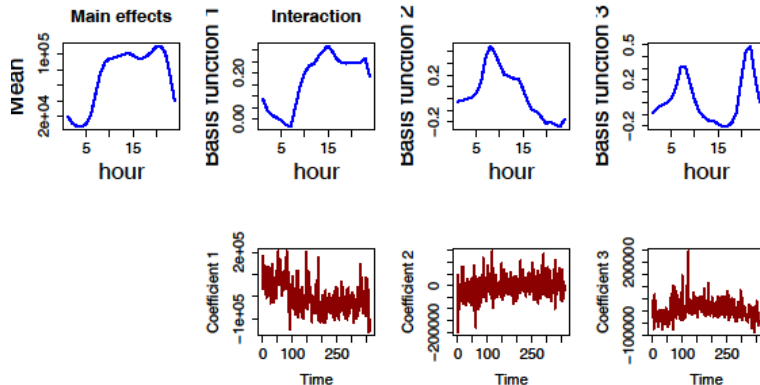

Figure 8. Hyndman & Shang functional PC scores method for number of users in service 1. Three basis function explaining 47%, 18% and 12% variability correspondingly
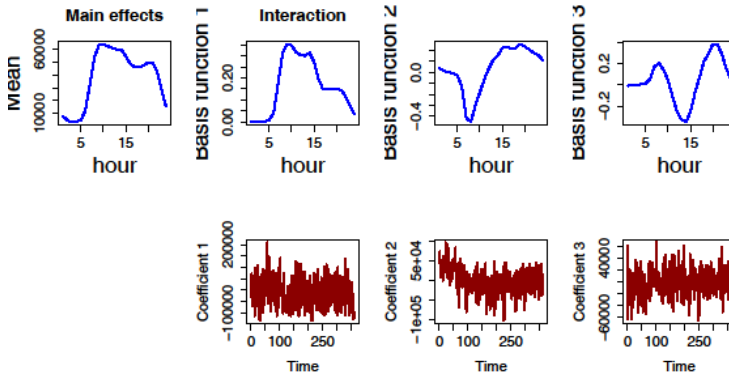
Figure 9. Hyndman & Shang functional PC scores method for number of users in service 2. Three basis function explaining 62%, 15% and 7% variability correspondingly
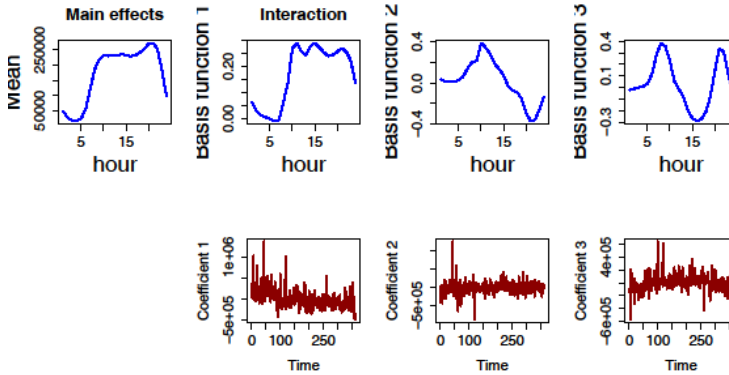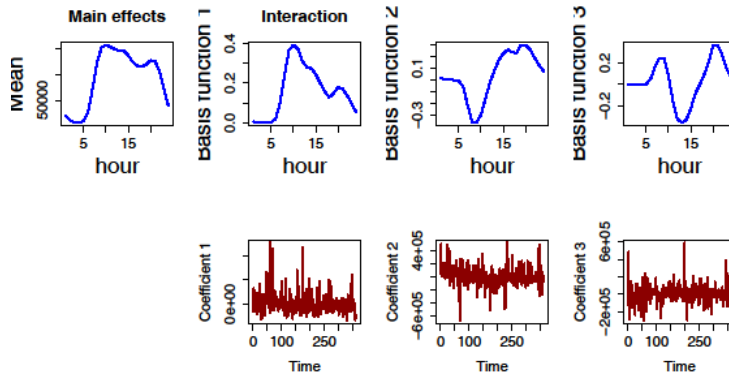

Figure 10. Hyndman & Shang functional PC scores method for number of views in service 1. Three basis function explaining 42%, 20% and 12% variability correspondingly

Figure 11. Hyndman & Shang functional PC scores method for number of views in service 2. Three basis function explaining 50%, 20% and 10% variability correspondingly

Fig. 12 – 13 present predictions for the considered examples using Hyndman and Shao method and ARIMA and linear exponential smoothing (ETS) for one-dimensional time series of principal component scores (see Hyndman et al., 2008). Fig. 14 – 15 present observed and predicted values of the number of users in the service 1 and the number of views in the service 1 using moving Frainman and Muniz median calculated from windows consisting of 50 functional observations. Fig. 16 presents observed and predicted values of the number of users in the service 1 calculated using fully linear regression model. Fig. 17 presents residuals in this regression model and Fig. 18 – 19 present an estimated coefficient function for this regression model.
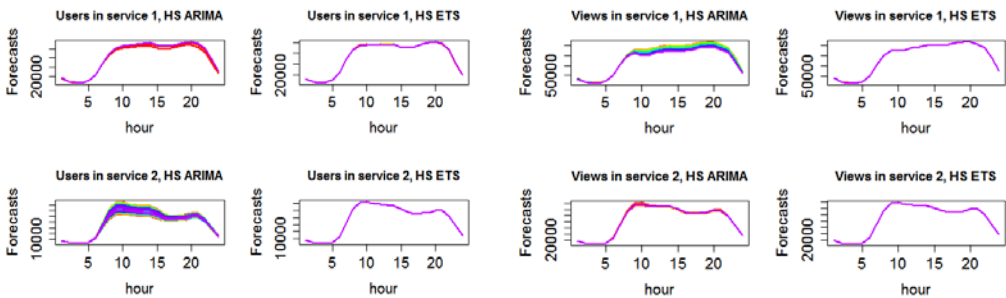

Figure 12. FTS prediction of number of users in the Internet services using Hyndman and Shao FTSA method

Figure 13. FTS prediction of number of page views in the Internet services using Hyndman and Shao FTSA method
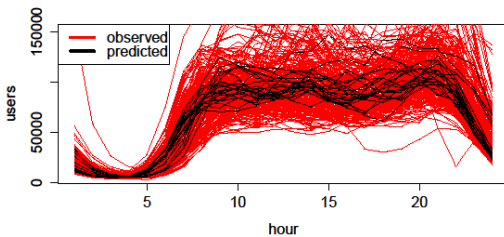

Figure 14. FTS prediction of number of users in the Internet service 1 using moving Frainman & Muniz median
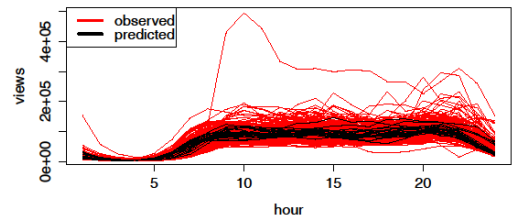
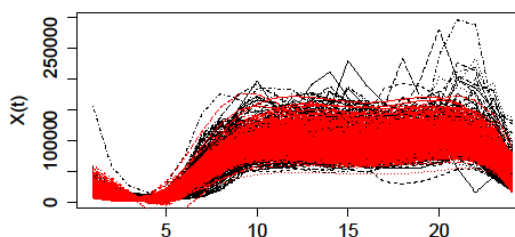**Figure 15.** FTS prediction of number of views in the Internet service 1 using moving Frainman & Muniz median

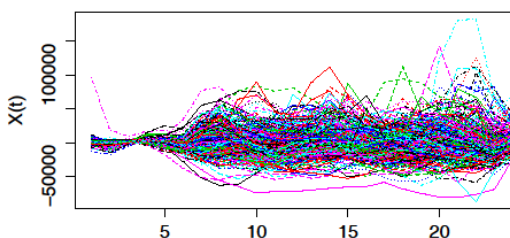**Figure 16.** Prediction of number of users in the Internet service 1 using f**ull regression model**

Figure 17. Prediction of number of users in the Internet service 1 using **full regression model** – functional residuals
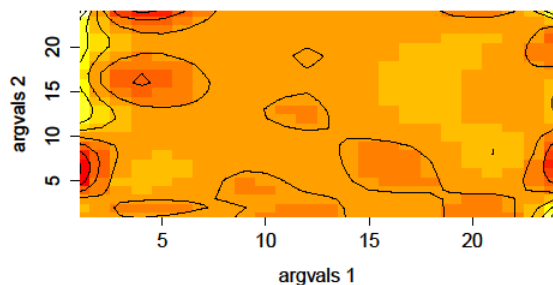

Figure 18. Contour plot: prediction of number of users in the Internet service 1 using full regression model – estimated regression parameters
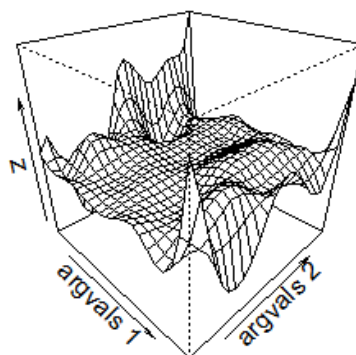

Figure 19. Perspective plot: – prediction of number of users in the Internet service 1 using full regression model – estimated regression parameters

For comparing the methods we divided the data set into two parts of equal sizes. We estimated prediction methods parameters using the first part of the data and tested them using the second part of the data. For testing the methods we used forecast accuracy measures proposed in Didieriksen et al. (2011) defined by formulas (9) and (10). According to our results the Hyndman and Shang method performed best, the moving Frainman and Muniz median performed the second best and the fully linear model was third. Surprisingly, the FAR(1) method as well as the kernel functional regression performed relatively poor in the case of our data set. This finding stays in a contrary to findings of Didieriksen et al. (2011),

where the simulation study was conducted. In the case of our data set, prediction effectiveness of Hyndman and Shang method (100%) in comparison to the moving Frainman and Muniz median and fully linear model was correspondingly as 100% to 91% to 87% in the case of the number of users prediction and as 100% to 99% to 96% in the case of page views prediction. In the case of simulation studies with data simulated from simple nonstationary models (based on models from Didieriksen et al. (2011) for which we changed the mean function and the covariance function) – Frainman and Muniz median performed best.

Additionally, Hyndman and Shang method exhibits the best properties in the context of economic interpretations. The estimated basis functions in a clear way decompose patterns of the Internet behaviour of users. We can easily notice components related to the Internet usage at work as well as the usage at home. The principal component scores time series show importance of the components within the considered period and may be effectively interpreted in a reference to certain political or social events. The eigenvalues corresponding to the eigenfunctions show importance of the particular components for the considered Internet service. We obtained the best predictions using linear exponential smoothing prediction for one-dimensional principal component scores.

In the case of abrupt changes of the data generating mechanism we recommend using moving Frainman and Muniz median which easily adapt the prediction device. It is easy to notice that methods which are based on estimated principal component functions brake down when the covariance operator changes.

Although fully functional model provides complex family of regression diagnostic and goodness of fit measures, its predictive power in the case of our example was below our expectations. Inspection of estimated coefficient function (Fig. 18 – 19) shows relative constant, as to the time arguments t and $s,$ dependency of 24 hour activity of the Internet users.

For all the considered methods, it is possible to calculate the prediction confidence bands. In this context, prediction confidence bands provided by Hyndman and Shang approach based on prediction bands for (uncorrelated) one-dimensional time series prediction seem to be the most informative.

4. Conclusions

The forecasting quality of functional autoregression, fully functional regression and Hyndman & Shang method strongly depend on the stationarity of the underlying functional time series, the choice of a basis system, smoothness of the considered functions, the PCA algorithm used. For the considered empirical example, in the context of prediction as well as explanation of the considered phenomenon Hyndman & Shang method performed best.

The moving Frainman and Muniz functional median performed best in the case of simulated processes containing additive outliers. Conceptually simple, the moving functional median seems to be the most promising in the context of

nonstationary functional time series analysis. The nonstationarity issues relate to our current and future studies.

Acknowledgements

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