

**THE GRAPH OF THE COSINE  
IS AN ELLIPSE****Antoni Smoluk***Consuetudo altera natura est*

(Habit is second nature)

**Abstract.** Periodic function – this is a proposition of the new definition – has as its domain the multiplicative group  $T = \{z \in \mathbb{C} : |z| = 1\}$  of complex numbers with module one. Therefore a graph of the function  $f: T \rightarrow \mathbb{R}$  lies on the cylinder  $T \times \mathbb{R}$ , and so the graph of cosine function is the ellipse  $\frac{x^2}{2} + y^2 = 1$ .

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A function whose domain is the set of natural numbers is called in brief a sequence. For the same reasons – of precision and deep content – periodic functions should be defined in a new way other than they used to be – more concisely and clearly. A function defined on a disk, i.e. a function whose domain is a group of disk,  $T = \{e^{it} : t \in \mathbb{R}\}$ , is called a periodic function. This is because a multiplicative group  $T$  is isomorphic with an additive quotient group  $\mathbb{R}/2\pi$  of real numbers  $\mathbb{R}$ , where naturally  $\mathbb{R}/2\pi$  is a family of equivalence classes  $x = y \pmod{2\pi}$ . The graph of function  $f: X \rightarrow Y$  is a subset of the product  $X \times Y$ . Similarly, the graph of a periodic function lies on the cylinder  $T \times \mathbb{R}$ , or more generally, on  $T \times Y$ . Periodic functions – considered traditionally – are invariant functions, i.e. automorphisms under the additive subgroup action  $\{2\pi k : k \in \mathbb{Z}\}$  for the group  $\mathbb{R}$ , where  $\mathbb{Z}$  is a group of integer numbers; an orbit of a periodic function is just one point.

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**Antoni Smoluk**

Department of Mathematics and Cybernetics, Wrocław University of Economics, Komandorska Street 118/120, 53-345 Wrocław, Poland.

E-mail: math@ue.wroc.pl

**Remark.** The graph of the cosine function  $\{(e^{it}, \cos t) : t \in R\}$  is the ellipse

$$\frac{x^2}{2} + y^2 = 1.$$

This can be seen almost directly. The function of two variables  $f: R^2 \rightarrow R$ , defined by formula  $f(x, y) = x$ , subject to  $x^2 + y^2 = 1$ , reaches its conditional extrema: the maximum at point  $A = (1, 0)$ , and the minimum at point  $B = (-1, 0)$ . Certainly,

$$\{(x, y) \in R^2 : x^2 + y^2 = 1\} = \{(\cos t, \sin t) : t \in R\};$$

the intersection of the plane  $z = x$ , or the set

$$\{(x, y, z) \in R^3 : (x, y) \in R^2\}$$

with the cylinder

$$\{(x, y, z) \in R^3 : (x, y) \in R^2, x^2 + y^2 = 1\};$$

in  $R^3$ , is precisely the graph of the cosine function, thus, the ellipse. Its equation written with new coordinates:

$$x' = \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z, \quad y' = y, \quad z' = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z,$$

is given above – without prime symbols; because it holds:  $z = x$ , so:  $x' = \sqrt{2}x$ ,  $y' = y$ ,  $z' = 0$ . Then, the parametric equations are given by the formulae:

$$x' = \sqrt{2} \cos t, \quad y' = \sin t, \quad z' = 0, \quad t \in R.$$

Similarly, examining the conditional extrema of function  $g(x, y) = y$ , subject to the same condition, we see that the graph of the sine function is an ellipse

$$x^2 + \frac{y^2}{2} = 1.$$

This is a graph of the cosine function rotated by  $\pi/2$ .

Functions  $f(x, y) = x$  and  $g(x, y) = y$  are clearly projections on the first and second axis. Function  $h(x, y) = \max\{x, y\}$  is not a projection, however, it deserves to be referred to as a semi–projection. This function is not smooth – at points of the line  $y = x$  there is no derivative. The four points:

$$A = (0, 1), \quad B = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad C = (1, 0), \quad D = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

are local conditional extrema of  $h$  on a set  $x^2 + y^2 = 1$ ; at  $A$  and  $C$  the function  $h$

reaches its maxima, whereas at  $B$  and  $D$  – its minima. The intersection of the cylinder with the graph of function  $h$  is a new periodic function – in half the sine and in half the cosine. The graph of this function is not a flat object. This is likewise if we take function  $j(x, y) = \min \{x, y\}$  instead of  $h$ . Such examples of non-smooth continuous periodic functions, spline functions, show that the modified definition of a periodic function is fruitful for teaching purposes. The continuous periodic function  $f \in C(T, R)$  is an intersection of the plane  $F \in C([-1, 1]^2, R)$  with a cylinder. The mapping of the Banach space  $C([-1, 1]^2, R)$  onto space  $C(T, R)$ , that assigns a restriction of  $F$  to the set  $T$ , is a linear surjection of norm 1; in both spaces we imply the Chebyshev norm.

I believe that these remarks about the graph of the sine function and, generally, the definition of periodic function will be useful to teachers, especially those who perceive mathematics as a science about the real world, not as a variety of an abstract, multidimensional game of chess. We only acquire knowledge of what exists. There is no science of nonentity. If we assume a group of disks as a domain of periodic functions, then we are able to explain the essence of all periodic courses in nature: biology, physics, economics, outer space. Indeed, beautiful maidens live in airy vortices – ethereal Vilas. Vortical motion is widespread because its essence is seeking an equilibrium. Time is of a circular nature – that which has been, is now, and that which is to be, has already been. If a purpose of school is to encourage, to transfer knowledge and to train in scientific culture, then exactly periodic functions should be treated this way. However, old habits are very stubborn – tradition is stable. A survey among mathematicians of various levels revealed that even if the sample was certainly biased, merely a dozen respondents, the idea presented here is absolutely unknown to current graduates of mathematics. A couple of full professors in mathematics – *nulla calamitas sola* – was not able to get the message about the cosine even after two days of detailed explanations. I do not know the reasons why the idea is so fiercely resisted as if it was a harmful and bad science, anyway it seems to unsettle the state of mind of current mathematicians. Maybe they are predominantly afraid of something new and simple? This is a frequent reaction of elderly people who prefer not to continue studying, they rely on routine. But my respondents were mostly young and very young people. *Durnowo krysty, win kryczy pusty* (if you are christening a stupid man, he always cries: leave me alone). This terse proverb of Ruthenians better corresponds with us, Poles, who used to abuse their freedom, than with proud Cossacks. We always have our own opinions. Perhaps this essay is also an

example. Besides, as far back as Aristotle, we know his opinion that *purus mathematicus* is also *purus stupidus* [a genuine mathematician is simply stupid]. Maybe he was right, as the saying goes about a mother of a famous Polish mathematician who was complaining to her neighbour – still, with some sort of pride: *I have three sons – two normal and one mathematician.*

*I know better – the sine is a wavy line and nothing more.* Wiseacres ready to hastily speak like that are many in our – Polish – mathematics. And yet, it is a deep-rooted idea of a disk being a domain of periodic functions that has been well known to experts in Fourier analysis. As examples, let me give two quotes. First, from a difficult, popular book by Hugo Steinhaus entitled *Kalejdoskop matematyczny* [Mathematical snapshots]. “If we wind paper round a candle, then cut it obliquely with a sharp knife, and then unwind the paper, we obtain a sinusoid” (Steinhaus 1954, p. 234). The second quotation is from an equally beautiful book – rigorously specialized, by Jean-Pierre Kahane; it is the first statement of his introduction to the work on absolutely convergent Fourier series (Kahane 1970). “The book deals with *A* class functions, i.e. continuous functions as a circle with absolutely convergent Fourier series”. Steinhaus unwinds an ellipse to show a sinusoid, Kahane winds a sinusoid and obtains an ellipse, though he does not explicitly write about it. Hence, there is a sinusoid – a wavy line, and there is an ellipse – a sine. A Banach space of periodic continuous functions on a line is isomorphic and isometric with a space of continuous functions on a circle. *Repetitio est mater studiorum.* A domain of a periodic function *ex definitione* is a group of disk  $T$ . This definition extends immediately on double-periodic functions defined on torus  $T^2$  and generally on multi-periodic functions whose domain is the product  $T^n$ . That’s all. It may be considered a proxy – speaking grandiloquently – of a new paradigm in the theory of periodic functions. Let us throw away a comfortable routine, then it will be better, more natural, more simple. Maybe a circular domain of periodic functions will gain most supporters if we present the idea during the lecture. The audience will decide which definition they assume. More models will emerge, theory will be enhanced, and knowledge will be wider. Yet, the goal of science is excellence.

August Kekulé, a German organic chemist, the founder of the theory of chemical structure, discovered the ring nature of benzene and put forward a classical formula for its structure. The formula appeared in a dream when he fell asleep at his desk. It was truly a revelation. Likewise, it was about the ellipse and the cosine. The ellipse being the graph of the cosine was a dream. Thus, it deserves attention also because of it. Dreams are sometimes

prophetic. This is a minor contribution – *toutes proportions gardées* – to the psychology of scientific discovery. Arguably the greatest monument of the sine is the Sky Tower building in Wrocław (Figure 1).

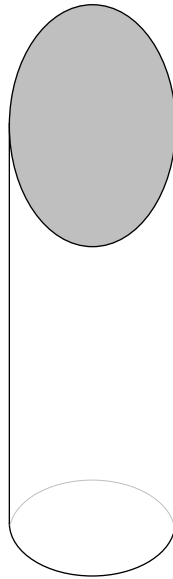


Fig. 1. A monument of the sine

Source: own work.

As a result of the battle for the cosine, Professor Bolesław Kopociński has concocted an epigram imitating epitaphs from Hrycianki, a village near Navahrudak. A well-timed joke is always appropriate – it meets the essence: sometimes it is supportive, at other times devastating. Thanks anyway on behalf of all sines and cosines.

#### Epitaph

Tut lażył Anton Smaljuk,  
 Prafiessor wsjakich nauk.  
 Na mięso nie miał wkusa,  
 A dumał o kasinuszach.  
 Ho! ho! ho!  
 Szczoż wsiakim da taho?

Professor Smoluk lies here,  
 Any science to him was dear.  
 He did not fancy meat,  
 But the cosine was his treat.  
 Ho, ho, ho!  
 Never mind, take care and go.

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