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A Note on Option Pricing with the Use of Discrete-Time Stochastic Volatility Processes

Anna Pajor*

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Abstract

In this paper we show that in the lognormal discrete-time stochastic volatility model with predictable conditional expected returns, the conditional expected value of the discounted payoff of a European call option is infinite. Our empirical illustration shows that the characteristics of the predictive distributions of the discounted payoffs, obtained using Monte Carlo methods, do not indicate directly that the expected discounted payoffs are infinite.

Key Words: option pricing, SV model, Bayesian forecasting

JEL Classification: C11, C22, C53

*Cracow University of Economics, e-mail eopajor@cyf-kr.edu.pl

1 Introduction

The classical Black-Scholes model assumes that asset prices follow a geometric Brownian motion, and the risk-free interest rate is constant. Thus, numerous studies on option pricing have modified the Black-Scholes model to allow for stochastic volatility of the underlying asset processes or stochastic interest rates. Hull and White (1987) studied option pricing in the case of constant interest rate and stochastic volatility, where volatility followed an independent geometric Brownian motion. Mahieu and Schotman (1998) estimated the lognormal discrete-time stochastic volatility model under the empirical measure P and applied directly the solution of Hull and White (1987) for option pricing. Amin and Ng (1993) built the option pricing model which incorporates both a stochastic interest rate and a stochastic volatility process for stock returns. Their results were used directly by Jiang and Sluis (1999) in context of a discrete-time bivariate stochastic volatility model. In this paper we show that in the case of the lognormal discrete-time stochastic volatility model (in which the conditional variance is lognormally distributed) the conditional expected value of the discounted payoff may be infinite under physical (real world) measure P. The Risk-Neutral Probability Measure or Local Risk Neutral Valuation Relationship (see Duan 1995, 1999), under which the expected payoffs on call options are finite, lead to the change of the assumption about the tails of the distribution of the underlying asset. The paper is organized as follows. Sections 2 presents the discrete-time stochastic volatility model and shows that the conditional expected value of the discounted payoff of a European call option is infinite. Section 3 contains an empirical illustration. Finally, section 4 concludes.

2 A discrete-time stochastic volatility process and European call option pricing

Let (Ω, ψ, P) be a probability space and $F = \{\psi_t, t = 0,\ 1, ...,\ T+s\}$ be an increasing, complete filtration satisfying $\psi_{T+s} = F$. Let $S_t > 0$, $t = 0, 1, ..., T + s$, be the price of a stock at time t. We assume that the process $\{S_t\}_{t=0,1,...,T+s}$ is adapted to the filtration F. The return process s_t is defined as:

$$s_t = \ln(S_t/S_{t-1}). \tag{1}$$

The discrete-time stochastic volatility model of s_t may be written as:

$$s_t = \mu_t + \sqrt{h_t}\varepsilon_t, \tag{2}$$

$$\ln h_t = \gamma + \phi \ln h_{t-1} + \sigma_h \eta_t, \quad t = 1, 2, ..., T + s, \tag{3}$$

where ε_t and η_t are assumed to be $iiN(0,1)$, and are mutually independent (a review of SV models is provided in Psychoyios, Skiadopoulos and Alexakis 2003). We assume

that the conditional mean of the return process μ_t is ψ_t - predictable (in Mahieu and Schotman 1998 μ_t is constant).
Now, we consider a European call option with maturity $T+s$. The payoff function is $(S_{T+s}-K)^+ = \max\{S_{T+s}-K, 0\}$, where S_{T+s} is the price of the underlying asset (no dividend being paid) at time $T+s$ and K is the exercise price. The discounted payoff considered at time T is

$$W_{T|T+s} = e^{-rs}(S_{T+s}-K)^+,$$

where r is the risk-free interest rate (assumed constant and known). The value of the option is given by

$$C_{T|T+s} = E_Q(W_{T|T+s}|\psi_T) = e^{-rs}E_Q((S_{T+s}-K)^+|\psi_T),$$

where Q is an equivalent martingale measure.
It is important to stress that the specification (2)-(3) relaxes only Black and Scholes constant volatility assumption (the volatility follows a separate process). In deterministic volatility models an investor incurs only the risk from a randomly evolving asset price. Subject to certain modelling assumptions (see Black and Scholes 1973) it is possible to perfectly replicate the payoff of the option through dynamic trading. Thus, there is unique preference independent price for the option. This price can be calculated as the discounted expected value under the equivalent martingale measure. In the SV model (presented above) there are two sources of risk: the risk from the asset price and from the volatility of the asset. It is clear that this model is incomplete. Due to the incompleteness of markets, the equivalent martingale measure is not unique. It is well known that if P corresponds to a complete market model then the equivalent martingale measure Q is uniquely defined by P. However, if P corresponds to the more realistic case of an incomplete market model (e.g. stochastic volatility model) then the assumption on P does not determine the martingale measure Q in a unique way. In the incomplete market model, such as the stochastic volatility model, there is no unique martingale measure. The choice of the martingale measure is parallel to defining the state price density (see Jiang 2007).
Many papers investigate option prices in a risk-neutral world and for the case that the volatility risk premium is zero. Note that in Jiang and Sluis (1999), under assumption that the risk premium in both interest rate and asset return processes as well as the conditional volatility processes are all zero (that is, the risk-neutral process is assumed to be the same as the objective underlying process), the conditional expected value of the discounted payoff is calculated via Monte Carlo simulation. It is important to stress that in the case of the SV model the conditional expected value of $W_{T|T+s}$ is infinite under the original measure P. The predictive distribution of $W_{T|T+s}$ has the same right tail as the right tail of the underlying distribution of S_{T+s}.
Since $S_t = e^{s_t}S_{t-1}$, for $t = 1, ..., T+s$, we have $E(S_t|\psi_{t-1}) = S_{t-1}E(e^{s_t}|\psi_{t-1})$. Under our assumption $s_t|h_t, \psi_{t-1} \sim N(\mu_t, h_t)$, thus $E(e^{s_t}|h_t, \psi_{t-1}) = e^{\mu_t+0.5h_t}$. Since

$\ln h_t|\psi_{t-1} \sim N(\gamma + \phi(\ln h_{t-1} - \gamma), \sigma_h^2)$, it follows that the conditional distribution of $e^{0.5h_t}$ given ψ_{t-1}, is log lognormal. Thus, it is easily to show that

$$E(e^{s_t}|\psi_{t-1}) = E(e^{\mu_t+0.5h_t}|\psi_{t-1}) = +\infty,$$

which implies that the conditional expected value of $W_{T|T+s}$ is infinite.
Similarly, using the law of iterated expectations, one can show that when the interest rate is stochastic and the conditional variance of the asset process is lognormally distributed, the conditional expected value of $W_{T|T+s}$ is infinite under measure P.
Note that when we assume that $\mu_t = r - 0.5h_t$ (now μ_t is not predictable), we have $E(S_t|\psi_{t-1}) = S_{t-1}e^r$. But when ε_t has a t-Student distribution with any degrees of freedom, it is easy to show that $E(e^{s_t}|\psi_{t-1}) = +\infty$. If in the risk-neutral measure the return also has a fat right tail such as the t-Student distribution, then the call option price (defined as an expected value of the discounted payoff under this measure) is also infinite.

3 An empirical illustration

We analyse the data from the Warsaw Stock Exchange. The growth rate of the WIG20 index ($y_t = 100s_t$, S_t is the index level at time t) is modelled using the discrete-time AR(1)-SV model, i.e. in the specification (1)-(3) $\mu_t = \delta + \rho(y_{t-1} - \delta)$. We use the Bayesian approach, which takes completely into account uncertainty, which comes from prediction and from the parameters, by construction of the predictive distribution of the discounted payoff. A measure of uncertainty can be attached to the option price by computing predictive quantiles. The parameters have the following prior structure:

$$p(\delta, \rho, \gamma, \phi, \sigma_h^2) = p(\delta)p(\rho)p(\gamma)p(\phi)p(\sigma_h^2),$$

where we use proper prior densities assuming the following distributions: $\delta \sim N(0,1)$, $\rho \sim U(-1,1)$,$\gamma \sim N(0,100)$,$\phi \sim N(0,100)I_{(-1,1)}(\phi)$, $\sigma_h^{-2} \sim G(1, 0.005)$.
The prior distribution for δ is standardized normal, $U(-1,1)$ denotes the uniform distribution over (-1,1). The prior distribution for ϕ is normal, truncated by the restriction that the absolute value of ϕ is less than one ($I_{(-1,1)}(.)$ denotes the indicator function of the interval $(-1,1)$, which is the region of stationarity of lnh_t). The symbol $G(v_0, s_0)$ denotes the Gamma distribution with mean v_0/s_0 and variance v_0/s_0^2 (thus σ_h^{-2} has a Gamma prior with mean 200 and standard deviation 200; see Jacquier Polson and Rossi 2004). The initial condition h_0 is equal to y_0^2. These assumptions reflect rather weak prior knowledge about the parameters.
We use daily observations (closing quotes) of the WIG20 index over the period from January 2, 2001 to February 6, 2008. The data was downloaded from www.money.pl. The dataset of the daily logarithmic growth rates (expressed in percentage points), y_t, consists of 1782 observations. The first observation is used to construct initial conditions, thus $T = 1781$ (number of modelled observations). The data are plotted

Table 1: Quantiles of the discounted payoff of the European call options for $s = 31$

	$\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$ for $k = 1, \ldots, s$ quantile of order					$\mu_{T+k} = r_d - 0.5\omega_{T+k}$ for $k = 1, \ldots, s$ quantile of order				
K	0.05	0.25	0.5	0.75	0.95	0.05	0.25	0.5	0.75	0.95
2400	63.55	373.86	597.68	841.03	1251.47	31	326.12	537.23	765.39	1141.73
2500	0	274.66	498.48	741.83	1151.96	0	226.92	438.03	666.19	1042.22
2600	0	175.46	399.28	642.63	1052.76	0	127.41	338.83	566.68	943.02
2700	0	75.95	299.77	543.43	953.56	0	28.21	239.63	467.48	843.82
2800	0	0	200.57	443.92	854.36	0	0	140.12	368.28	744.62
2900	0	0	101.37	344.72	754.85	0	0	40.92	269.08	645.11
3000	0	0	2.17	245.52	655.65	0	0	0	169.57	545.91
3100	0	0	0	146.32	556.45	0	0	0	70.37	446.71
3200	0	0	0	46.81	457.25	0	0	0	0	347.2
3300	0	0	0	0	357.74	0	0	0	0	248
3400	0	0	0	0	258.54	0	0	0	0	148.8
3500	0	0	0	0	159.34	0	0	0	0	49.6
3600	0	0	0	0	60.14	0	0	0	0	0
3700	0	0	0	0	0	0	0	0	0	0
3800	0	0	0	0	0	0	0	0	0	0
3900	0	0	0	0	0	0	0	0	0	0
4000	0	0	0	0	0	0	0	0	0	0
4100	0	0	0	0	0	0	0	0	0	0
4200	0	0	0	0	0	0	0	0	0	0
4300	0	0	0	0	0	0	0	0	0	0
4400	0	0	0	0	0	0	0	0	0	0
4500	0	0	0	0	0	0	0	0	0	0

in Figure 1 and 2 (see the Appendix). It can be seen from the graph that the growth rates seem to be centered around zero, with changing volatility and the presence of outliers.

Table 2: Quantiles of the discounted payoff of the European call options for $s = 92$

	$\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$ for $k = 1, \ldots, s$ quantile of order					$\mu_{T+k} = r_d - 0.5\omega_{T+k}$ for $k = 1, \ldots, s$ quantile of order				
K	0.05	0.25	0.5	0.75	0.95	0.05	0.25	0.5	0.75	0.95
2400	0	353.4	712.38	1120.03	1844.81	0	216.38	542.19	905.2	1542.87
2500	0	255.44	614.42	1022.38	1747.16	0	118.42	444.23	807.24	1444.91
2600	0	157.48	516.46	924.42	1648.89	0	20.77	346.27	709.28	1347.26
2700	0	59.83	418.5	826.46	1551.24	0	0	248.62	611.32	1249.3
2800	0	0	320.85	728.5	1453.59	0	0	150.66	513.67	1151.34
2900	0	0	222.89	630.85	1355.32	0	0	52.7	415.71	1053.69
3000	0	0	222.89	630.85	1355.32	0	0	52.7	415.71	1053.69
3100	0	0	26.97	434.93	1159.71	0	0	0	219.79	857.77
3200	0	0	0	336.97	1061.75	0	0	0	121.83	759.81
3300	0	0	0	239.32	963.79	0	0	0	24.18	662.16
3400	0	0	0	141.36	865.83	0	0	0	0	564.2
3500	0	0	0	43.4	768.18	0	0	0	0	466.24
3600	0	0	0	0	670.22	0	0	0	0	368.28
3700	0	0	0	0	572.57	0	0	0	0	270.63
3800	0	0	0	0	474.61	0	0	0	0	172.67
3900	0	0	0	0	376.65	0	0	0	0	74.71
4000	0	0	0	0	279	0	0	0	0	0
4100	0	0	0	0	181.04	0	0	0	0	0
4200	0	0	0	0	83.08	0	0	0	0	0
4300	0	0	0	0	0	0	0	0	0	0

We consider all European options on the WIG20 index, which were quoted on the Warsaw Stock Exchange (WSE) on February 6, 2008 (at the end of observed sample). The exercise dates are March 20, 2008 (i.e. $s = 31$ trading days) or June 20, 2008 (i.e. $s = 92$ trading days). It was assumed that the risk-free interest rate is 5.84% per annum (r is equal to the 6 month WIBOR rate quoted on February 6). The em-

pirical results presented in Pajor (2007) allowed us to infer that a stochastic interest rate is not very important for forecasting of the discounted payoff. The predictive distributions of the discounted payoff have such huge dispersions that in practice the differences are negligible.
We compare the predictive distributions of the discounted payoff obtained assuming that in the forecast horizon $\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$ for $k = 1, ..., s$ and those obtained with $\mu_{T+k} = r_d - 0.5\omega_{T+k}$, where $r_d = r/251$, $\omega_{T+k} = 0.01h_{T+k}$, $k = 1, ..., s$. In the first case (i.e. $\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$ for $k = 1, ..., s$), μ_{T+k} is ψ_{T+k} - predictable, and the conditional expected value of the discounted payoff of a European call option is infinite. In the second case (i.e. $\mu_{T+k} = r_d - 0.5\omega_{T+k}$), μ_{T+k} is not predictable, but the expected rate of return on S_t is the risk-free interest rate, and we have $E(S_{T+s}|\psi_T) = S_T e^{sr_d/100}$. Consequently, the conditional expected value of the discounted payoff of the European call option is finite and can be written as the expected Black and Scholes price, where the expectation is taken over the conditional distribution of the mean volatility (see Hull and White 1987).

Table 3: True value of the discounted payoff of the European call options (column A), prices of the options (column B) and reference prices (column C) on February 6, 2008. Data were downloaded from http://bossa.pl

	s =31			s =92		
K	A	B	C	A	B	C
2400	466.6222	543.8	577.45	323.99	-	649.7
2500	367.3409	-	484.85	226.1079	-	571.55
2600	268.0596	373.55	397.5	128.2257	-	498.8
2700	168.7782	285	317.4	30.34348	-	431.85
2800	69.49692	215	246.3	0	-	370.9
2900	0	160	185.55	0	-	316.15
3000	0	122	135.65	0	-	316.15
3100	0	88	96.05	0	152.95	224.75
3200	0	68	66.1	0	135	187.45
3300	0	43	44.12	0	91.95	155.35
3400	0	27	28.61	0	69.95	128
3500	0	15	18.05	0	-	104.85
3600	0	10.88	11.08	0	-	85.4
3700	0	6	6.63	0	-	69.2
3800	0	2.6	3.88	0	-	55.75
3900	0	2	2.22	0	-	44.74
4000	0	2.1	1.24	0	-	35.74
4100	0	1	0.68	0	-	28.43
4200	0	-	0.37	0	-	22.53
4300	0	-	0.2	0	-	17.79
4400	0	-	0.1	-	-	-
4500	0	0.11	0.05	-	-	-

We report in Table 1 and 2 the main characteristics of the predictive distributions of the discounted payoff for the European call option on the WIG20 index. All presented results were obtained with the use of the Metropolis and Hastings algorithm within the Gibbs sampler using 10^5 iterations after $5{\cdot}10^4$ burn-in Gibbs steps (see Pajor 2003 for details).
The settlement prices for derivative securities were equal to 2870 (for $s = 31$) and 2731 (for $s = 92$), while the last observed value of the WIG20 index was equal to 2941.65. Consequently, the options with the exercise price above 2800 (2900 or more)

for $s = 31$ and 2700 (2800 or more) for $s = 92$ were not executed. The true values of the discounted payoff are located between the medians and the quantiles of order 0.75 or between the quantiles of order 0.25 and the medians. However, the observed market prices and reference prices of the options are in more cases located above the quantiles of order 0.75 or 0.95.

Table 4: Sample means and standard deviations computed using draws from the predictive distributions of the discounted payoffs. Case A relates to $\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$, while case B relates to $\mu_{T+k} = r_d - 0.5\omega_{T+k}$

	s =31				s =92			
	Case A		Case B		Case A		Case B	
K	*mean*	*st. dev.*	*mean*	*st. dev.*	*mean*	*st. dev.*	*mean*	*st. dev.*
2400	624.31	356.94	560.68	331.29	787.62	586.18	614.80	506.79
2500	529.61	349.53	467.09	322.44	699.34	573.92	531.53	490.07
2600	438.75	337.78	378.30	308.54	614.99	558.27	453.68	469.60
2700	353.79	320.72	296.74	288.74	535.38	539.20	382.25	445.55
2800	276.92	298.12	224.62	263.41	461.36	516.76	317.89	418.53
2900	210.11	270.64	163.89	233.68	393.52	491.39	261.01	389.30
3000	154.54	239.78	115.34	201.60	332.38	463.58	211.80	358.70
3100	110.32	207.58	78.47	169.50	278.19	433.99	169.98	327.69
3200	76.59	176.08	51.71	139.44	230.71	403.49	135.02	297.14
3300	51.92	146.80	33.25	112.67	189.81	372.75	106.34	267.75
3400	34.49	120.76	20.94	89.82	155.05	342.45	83.15	240.04
3500	22.55	98.38	12.95	70.95	125.86	313.15	64.65	214.37
3600	14.58	79.64	7.94	55.70	101.62	285.28	50.03	190.95
3700	9.34	64.25	4.83	43.58	81.74	259.13	38.64	169.82
3800	5.93	51.86	2.93	34.03	65.58	234.88	29.79	150.94
3900	3.76	42.02	1.77	26.58	52.52	212.59	22.96	134.20
4000	2.39	34.26	1.06	20.81	42.03	192.28	17.70	119.44
4100	1.55	28.15	0.65	16.34	33.65	173.87	13.67	106.49
4200	1.03	23.28	0.40	12.84	26.99	157.23	10.56	95.16
4300	0.68	19.35	0.24	10.13	21.67	142.26	8.20	85.29
4400	0.46	16.21	0.15	8.02	-	-	-	-
4500	0.32	13.68	0.10	6.34	-	-	-	-

In Figure 3 (see the Appendix) we present histograms of the predictive distributions of the discounted payoff of the European options with the exercise price (K) equal to 2700 index points. The first bars of graphs denote probabilities that the options will not be exercised. The little gray squares represent the true values of the discounted payoff. The predictive histograms are characterized by huge dispersion and thick tails, thus uncertainty about the future payoff was very big *ex-ante*. However, the predictive distributions produced by the AR(1)-SV model are more spread (see quantiles in Table 1 and 2) and have thicker right tails than the predictive distributions produced by the SV model with $\mu_{T+k} = r_d - 0.5\omega_{T+k}$ for $k = 1, ..., s$. It can be seen from the graphs that in SV model with $\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$ for $k = 1, ..., s$ the last bars of the histograms are higher. It is in accordance with our theoretical results.

Finally, it is important to stress that the characteristics of the predictive distributions of the discounted payoffs obtained using the numerical methods do not indicate directly that the expected discounted payoffs are infinite. When $\mu_{T+k} = \delta + \rho(y_{T+k-1} - \delta)$, the empirical (sample) means, computed using draws from the predictive distributions of the discounted payoffs, are only higher than those in the SV model with $\mu_{T+k} = r_d - 0.5\omega_{T+k}$ (see Table 4). But these sample means cannot be used as approximations of the expected discounted payoffs.

4 Conclusions

In this paper we show that in the lognormal stochastic volatility model with predictable conditional expected returns the conditional expected value of the discounted payoff of a European call option is infinite. This is due to the fact that the "underlying asset price tomorrow" has an infinite expectation. The assumptions under which the expected payoff is finite lead to the change of the thickness of the tails.

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References

[1] Amin K.I., Ng V.K., (1993), Option Valuation with Systematic Stochastic Volatility, *The Journal of Finance* 48, 881-910.

[2] Black F., Scholes M., (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy* 81, 637-654.

[3] Duan J. -C., (1995), The GARCH Option Pricing Model, *Mathematical Finance* 5, 13-32.

[4] Duan J. -C., (1999), Conditionally Fat-Tailed Distributions and the Volatility Smile in Options, *working paper*, http://www.bm.ust.hk/~jeduan.

[5] Hull J., White A., (1987), The Pricing of Options on Assets with Stochastic Volatilities, *Journal of Finance* 42, 281-300.

[6] Jacquier E., Polson N., Rossi P., (2004), Bayesian Analysis of Stochastic Volatility Models with Fat-Tails and Correlated Errors, *Journal of Econometrics* 122, 185-212.

[7] Jiang G.J., (2007), Stochastic Volatility and Option Pricing, [in:] *Forecasting Volatility in the Financial Market*, [ed.:] J.L. Knight, S.E. Satchell, Third edition, London: Butterworth Hernemann, 131-171

[8] Jiang G.J., van der Sluis P.J., (1999), Index Option Pricing Models with Stochastic Volatility and Stochastic Interest Rates, *European Finance Review* 3, 273-310.

[9] Mahieu R.J., Schotman P.C., (1998), An Emprical Application of Stochastic Volatility Models, *Journal of Applied Econometrics* 13, 333-360.

[10] Pajor A., (2003), *Procesy zmienności stochastycznej w bayesowskiej analizie finansowych szeregów czasowych (Stochastic Volatility Processes in Bayesian Analysis of Financial Time Series)*, doctoral dissertation (in Polish), published by Cracow University of Economics, Kraków.

[11] Pajor A., (2007), Bayesian Option Pricing with Stochastic Volatility and Stochastic interest Rate, [in:] *33-rd International Conference MACROMODELS'06*, [ed.:] W. Welfe, A. Welfe, Absolwent, Łódź, 65-88.

[12] Psychoyios D., Skiadopoulos G., Alexakis P., (2003), A Review of Stochastic Volatility Processes: Properties and Implications, *Journal of Risk Finance* 4, 43-59.

Appendix - Figures

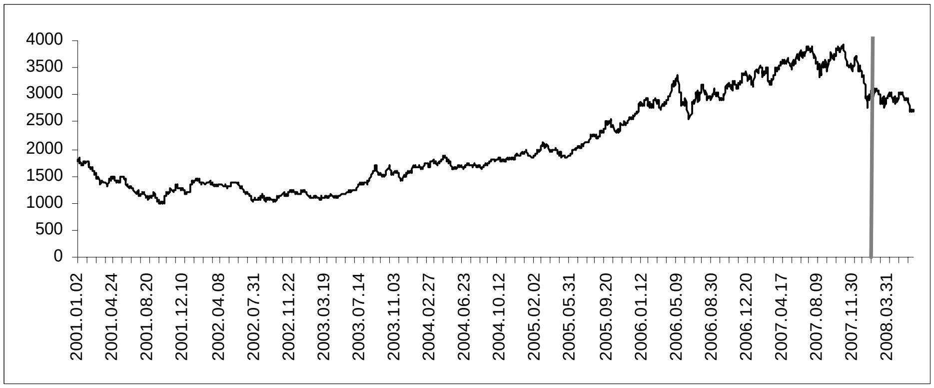

Figure 1: Daily quotations of WIG20 index (January 2, 2001 – June 20, 2008). The vertical line represents the end of the observed sample (February 6, 2008)

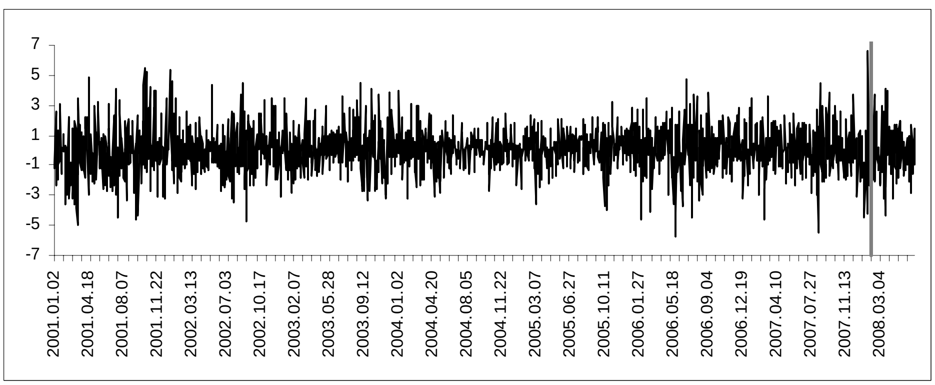

Figure 2: Daily growth rates of WIG20 index (January 2, 2001 – June 20, 2008). The vertical line represents the end of the observed sample (February 6, 2008)

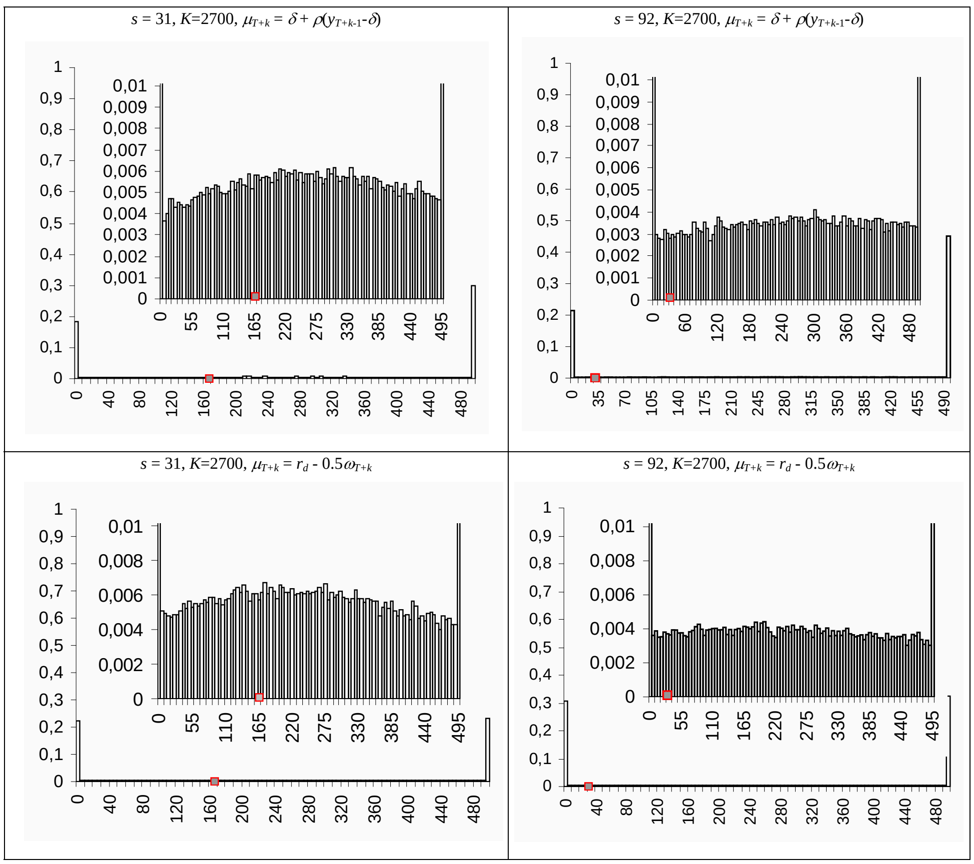

Figure 3: Histograms of the predictive distributions of the discounted payoff. The small graphs represent the same predictive distributions as the big graphs, but in a different scale. The gray square represents the true value of the discounted payoff