PROPORTIONAL REINSURANCE FOR A FRACTIONAL BROWNIAN RISK MODEL

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Abstract: This paper investigates the ruin probabilities for a two-dimensional fractional Brownian risk model with a proportional reinsurance scheme. The author focused on the joint and simultaneous ruin probabilities in a finite-time horizon. The risk processes of both insurance and reinsurance companies are composed of a large number of i.i.d. sub-risk processes, representing independent businesses. The asymptotics were derived as the initial capital tends to infinity.

Keywords: fractional Brownian motion; asymptotics; ruin probability; two-dimensional risk model; proportional reinsurance.

1. Introduction

Consider a two-dimensional risk model with a proportional reinsurance scheme. Suppose that two companies, insurance and reinsurance, share claims in proportions $\sigma_1, \sigma_2 > 0$, where $\sigma_1 + \sigma_2 = 1$, and receive premiums at rates $c_1, c_2 > 0$, respectively. Let R_i denote the risk process of i-th company

$$R_i(t) := a_i + c_i t - \sigma_i X(t), t \geq 0,$$

where $X(t)$ describes the accumulated claims up to time t, $a_i > 0$ is the initial capital and $c_i > 0$ is the premium rate, $i = 1,2$.

In the literature, various processes of accumulated claims are investigated, with particular emphasis on both the Lévy and Gaussian

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processes. The study of Gaussian processes in risk theory was initiated in the fundamental work of Iglehart (1969), where $X(t)$ is a standard Brownian motion and appears as the limit in the so-called diffusion approximation regime. In an important work by Michna (1998), it was argued that the class of *fractional Brownian motions* can serve as a right approximation of the accumulated claims process.

The modern risk theory focuses on the ruin probability in multi-dimensional risk models. The exact distribution of ruin probabilities are known only in a few specific cases in dimension two: the Brownian motion (Kępczyński, 2020) and spectrally one-sided Lévy processes (Avram, Palmowski, and Pistorius, 2008a; Avram, Palmowski, and Pistorius, 2008b; Michna, 2021). This is the motivation to study the asymptotic properties, bounds and Laplace transform of the ruin probability.

Having introduced the risk processes, one can distinguish the following ruin types:

- *Simultaneous ruin* occurs when there exists $t \in [0, T]$ such that both companies are ruined at time t

$$\pi_{sim}^{T}(a_1, a_2) = \mathbb{P}\left(\inf_{t\in[0,T]}(R_1(t), R_2(t)) < (0,0)\right).$$

- *Joint ruin* occurs when both companies are ruined in time interval $[0, T]$, not necessarily at the same moment

$$\pi_{and}^{T}(a_1, a_2) = \mathbb{P}\left(\inf_{s\in[0,T]} R_1(s) < 0 \text{ and } \inf_{t\in[0,T]} R_2(t) < 0\right).$$

- *'At least one' ruin* occurs when at least one insurance company is ruined in time interval $[0, T]$

$$\pi_{or}^{T}(a_1, a_2) = \mathbb{P}\left(\inf_{s\in[0,T]} R_1(s) < 0 \text{ or } \inf_{t\in[0,T]} R_2(t) < 0\right).$$

One can refer to [Avram et al., 2008a; Avram et al., 2008b; Dębicki, Ji, and Rolski, 2019, 2020a; Dębicki, Hashorva, and Krystecki, 2020; Dębicki, Hashorva, and Michna, 2018; Dębicki, Kosiński, Mandjes, and Rolski, 2010; Foss, Korshunov, Palmowski, and Rolski, 2017; Ji and Robert, 2018; Kępczyński, 2020; Michna, 2021) for relevant recent discussions about two-dimensional risk models. For example, models with Gaussian claim processes are considered in (Dębicki et al., 2019; Dębicki, Ji, and Rolski, 2020; Dębicki, Hashorva, and Krystecki, 2020; Dębicki et al., 2018; Dębicki et al., 2010; Ji and Robert, 2018; Kępczyński, 2020] while Lévy claim processes are investigated in (Avram et al., 2008a, 2008b; Dębicki, Hashorva, and Michna, 2018; Foss et al., 2017; Michna, 2021). The above papers consider mainly asymptotics and, in Lévy claims

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case, also contain exact distributions and Laplace transforms of ruin probabilities.

In this study the two-dimensional fractional Brownian risk model was examined, i.e. suppose that $X(t)$ is a fractional Brownian motion $B_H(t)$, that is, a centered Gausssian process with stationary increments, covariance function $r(s,t)=\frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H})$ and $B_H(0)=0$ a.s. We focus on joint and simultaneous ruin probabilities in cases when the risk processes of both insurance and reinsurance companies are composed of a large number of i.i.d. sub-risk processes $R_i^{(k)}$, representing independent businesses. Thus we investigate

$$\pi_{sim}^T(N):=\mathbb{P}\left(\exists t\in[0,T]\colon \sum_{k=1}^{N} R_1^{(k)}(t)<0, \sum_{k=1}^{N} R_2^{(k)}(t)<0\right) \quad (1.1)$$

and

$$\pi_{and}^T(N):=\mathbb{P}\left(\inf_{s\in[0,T]}\sum_{k=1}^{N} R_1^{(k)}(s)<0, \inf_{t\in[0,T]}\sum_{k=1}^{N} R_2^{(k)}(t)<0\right), \quad (1.2)$$

where $R_i^{(k)}(t)=a_i+c_it-\sigma_iB_H^{(k)}(t), k=1,\dots,N.$, concentrating on the asymptotic behaviour of ruin probabilities (1.1) and (1.2), as $N\to\infty$. In Theorem 2.1, which contains the main contribution of this paper, one finds exact asymptotics of the simultaneous ruin probability. In Lemma 2.2 and Theorem 2.2, asymptotics of the joint ruin probability, logarithmic and exact, were studied.

Let us briefly mention the following standard notation for two given positive functions $f(\cdot)$ and $g(\cdot)$ which are use in this paper. One can write $f(x)=g(x)(1+o(1))$ if $\lim_{x\to\infty} f(x)/g(x)=1$ and $f(x)=o(g(x))$ if $\lim_{x\to\infty}\frac{f(x)}{g(x)}=0$, and $1-\Phi(x)=\Psi(x):=\mathbb{P}(\mathcal{N}>x)$, where $\mathcal{N}$ is a standard normal random variable.

The remainder of the paper is organized as follows. In Section 2 the author formalizes the problem and presents the main results of this study. Section 3 contains auxiliary facts and proofs.

2. Main results

Let us begin with several observations and assumptions.

- Using those, for $\sigma_1,\sigma_2>0$

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$$\begin{aligned}\pi_{sim}^{T}(a_1,a_2) &= \mathbb{P}\left(\inf_{t\in[0,T]}(a_1+c_1t-\sigma_1X(t),a_2+c_2t-\sigma_2X(t))<(0,0)\right)\\ &= \mathbb{P}\left(\inf_{t\in[0,T]}\left(\frac{a_1}{\sigma_1}+\frac{c_1}{\sigma_1}t-X(t),\frac{a_2}{\sigma_2}+\frac{c_2}{\sigma_2}t-X(t)\right)<(0,0)\right)\end{aligned}$$

and

$$\pi_{and}^{T}(a_1,a_2)=\mathbb{P}\left(\inf_{s\in[0,T]}\left(\frac{a_1}{\sigma_1}+\frac{c_1}{\sigma_1}s-X(s)\right)<0\right.$$

and

$$\left.\inf_{t\in[0,T]}\left(\frac{a_2}{\sigma_2}+\frac{c_2}{\sigma_2}t-X(t)\right)<0\right)$$

without loss of generality we shall suppose that $\sigma_i=1, i=1,2$.

- Note that $\sum_{k=1}^{N}B_H^{(k)}(t)=_d\sqrt{N}B_H(t)$, where $=_d$ denotes equality in distribution. Thus one can rewrite the ruin probabilities (1.1) and (1.2) as

$$\pi_{sim}^{T}(N)=\mathbb{P}(\exists t\in[0,T]:\left(B_H(t)-c_1\sqrt{N}t\right)>$$

$$a_1\sqrt{N},\left(B_H(t)-c_2\sqrt{N}t\right)>a_2\sqrt{N}),$$

and

$$\pi_{and}^{T}(N)=\mathbb{P}\left(\sup_{t\in[0,T]}\left(B_H(t)-c_1\sqrt{N}t\right)>a_1\sqrt{N},\right.$$

$$\left.\sup_{t\in[0,T]}\left(B_H(t)-c_2\sqrt{N}t\right)>a_2\sqrt{N}\right).$$

- Bearing in mind the application in reinsurance mathematics, one can focus on the case with positive drifts $c_1,c_2>0$. Observe that the assumption $c_1=c_2$ leads to classical ruin probability with the initial capital equal $\max(a_1,a_2)$. Due to the symmetry of the two-dimensional problem, in the rest of the paper without loss of generality one assumes that $c_1>c_2>0$.
- Note that if $a_1\geq a_2>0$, then

$$\begin{gathered}\left\{\exists t\in[0,T]:\left(B_H(t)-c_1\sqrt{N}t\right)>a_1\sqrt{N},\left(B_H(t)-c_2\sqrt{N}t\right)>a_2\sqrt{N}\right\} = \\ \left\{\sup_{t\in[0,T]}\left(B_H(t)-c_1\sqrt{N}t\right)>a_1\sqrt{N},\sup_{t\in[0,T]}\left(B_H(t)-c_2\sqrt{N}t\right)>a_2\sqrt{N}\right\} = \\ \left\{\sup_{t\in[0,T]}\left(B_H(t)-c_1\sqrt{N}t\right)>a_1\sqrt{N}\right\}\end{gathered}$$

and this problem degenerates to the one-dimensional ruin, i.e.

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$$\pi_{sim}^T(N) = \pi_{and}^T(N) = \mathbb{P}\left(\sup_{t\in[0,T]}\left(B_H(t) - c_1\sqrt{N}t\right) > a_1\sqrt{N}\right),$$

which was considered in (Dębicki and Mandjes, 2003). To avoid dimension-reduction, we shall assume $0 < a_1 < a_2$.

Denote $t^* := \frac{a_2-a_1}{c_1-c_2}$. In the rest of the paper the author focused on the case $t^* < T$, which is the only one that leads to non-degenerated scenarios.

- It follows from the general theory on extremes of Gaussian processes that in the one-dimensional case, the point that maximizes variance of $\frac{B_H(t)}{a_i+c_it}$ corresponds to the logarithmic asymptotics; see e.g. (Piterbarg, 1996). That is

$$\lim_{N\to\infty}\frac{\log\mathbb{P}\left(\sup_{t\geq 0}\left(B_H(t) - c_i\sqrt{N}t\right) > a_i\sqrt{N}\right)}{N} =$$

$$-\frac{1}{2}\left[\sup_{t\geq 0}\mathbb{V}ar\left(\frac{B_H(t)}{a_i + c_it}\right)\right]^{-1}.$$

Elementary calculations show that

$$t_i := \operatorname{argsup}_{t\geq 0}\mathbb{V}ar\left(\frac{B_H(t)}{a_i + c_it}\right) = \frac{a_iH}{c_i(1-H)}, \text{ for } i = 1,2.$$

Additionally, by assumptions $0 < a_1 < a_2$ and $c_1 > c_2 > 0$ it holds that

$$t_1 < t_2.$$

It turns out that points t_1 and t_2 also play an important role in two-dimensional case. As shown later, the order between t_1, t_2 and t^* affects the asymptotics of $\pi_{sim}^T(N)$ and $\pi_{and}^T(N)$, as $N \to \infty$.

Let us introduce some constants that play a crucial role in the main results of this study. First we define the classical *Pickands constant*

$$\mathcal{H}_{2H} = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_H(t) - t^{2H}\right)\right)\right).$$

It is known that $\mathcal{H}_{2H} \in (0,\infty)$ for $H \in (0,1]$ and $\mathcal{H}_1 = 1, \mathcal{H}_2 = 1/\sqrt{\pi}$; see e.g. (Dębicki and Mandjes, 2003; Ji and Robert, 2018; Piterbarg, 1996).

Furthermore, for any continuous function $d(\cdot)$ such that $d(0) = 0$, define

$$\widetilde{\mathcal{H}}_1^d = \lim_{T\to\infty} \mathbb{E}\left(\exp\left(\sup_{t\in[-T,T]}\left(\sqrt{2}B_{1/2}(t) - |t| - d(t)\right)\right)\right)$$

whenever the limit exists; see e.g. (Ji and Robert, 2018).

2.1. Simultaneous ruin

This section contains the exact asymptotics of the simultaneous ruin probability. Ji and Robert (2018) considered a similar problem in the infinite-time horizon. The author used a similar argument to extend Theorem 3.1 in (Ji and Robert, 2018) to the case $T \in (0, \infty)$.
First we recall the asymptotics of

$$\psi^T(N; a, c) := \mathbb{P}\left(\sup_{t\in[0,T]}\left(B_H(t) - c\sqrt{N}t\right) > a\sqrt{N}\right),$$

which play an important role in the further analysis.

The proof of the following lemma can be found in (Dębicki and Mandjes, 2003; Proposition 4.1).

Lemma 2.1. Suppose that $H \in (0,1]$ and $a, c, T > 0$. Let $m: = m(a, c, H) = \left(\frac{a}{1-H}\right)^{1-H}\left(\frac{c}{H}\right)^{H}$.

(i) If $T > \frac{H}{1-H}\frac{a}{c}$, then, as $N \to \infty$,

$$\psi^T(N; a, c) = \frac{\mathcal{H}_{2H}}{\pi}\frac{1}{\sqrt{H(1-H)}}\left(\frac{m}{\sqrt{2}}\right)^{\frac{1}{H}-1} N^{\frac{H-1}{2}}\frac{1}{\sqrt{2\pi}\sqrt{N}}\frac{1}{m}e^{-\frac{m^2}{2}N}\left(1 + o(1)\right).$$

(ii) If $T = \frac{H}{1-H}\frac{a}{c}$, then, as $N \to \infty$,

$$\psi^T(N; a, c) = \frac{\mathcal{H}_{2H}}{2\pi}\frac{1}{\sqrt{H(1-H)}}\left(\frac{m}{\sqrt{2}}\right)^{\frac{1}{H}-1} N^{\frac{H-1}{2}}\frac{1}{\sqrt{2\pi}\sqrt{N}}\frac{1}{m}e^{-\frac{m^2}{2}N}\left(1 + o(1)\right).$$

(iii, a) If $T < \frac{H}{1-H}\frac{a}{c}$ and $H \in (0, \frac{1}{2})$ then, as $N \to \infty$,

$$\psi^T(N; a, c) =$$

$$\mathcal{H}_{2H}\frac{T^{2H-1}(a+cT)^{\frac{1}{H}-1}}{cT - H(a+cT)}\frac{N^{\frac{H-2}{2}}}{2^{\frac{H}{2}}}\frac{1}{\sqrt{2\pi}\sqrt{N}}\frac{T^H}{a+cT}e^{-\frac{(a+cT)^2}{2T^{2H}}N}\left(1 + o(1)\right).$$

(iii, b) If $T < \frac{H}{1-H}\frac{a}{c}$ and $H = \frac{1}{2}$ then, as $N \to \infty$,

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$$\psi^T(N;a,c)=\frac{1}{\sqrt{2\pi}\sqrt{N}}\frac{2a\sqrt{T}}{(a-cT)(a+cT)}e^{-\frac{(a+cT)^2}{2T}N}\big(1+o(1)\big).$$

(iii,c) If $T<\frac{H}{1-H}\frac{a}{c}$ and $H\in\left(\frac{1}{2},1\right)$ then, $N\to\infty$,

$$\psi^T(N;a,c)=\frac{1}{\sqrt{2\pi}\sqrt{N}}\frac{T^H}{(a+cT)}e^{-\frac{(a+cT)^2}{2T^{2H}}N}(1+o(1)).$$

Since in several cases the asymptotics of two-dimensional ruin probabilities is reduced to a one-dimensional one, for the sake of brevity the main result of this section is given in the language of one-dimensional ruin probability in Lemma 2.1. Let us denote

$$A_i=\frac{|(a_i+c_it^*)H-c_it^*|}{(a_i+c_it^*)t^*},i=1,2.$$

Theorem 2.1. Suppose that $t^*<T$.
(i) If $t^*<t_1$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\psi^T(N;a_1,c_1)\big(1+o(1)\big).$$

(ii) If $t^*=t_1$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\frac{1}{2}\psi^T(N;a_1,c_1)\big(1+o(1)\big).$$

(iii,a) If $t_1<t^*<t_2$ and $H<1/2$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\frac{A_1+A_2}{2^{1/(2H)}t^*A_1A_2}\mathcal{H}_{2H}\left(\frac{a_1+c_1t^*}{(t^*)^H}\sqrt{N}\right)^{1/H-2}\Psi\left(\frac{a_1+c_1t^*}{(t^*)^H}\sqrt{N}\right)$$
$$\big(1+o(1)\big).$$

(iii,b) If $t_1<t^*<t_2$ and $H=1/2$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\widetilde{\mathcal{H}}_1^d\Psi\left(\frac{a_1+c_1t^*}{(t^*)^H}\sqrt{N}\right)(1+o(1)),$$

with $d(t)=2t^*A_2|t|1\{t<0\}+2t^*A_1|t|1\{t\geq 0\}$.

(iii,c) If $t_1<t^*<t_2$ and $H>1/2$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\Psi\left(\frac{a_1+c_1t^*}{(t^*)^H}\sqrt{N}\right)\big(1+o(1)\big).$$

(iv) If $t_1<t_2=t^*$, then, as $N\to\infty$,

$$\pi_{sim}^T(N)=\frac{1}{2}\psi^T(N;a_2,c_2)\big(1+o(1)\big).$$

(v) If $t_1<t_2<t^*$, then, as $N\to\infty$,

$$\pi_{sim}^{T}(N) = \psi^{T}(N; a_2, c_2)(1 + o(1)).$$

Remark 2.1. Ji and Robert (2018) showed that for function $d(t) = 2t^* A_2 |t| 1\{t < 0\} + 2t^* A_1 |t| 1\{t \geq 0\}$ the constant $\widetilde{\mathcal{H}}_1^d$ is well-defined, positive and finite.

2.2. Joint ruin

This section contains the logarithmic and exact asymptotics of the joint ruin probability. Kępczyński (2020) and Lieshout and Mandjes (2007) considered the related problems for a standard Brownian motion in finite-time and infinite-time horizons, respectively.

The following lemma gives the logarithmic asymptotics of the joint ruin probability.

Lemma 2.2. Let $H \in (0,1]$. Then, as $N \to \infty$,

$$\lim_{N\to\infty} \frac{\log\left(\pi_{and}^{T}(N)\right)}{N} = -\frac{1}{2} \inf_{0 \leq s,t \leq T} \frac{1}{\min\left(\sigma_1^2(s), \sigma_2^2(t)\right)} \left(1 + \frac{(c(s,t) - r(s,t))^2}{1 - r^2(s,t)} 1r(s,t) < c(s,t)\right),$$

where $r(s,t) := \mathbb{C}orr\left(\frac{B_H(s)}{a_1+c_1 s}, \frac{B_H(t)}{a_2+c_2 t}\right) = \frac{1}{2 s^H t^H}(t^{2H} + s^{2H} - |t-s|^{2H})$, $c(s,t) := \min\left(\frac{\sigma_2(t)}{\sigma_1(s)}, \frac{\sigma_1(s)}{\sigma_2(t)}\right)$ and $\sigma_i^2(t) := \mathbb{V}ar\left(\frac{B_H(t)}{a_i + c_i t}\right) = \frac{t^{2H}}{(a_i+c_i t)^2}$, $i = 1,2$.

The following proposition considers the special case $H = 1$.

Proposition 2.2. Suppose that $H = 1$.
(i) If $t^* \leq T$, then

$$\pi_{and}^{T}(N) = \Psi\left(\frac{a_1 + c_1 T}{T}\sqrt{N}\right).$$

(ii) If $t^* > T$, then

$$\pi_{and}^{T}(N) = \Psi\left(\frac{a_2 + c_2 T}{T}\sqrt{N}\right).$$

Remark 2.3. Suppose that $H = 1$.
(i) If $t^* \leq T$, then, as $N \to \infty$,

$$\pi_{and}^{T}(N) = \frac{1}{\sqrt{2\pi}\sqrt{N}} \frac{T}{a_1 + c_1 T} e^{-\frac{(a_1+c_1 T)^2}{2T^2}N}\left(1 + o(1)\right).$$

(ii) If $t^* > T$, then, as $N \to \infty$,

$$\pi_{and}^{T}(N) = \frac{1}{\sqrt{2\pi}\sqrt{N}} \frac{T}{a_2 + c_2 T} e^{-\frac{(a_2+c_2 T)^2}{2T^2}N}(1 + o(1)).$$

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Theorem 2.2. Suppose that $t^* < T$.
(i) If $t^* < t_1$ then, as $N \to \infty$,

$$\pi_{and}^T(N) = \psi^T(N; a_1, c_1)\big(1 + o(1)\big).$$

(ii) If $t_1 < t_2 < t^* < T$, then, as $N \to \infty$,

$$\pi_{and}^T(N) = \psi^T(N; a_2, c_2)(1 + o(1)).$$

Remark 2.4. Theorem 2.2 gives exact asymptotics, but only in cases which lead to a dimension-reduction scenario. In the remaining case $t_1 \leq t^* \leq t_2$ the analysis *of* $\pi_{and}^T(N)$ goes beyond the approach presented in this contribution and thus one can obtain only logarithmic asymptotics as in Lemma 2.2. The author refered to a recent contribution by Kępczyński (2020) where case $t_1 \leq t^* \leq t_2$ was solved for $H = 1/2$.

3. Proofs

Proof of Theorem 2.1. We divide the proof into the following three cases: $t^* < t_1, t_1 \leq t^* \leq t_2$ and $t_1 < t_2 < t^*$.

Case (i): $t^* < t_1$. We have

$$\pi_{sim}^T(N) \geq \mathbb{P}\big(\exists t \in [t^*, T]: \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}, \big(B_H(t) - c_2\sqrt{N}t\big) > a_2\sqrt{N}\big) = \mathbb{P}\left(\sup_{t\in[t^*,T]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right) \geq \mathbb{P}\left(\sup_{t\in[0,T]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right) - \mathbb{P}\left(\sup_{t\in[0,t^*]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right)$$

and

$$\pi_{sim}^T(N) \quad \leq \mathbb{P}\left(\sup_{t\in[0,T]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right).$$

From Lemma 2.1 and the fact that $\frac{(a_1+c_1t^*)^2}{(t^*)^{2H}} > \frac{(a_1+c_1T)^2}{T^{2H}} \geq \left(\left(\frac{a_1}{1-H}\right)^{1-H}\left(\frac{c_1}{H}\right)^H\right)^2 = \frac{(a_1+c_1t_1)^2}{t_1^{2H}}$ we obtain, as $N \to \infty$,

$$\mathbb{P}\left(\sup_{t\in[0,t^*]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right) = o\left(\mathbb{P}\left(\sup_{t\in[0,T]} \big(B_H(t) - c_1\sqrt{N}t\big) > a_1\sqrt{N}\right)\right).$$

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Thus, as $N \to \infty$,

$$\pi_{sim}^{T}(N) = \mathbb{P}\left(\sup_{t\in[0,T]}\left(B_H(t) - c_1\sqrt{N}t\right) > a_1\sqrt{N}\right)(1 + o(1)).$$

Case $(ii - iv)$: $t_1 \leq t^* \leq t_2$. Let

$$Z(t) = \frac{B_H(t)}{g(t)}, \text{ where } g(t) = \max(a_1 + c_1 t, a_2 + c_2 t)$$

and

$$\sigma_Z^2(t) = \mathbb{V}ar(Z(t)) = \min\left(\mathbb{V}ar\left(\frac{B_H(t)}{a_1 + c_1 t}\right), \mathbb{V}ar\left(\frac{B_H(t)}{a_2 + c_2 t}\right)\right) = \frac{t^{2H}}{g^2(t)}, t \geq 0.$$

We have

$$\pi_{sim}^{T}(N) = \mathbb{P}\left(\sup_{t\in[0,T]} Z(t) > \sqrt{N}\right).$$

Elementary calculations show that

$$t_{opt} = \operatorname{argsup}_{t\geq 0} \sigma_Z^2(t) = t^*$$

is the unique point that maximizes $\sigma_Z^2(t)$ over $[0, \infty)$.

We have a lower bound

$$\pi_{sim}^{T}(N) \geq \mathbb{P}\left(\sup_{t\geq 0} Z(t) > \sqrt{N}\right) - \mathbb{P}\left(\sup_{t\geq T} Z(t) > \sqrt{N}\right)$$

and an upper bound

$$\pi_{sim}^{T}(N) \leq \mathbb{P}\left(\sup_{t\geq 0} Z(t) > \sqrt{N}\right).$$

Since $\lim_{t\to\infty} Z(t) = 0$ a.s., the process $\{Z(t): t \geq 0\}$ has bounded sample paths. Hence from the Borell-TIS inequality (see Theorem 2.6.1 in (Adler and Taylor, 2009)) we obtain that for all sufficiently large N, where $C_0 = \mathbb{E}\left(\sup_{t\geq T} Z(t)\right) < \infty$ it holds

$$\mathbb{P}\left(\sup_{t\geq T} Z(t) > \sqrt{N}\right) \leq \exp\left(-\frac{1}{2\sup_{t\geq T}\sigma_Z^2(t)}\left(\sqrt{N} - C_0\right)^2\right).$$

Note that $t_{opt} = t^* \in [0, T)$ and $\sup_{t\geq T} \sigma_Z^2(t) < \sigma_Z^2(t_{opt})$. Hence we obtain, as $N \to \infty$,

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$$\mathbb{P}\left(\sup_{t\geq T} Z(t) > \sqrt{N}\right) = o\left(\mathbb{P}\left(\sup_{t\geq 0} Z(t) > \sqrt{N}\right)\right).$$

Thus, as $N \to \infty$,

$$\pi_{sim}^{T}(N) = \mathbb{P}\left(\sup_{t\geq 0} Z(t) > \sqrt{N}\right)\left(1 + o(1)\right) =$$

$$\mathbb{P}\left(\exists t \geq 0: (B_H(t) - c_1 t) > a_1 N^{\frac{1}{2(1-H)}}, (B_H(t) - c_2 t) > a_2 N^{\frac{1}{2(1-H)}}(1 + o(1)\right)$$

and the thesis follows from Theorem 3.1 in (Ji and Robert, 2018).

Case (v): $t_1 < t_2 < t^*$. We have

$$\pi_{sim}^{T}(N) \geq \mathbb{P}\left(\exists t \in [0, t^*]: \left(B_H(t) - c_1\sqrt{N}t\right) > a_1\sqrt{N}, \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right) = \mathbb{P}\left(\sup_{t\in[0,t^*]} \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right)$$

and

$$\pi_{sim}^{T}(N) \leq \mathbb{P}\left(\sup_{t\in[0,T]} \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right).$$

Since $t_2 < t^* < T$, Lemma 2.1 implies, as $N \to \infty$,

$$\mathbb{P}\left(\sup_{t\in[0,t^*]} \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right) =$$

$$\mathbb{P}\left(\sup_{t\in[0,T]} \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right)(1 + o(1)).$$

Thus, as $N \to \infty$,

$$\pi_{sim}^{T}(N) = \mathbb{P}\left(\sup_{t\in[0,T]} \left(B_H(t) - c_2\sqrt{N}t\right) > a_2\sqrt{N}\right)(1 + o(1)).$$

This completes the proof.

The following lemma provides logarithmic asymptotics of a joint survival function for supremum of two centered and bounded Gaussian processes. Its proof can be found in (Dębicki et al., 2010; Remark 5).

Lemma 3.1. Let $\{X_1(s): s \in \mathcal{T}_1\}$ and $\{X_2(t): t \in \mathcal{T}_2\}$ be two centered and bounded $\mathbb{R}$-valued Gaussian processes. Then, for $q_1, q_2 > 0$, *as* $N \to \infty$,

$$\frac{\log\mathbb{P}\left(\sup_{s\in\mathcal{T}_1} X_1(s) > q_1\sqrt{N}, \sup_{t\in\mathcal{T}_2} X_2(t) > q_2\sqrt{N}\right)}{N} =$$

$$-\frac{1}{2} \inf_{s\in\mathcal{T}_1, t\in\mathcal{T}_2} \frac{1}{\min(\sigma_1(s)/q_1, \sigma_2(t)/q_2)^2}$$

$$\left(1+\frac{(c_q(s,t)-r(s,t))^2}{1-r^2(s,t)}1_{\{r(s,t)<c_q(s,t)\}}\right)(1+o(1))$$

where $\sigma_i(t)=\sqrt{\mathbb{V}ar(X_i(t))}$, $r(s,t)=\mathbb{C}orr(X_1(s),X_2(t))$ and $c_q(s,t)=\min\left(\frac{q_1}{\sigma_1(s)}\frac{\sigma_2(t)}{q_2},\frac{\sigma_1(s)}{q_1}\frac{q_2}{\sigma_2(t)}\right)$.

Proof of Lemma 2.2. It is sufficient to observe that $\left\{\frac{B_H(t)}{a_i+c_it}:t\geq 0\right\}$ is centered and bounded $\mathbb{R}$-valued Gaussian processes, for $i=1,2$. We note that

$$\sigma_i(t)=\frac{t^H}{a_i+c_it}\text{ and }\mathcal{T}_i=[0,T], c(s,t)=\min\left(\frac{\sigma_2(t)}{\sigma_1(s)},\frac{\sigma_1(s)}{\sigma_2(t)}\right)$$

and

$$r(s,t)=\mathbb{C}orr\left(\frac{B_H(s)}{a_1+c_1s},\frac{B_H(t)}{a_2+c_2t}\right)=\frac{1}{2s^Ht^H}(t^{2H}+s^{2H}-|t-s|^{2H}).$$

Hence Lemma 3.1 implies the thesis.

Proof of Proposition 2.2. It is well-known that $B_1(t)=t\mathcal{N}$, with $\mathcal{N}\sim\mathcal{N}(0,1)$; see e.g. (Piterbarg, 1996). We have that

$$\pi_{and}^T(N)=\mathbb{P}\left(\sup_{s\in[0,T]}\left(\mathcal{N}s-c_1\sqrt{N}s\right)>a_1\sqrt{N},\sup_{t\in[0,T]}\left(\mathcal{N}t-c_2\sqrt{N}t\right)>a_2\sqrt{N}\right),$$

where $\mathcal{N}\sim\mathcal{N}(0,1)$.

Observe that, for $i=1,2$, we have $\left\{\sup_{t\in[0,T]}(\mathcal{N}t-c_i\sqrt{N}t)>a_i\sqrt{N}\right\}=\left\{\mathcal{N}T-c_i\sqrt{N}T>a_i\sqrt{N}\right\}$.

Hence

$$\left\{\sup_{s\in[0,T]}\left(\mathcal{N}s-c_1\sqrt{N}s\right)>a_1\sqrt{N},\sup_{t\in[0,T]}\left(\mathcal{N}t-c_2\sqrt{N}t\right)>a_2\sqrt{N}\right\}=$$

$$\left\{\mathcal{N}>\max\left(\frac{a_1+c_1T}{T},\frac{a_2+c_2T}{T}\right)\sqrt{N}\right\}.$$

Finally, we obtain that

$$\pi_{and}^T(N)=\mathbb{P}\left(\mathcal{N}>\max\left(\frac{a_1+c_1T}{T},\frac{a_2+c_2T}{T}\right)\sqrt{N}\right).$$

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Proof of Remark 2.3. The proof follows straightforwardly from Proposition 2.2 and the fact that, as $x \to \infty$,

$$\mathbb{P}(\mathcal{N} > x) = \frac{1}{\sqrt{2\pi}x} e^{-\frac{1}{2}x^2}(1 + o(1)).$$

Proof of Theorem 2.2. Recall that $\psi^T(N; a, c) = \mathbb{P}\left(\sup_{t\in[0,T]}\left(B_H(t) - c\sqrt{N}t\right) > a\sqrt{N}\right)$. The proof immediately follows by a combination of

$$\pi_{sim}^T(N) \quad \leq \quad \pi_{and}^T(N) \leq \min(\psi^T(N; a_1, c_1), \psi^T(N; a_2, c_2))$$

with Lemma 2.1 and Theorem 2.1.

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REASEKURACJA PROPORCJONALNA W UŁAMKOWO BROWNOWSKIM MODELU RYZYKA

Streszczenie: Artykuł bada prawdopodobieństwa ruiny w dwuwymiarowym ułamkowo brownowskim modelu ryzyka w schemacie reasekuaracji proporcjonalnej. Autor skupił się na prawdopodobieństwach ruin łącznej oraz symultanicznej w skończonym horyzoncie czasu. Procesy ryzyka firm ubezpieczeniowej oraz reasekuracyjnej składają się z dużej liczby i.i.d procesów podryzyka reprezentujących niezależne biznesy. W pracy zostały wyznaczone asymptotyki, gdy kapitał początkowy dąży do nieskończoności.

Słowa kluczowe: ułamkowy ruch Browna, asymptotyki, prawdopodobieństwo ruiny, dwuwymiarowy model ryzyka, ubezpieczenia proporcjonalne.