h-PREINVEX FUZZY PROCESSES

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Summary: We introduce the notion of *h*-preinvex fuzzy processes. We study their properties and give some inequalities of Hadamard-type for *h*-convex fuzzy processes.

Keywords: *h*-preinvex fuzzy process, Hadamard's inequalities.

1. Introduction

Multifunctions have many diverse and interesting applications in control problems and the theory of contingent equations, in mathematical economics, and in various branches of analysis (for example, see [Aumann 1965; Brunovsky 1968; Debreu 1983; Hermes 1968]).

The special class of the multifunctions is called convex processes. Convex processes were studied first by R.T. Rockafellar who was interested in extending properties of linear transformations to a large class of maps preserving convexity and which arise naturally in economic theory. He elaborated a duality theory for these convex processes and developed his study in his book [Rockafellar 1970]. P. Coutat studied continuous closed convex sets and gave many new properties of convex processes [Coutat 1996].

The extension of this notion to the fuzzy framework was done by M. Matłoka [2000]. Y.R. Syau, C.Y. Low and T.H. Wu [2002] observed that the Matłoka definition is very strict. Therefore, they give another definition that extends the Matłoka definition. In recent years, many generalizations of convexity have appeared in the literature aiming at applications to duality theory and optimality conditions. In 1997 R. Pini and Ch. Singh [20] introduced (Φ_1, Φ_2)-convex functions and studied some of their properties [Pini, Singh 1997]. They showed that some of the well-known classes of generalized convex functions (e.g. B-vex functions [Bector, Singh 1991], invex functions [Hanson 1981]) form subclasses of the class of (Φ_1, Φ_2)-convex functions. In 1999 E.A. Youness showed that many results for convex functions actually hold for a wider class of functions, called E-convex functions [Youness 1999]. In 1978,

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W.W. Breckner introduced s-convex functions as a generalization of convex functions [Breckner 1978] and in 1993 studied the set valued version [Breckner 1993]. The extension of this notions to the fuzzy framework was done by M. Matłoka [1999, 2012], and Y. Chalco-Cano and M.A. Rojas-Medar [2004]. In 2007 S. Varošanec [26] introduced a large class of functions, the so-called h-convex functions [Varošanec 2007]. This class contains several well-known classes of functions such as convex functions and s-convex functions.

In this work, we introduce a new class of fuzzy processes, the so-called h-preinvex fuzzy processes. This class contains several well-known classes of fuzzy processes such as convex and s-convex.

The plan of the paper is as follows. In Section 2, we introduce the notations and definitions used throughout the paper. In Section 3, we establish the main results and finally, in Section 4, we show some, properties and the connection with fuzzy integral for fuzzy set-valued maps.

2. Preliminaries

Let R^n denote the n-dimensional Euclidean space.

A fuzzy subset of R^n is a function $u: R^n \to [0,1]$. Let $\mathcal{F}(R^n)$ denote the set of all nonempty fuzzy sets in R^n. A fuzzy set u is called convex [Lowen 1980], if

$$u(ty_1 + (1-t)y_2) \geq \min\{u(y_1), u(y_2)\},$$

for all $y_1, y_2 \in \text{supp}\,(u) = \overline{\{y: u(y) > 0\}}$ and $t \in (0,1)$.

We shall define addition and scalar multiplication on $\mathcal{F}(R^n)$ by the usual extension principle, i.e. for $u, v \in \mathcal{F}(R^n)$ and $\lambda \in R$, $u + v$ and $\lambda \cdot u$ are defined for any $y \in R^n$ by

$$(u+v)(y) = \sup_{y_1, y_2: y_1 + y_2 = y} \min(u(y_1), v(y_2))$$

and

$$(\lambda u)(y) = \begin{cases} u\left(\frac{y}{\lambda}\right) \text{ if } \lambda \neq 0, \\ \mathcal{X}_{\{0\}}(y) \text{ if } \lambda = 0, \end{cases}$$

where for any subset $A \subseteq R^n$, $\mathcal{X}_A$ denotes the characteristic function of A.

We can define an order $\subseteq$ on $\mathcal{F}(R^n)$ by setting

$$u \subseteq v \Leftrightarrow u(y) \leq v(y), \forall y \in R^n.$$

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Let $u \in \mathcal{F}(R^n)$. For $0 < \alpha \leq 1$, we denote $[u]^\alpha = \{y \in R^n : u(y) \geq \alpha\}$ the α-level set of u. $[u]^0 = \text{supp}(u) = \overline{\{y : u(y) > 0\}}$ is called the support of u.

A fuzzy set $u: R^n \to [0,1]$ is said to be a fuzzy compact set if $[u]^\alpha$ is compact for all $\alpha \in [0,1]$. For any fuzzy compact sets u and v it is verified that

- $u \subseteq v \Leftrightarrow [u]^\alpha \subseteq [v]^\alpha, \forall \alpha \in [0,1]$,
- $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \lambda$ and $\forall \alpha \in [0,1]$,
- $[u+v]^\alpha = [u]^\alpha + [v]^\alpha, \forall \alpha \in [0,1]$.

Definition 2.1. *A fuzzy set* $u: R^n \to [0,1]$ *is called h-convex fuzzy set iff*

$$u(h(t)x_1 + h(1-t)x_2) \geq \min\{u(x_1), u(x_2)\}$$

for all $x_1, x_2 \in R^n$, $t \in (0,1)$ *where* $h: J \to R$ *is a non-negative function*, $h \not\equiv 0$ *and* J *is an interval*, $(0,1) \subset J$.

Definition 2.2. [Weir, Mond 1988] *Let* X *be a nonempty set in* R^n *and* $x \in X$.

Then the set X *is said to be invex at* x *with respect to* $\eta: X \times X \to R^n$, *if*

$$x + t\eta(y,x) \in X$$

for all $y \in X$ *and* $t \in [0,1]$.

X is said to be an invex set with respect to η, if X is invex at each $x \in X$.

Definition 2.3. *A fuzzy set* $u: X \times R^n \to [0,1]$ *is called* (η, h)-*convex fuzzy set iff*

$$u(x_1 + t\eta(x_2,x_1), h(t)y_1 + h(1-t)y_2) \geq \min\{u(x_1,y_1), u(x_2,y_2)\}$$

for all $x_1, x_2 \in X$, $y_1, y_2 \in R^n$ *and* $t \in [0,1]$.

Definition 2.4. *The graph of a mapping* $F: X \to \mathcal{F}(R^n)$, *denoted by* G_F, *is a fuzzy set in* $X \times R^n$ *such that for any* $(x,y) \in X \times R^n$

$$G_F(x,y) = F(x)(y).$$

Definition 2.5. *The function* f *on the invex set* X *is said to be preinvex with respect to* η, *if*

$$f\big(x + t\eta(y,x)\big) \leq h(t)f(x) + h(1-t)f(y)$$

for each $x, y \in X$ *and* $t \in (0,1)$.

Let us note that:

- if $\eta(y,x) = y - x$ then we get the definition of h-convex function introduced by S. Varošanec [2007],
- if $h(t) = t$ then our definition reduces to the definition of preinvex function [Weir, Mond 1988],
- if $\eta(y,x) = y - x$ and $h(t) = t$ then we obtain the definition of convex function,
- if $\eta(y,x) = y - x$ and $h(t) = t^s$ then our definition reduces to the definition of s-convex function [Breckner 1978].

3. *h*-Preinvex fuzzy processes

In this section, we present some basic properties of h-preinvex fuzzy processes.

Definition 3.1. *A mapping F from invex set $X \subset R^m$ into $\mathcal{F}(R^n)$ is called h-preinvex fuzzy process iff for any $x_1, x_2 \in X$ and $t \in (0,1)$ it satisfies the condition*

$$h(t)F(x_1) + h(1-t)F(x_2) \subseteq F(x_1 + t\eta(x_2, x_1)),$$

where $h: J \to R$ is a positive function and J is an interval, $(0,1) \subset J$.

If $\eta(x_2, x_1) = x_2 - x_1$ then such a mapping we will call h – convex fuzzy process.

Remark 3.1. If $\eta(x_2, x_1) = x_2 - x_1$ and $h(t) = t^s$ then our definition reduces to the definition of s-convex fuzzy process defined by Y. Chalco--Cano and M.A. Rojas-Medar [2004].

Example 3.1. Consider $F: X \to \mathcal{F}(R)$ defined by

$$F(x) = \chi_{[f(x),\infty]}$$

where $f(x)$ is h-preinvex function from X into R. Then F is h-preinvex process.

Theorem 3.1. A mapping $F: X \to \mathcal{F}(R^n)$ is a h-preinvex fuzzy process if and only if

$$F(x_1 + t\eta(x_2, x_1))(y) \geq \sup_{y_1, y_2 : h(t)y_1 + h(1-t)y_2 = y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}$$

for all $x_1, x_2 \in X$ and $y \in R^n$.

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Proof. Suppose that F is h-preinvex fuzzy process. Let $x_1, x_2 \in X$, $t \in (0,1)$ and $y \in R^n$. Then, from Definition 3.1 we have

$$\begin{aligned} F\big(x_1 + t\eta(x_2,x_1)\big)(y) &\geq \big(h(t)F(x_1) + h(1-t)F(x_2)\big)(y) \\ &= \sup_{y_1,y_2: y_1+y_2=y} \min\{h(t)F(x_1)(y_1), h(1-t)F(x_2)(y_2)\} \\ &= \sup_{y_1,y_2: y_1+y_2=y} \min\left\{F(x_1)\left(\frac{y_1}{h(t)}\right), F(x_2)\left(\frac{y_2}{h(1-t)}\right)\right\} \\ &= \sup_{y_1,y_2: h(t)y_1+h(1-t)y_2=y} \min\{F(x_1)(y_1), F(x_2)(y_2)\}. \end{aligned}$$

Conversely, let us suppose that the inequality from the theorem is satisfied. Then for all $x_1, x_2 \in X$, $t \in (0,1)$ and $y \in R^n$, we have

$$\begin{aligned} &F\big(x_1 + t\eta(x_2,x_1)\big)(y) \\ &\geq \sup_{y_1,y_2: h(t)y_1+h(1-t)y_2=y} \min\{F(x_1)(y_1), F(x_2)(y_2)\} \\ &= \sup_{y_1',y_2': y_1'+y_2'=y} \min\left\{F(x_1)\left(\frac{y_1'}{h(t)}\right), F(x_2)\left(\frac{y_2'}{h(1-t)}\right)\right\} \\ &= \sup_{y_1',y_2': y_1'+y_2'=y} \min\{h(t)F(x_1)(y_1'), h(1-t)F(x_2)(y_2')\} \\ &= \big(h(t)F(x_1) + h(1-t)F(x_2)\big)(y) \end{aligned}$$

which implies that F is h-preinvex fuzzy process.

Theorem 3.2. Let $F: R^m \to \mathcal{F}(R^n)$ be a mapping such that

(1) $F(x_1 + x_2) \supseteq F(x_1) + F(x_2)$, $\quad \forall x_1, x_2 \in R^m$,
(2) $F(tx) = h(t)F(x)$, $\forall t > 0$ and $\forall x \in R^m$.

Then, F is h-preinvex fuzzy process.

Proof. Let $x_1, x_2 \in R^m$, $t \in (0,1)$ and $y \in R^n$. Then, from the addition and scalar multiplication on $\mathcal{F}(R^n)$, and from the conditions (1) and (2), we have

$$\begin{aligned} F(tx_1 + (1-t)x_2)(y) &\geq \Big(F(tx_1) + F\big((1-t)x_2\big)\Big)(y) \\ &= \sup_{y_1,y_2: y_1+y_2=y} \min\{F(tx_1)(y_1), F((1-t)x_2)(y_2)\} \\ &= \sup_{y_1',y_2':\, h(t)y_1'+h(1-t)y_2'=y} \min\{F(tx_1)(h(t)y_1'), F((1 \\ &\quad - t)x_2)(h(1-t)y_2')\} \end{aligned}$$

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$$= \sup_{y_1', y_2' :\, h(t)y_1' + h(1-t)y_2' = y} \min\{(h(t)F(x_1))(h(t)y_1'), (h(1-t)F(x_2))(h(1-t)y_2')\}$$

$$= \sup_{y_1', y_2' :\, h(t)y_1' + h(1-t)y_2' = y} \min\{F(x_1)(y_1'), F(x_2)(y_2')\}$$

which means that F satisfies the inequality from the Theorem 1. Therefore, F is h-preinvex fuzzy process.

Theorem 3.3. If $F: R^m \to \mathcal{F}(R^n)$ is a h-preinvex fuzzy process and for any $t \in (0,1)$ and $x \in X$ $x = x + t\eta(x,x)$ then for any $x \in X$, $F(x)$ is a h-convex fuzzy set.

Proof. Let $x \in X$. Since $x = x + t\eta(x,x)$ for all $t \in (0,1)$, by the definition of h-preinvex fuzzy processes, we have for all $y_1, y_2 \in R^n$ and $t \in (0,1)$,

$$F(x)(h(t)y_1 + h(1-t)y_2) = F(x + t\eta(x,x))(h(t)y_1 + h(1-t)y_2)$$

$$\geq \sup_{\substack{y_1', y_2':\\ h(t)y_1' + h(1-t)y_2' = h(t)y_1 + h(1-t)y_2}} \min\{F(x)(y_1'), F(x)(y_2')\}$$

$$\geq \min\{F(x)(y_1), F(x)(y_2)\}.$$

Hence, $F(x)$ is a h-convex fuzzy set. This completes the proof.

Theorem 3.4. The graph of h-preinvex fuzzy process is a (η, h)-convex fuzzy set in $X \times R^n$.

Proof. Let $x_1, x_2 \in X$ and $y_1, y_2 \in R^n$ and $t \in (0,1)$.

Then we have

$$G_F(x_1 + t\eta(x_2,x_1), h(t)y_1 + h(1-t)y_2))$$

$$= F(x_1 + t\eta(x_2,x_1))(h(t)y_1 + h(1-t)y_2)$$

$$\geq \sup_{\substack{y_1', y_2':\\ h(t)y_1' + h(1-t)y_2' = h(t)y_1 + h(1-t)y_2}} \min\{F(x_1)(y_1'), F(x_2)(y_2')\}$$

$$\geq \min\{F(x_1)(y_1), F(x_2)(y_2)\} = \min\{G_F(x_1,y_1), G_F(x_2,y_2)\}.$$

So, G_F is a (η, h)-convex fuzzy set in $X \times R^n$.

Theorem 3.5. If F is a h-preinvex fuzzy process then for any $\alpha \in (0,1]$

$$[F(x_1 + t\eta(x_2,x_1))]^\alpha \supset h(t)[F(x_1)]^\alpha + h(1-t)[F(x_2)]^\alpha$$

for any $x_1, x_2 \in X$ and $t \in (0,1)$.

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Proof. Let $x_1, x_2 \in X$ and $t \in (0,1)$, and $\alpha \in (0,1]$. Then we have

$$[F(x_1 + t\eta(x_2, x_1))]^\alpha = \{y \in R^n : F(x_1 + t\eta(x_2, x_1))(y) \geq \alpha\},$$
$$[F(x_1)]^\alpha = \{y_1 \in R^n : F(x_1)(y_1) \geq \alpha\},$$
$$[F(x_2)]^\alpha = \{y_2 \in R^n : F(x_2)(y_2) \geq \alpha\}.$$

Moreover

$$h(t)[F(x_1)]^\alpha + h(1-t)[F(x_2)]^\alpha = \{y = h(t)y_1 + h(1-t)y_2 : y_1 \in [F(x_1)]^\alpha, y_2 \in [F(x_2)]^\alpha\}$$

and

$$F(x_1 + t\eta(x_2, x_1))(y) \geq \sup_{\substack{y_1, y_2: \\ h(t)y_1 + h(1-t)y_2 = y}} \min\{F(x_1)(y_1), F(x_2)(y_2)\}.$$

This means that if $y \in h(t)[F(x_1)]^\alpha + h(1-t)[F(x_2)]^\alpha$ then $y \in \left[F\left(x_1 + t\eta(x_2, x_1)\right)\right]^\alpha$, i.e.

$$[F(x_1 + t\eta(x_2, x_1))]^\alpha \supset h(t)[F(x_1)]^\alpha + h(1-t)[F(x_2)]^\alpha$$

what completes the proof.

4. Hadamard's inequality

In this section, we present some inequalities of Hadamard type for *h*-preinvex fuzzy processes.

Let $\mathcal{F}_K(R^n)$ $\left(\mathcal{F}_C(R^n)\right)$ denote the set of all fuzzy compact (compact and convex) sets and $K(R^n)$ the set of all nonempty, compact (crisp) subsets of R^n.

R.L. Aumann [1965] introduced the following definition of the integral of set-valued function.

Definition 4.1. The integral of a measurable set-valued function $G: [a,b] \to K(R^n)$ is defined by

$$\int_a^b G dt = \left\{\int_a^b g(t)dt : g \in S(G)\right\},$$

where $\int_a^b g(t)dt$ is the Bochner-integral and $S(G)$ is the set of all integrable selectors of G, i.e.,

$$S(G) = \{g \in L^1([a,b], R^n) : g(t) \in G(t) \ a.e.\}.$$

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For a mapping $F:[a,b]\to\mathcal{F}_K(R^n)$ let us define a mapping $F_\alpha:[a,b]\to K(R^n)$ by $F_\alpha(x)=[F(x)]^\alpha$, $\forall\alpha\in[0,1]$.

Definition 4.2 [Puri, Ralescu 1987]. *A mapping F is called measurable (integrably bounded) if* F_α is *measurable (integrably bounded) for all* $\alpha\in[0,1]$.

Proposition 4.1 [Puri, Ralescu 1987]. If $F:[a,b]\to\mathcal{F}_K(R^n)$ is integrably bounded, then there exists a unique fuzzy set $u\in\mathcal{F}_K(R^n)$ such that $[u]^\alpha=\int_a^b F_\alpha dt$, $\forall\alpha\in[0,1]$.

The fuzzy set $u\in\mathcal{F}_K(R^n)$ in the above Proposition defines integral of F by $\int_a^b Fdt=u$ iff $[u]^\alpha=\int_a^b F_\alpha dt$, for every $\alpha\in[0,1]$.

Many important inequalities have been established for the class of convex functions, but the most famous is the Hermite–Hadamard's inequality. These double inequalities are stated as:

(a) for convex function

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{2},$$

(b) for *s* – convex function (see [Dragomir, Fitzpatrick 1999])

$$2^{s-1}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\frac{f(a)+f(b)}{s+1},$$

(c) for *h*-convex function (see [Sarikaya et al. 2008])

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)dx\le\int_0^1 h(t)dt[f(a)+f(b)],$$

(d) for preinvex function (see [Noor 2009])

$$f\left(a+\frac{1}{2}\eta(b,a)\right)\le\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx\le\frac{f(a)+f(b)}{2}.$$

Now, we prove an inequality of Hermite–Hadamard type for *h*-preinvex fuzzy processes. But first we need the following assumption regarding the function η which is due to S.R. Mohan and S.K. Neogy [1995]:

Condition C. Let $X\subseteq R$ be an open invex subset with respect to η. For any $x,y\in X$ and any $t\in[0,1]$

$$\eta(y,y+t\eta(x,y))=-t\eta(x,y)$$

$$\eta(x,y+t\eta(x,y))=(1-t)\eta(x,y).$$

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Note that every $x, y \in X$ and every $t_1, t_2 \in [0,1]$ from the above Condition C, we have

$$\eta(y + t_2\eta(x,y), y + t_1\eta(x,y)) = (t_2 - t_1)\eta(x,y).$$

Theorem 4.1. Let $F : [a, a + \eta(b,a)] \to \mathcal{F}_K(R^n)$ be h-preinvex and integrably bounded fuzzy process, with $a < a + \eta(b,a)$. Then, if $h\left(\frac{1}{2}\right) \neq 0$ and the Condition C is fulfilled we have

$$\{F(a) + F(b)\}\int_0^1 h(t)dt \subseteq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} F(x)dx \subseteq \frac{1}{2h\left(\frac{1}{2}\right)} F\big(a + \eta(b,a)\big).$$

Proof. Since F is h-preinvex fuzzy process we have that

$$h(t)F(a) + h(1-t)F(b) \subseteq F(a + t\eta(b,a))$$

for all $t \in (0,1)$. Integrating on $[0,1]$, we obtain

$$\int_0^1 F(a + \eta(b,a))dt \supseteq \int_0^1 h(t)F(a) + h(1-t)F(b)dt = \{F(a) + F(b)\} \cdot \int_0^1 h(t)dt.$$

Making the change of variable $x = a + t\,\eta(b,a)$ we obtain

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} F(x)dx \supseteq \{F(a) + F(b)\}\int_0^1 h(t)dt.$$

To prove the second inclusion, we observe that from the Condition C it follows that

$$F\left(a + \frac{1}{2}\eta(b,a)\right) = F\left(a + t\,\eta(b,a) + \frac{1}{2}\eta(a + (1-t)\eta(b,a), a + t\,\eta(b,a)\right)$$
$$\supseteq h\left(\frac{1}{2}\right) F\big(a + t\,\eta(b,a)\big) + F\big(a + (1-t)\eta(b,a)\big).$$

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Integrating this inclusion on [0,1] we get

$$\int_0^1 F\left(a+\frac{1}{2}\eta(b,a)\right)dt \supseteq h\left(\frac{1}{2}\right)\left[\int_0^1 F\left(a+\frac{1}{2}\eta(b,a)\right)dt + \int_0^1 F\,(a+(1-t)\eta(b,a))dt\right].$$

Since

$$\int_0^1 F\,(a+t\,\eta(b,a))dt = \int_0^1 F\,(a+(1-t)\eta(b,a))dt = \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} F(x)dx,$$

it follows that

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} F(x)dx \subseteq \frac{1}{2h\left(\frac{1}{2}\right)} F\left(a+\frac{1}{2}\eta(b,a)\right)$$

what completes the proof.

Remark 4.1. If $\eta(b,a)=b-a$ and $h(t)=t^s$ then our inequality reduces to the inequality for *s*-convex fuzzy processes obtained by R. Osuna-Gomez et al. [2004].

Now, for a mapping $f\colon [a,b]\to R$ let us define a mapping

$$H(t)=\frac{1}{b-a}\int_a^b f\left(t\,x+(1-t)\frac{a+b}{2}\right)dx.$$

Some properties of this mapping for convex function are given in [Dragomir 1992]. We extend these properties for *h*-convex fuzzy processes. Let $F:[a,b]\to \mathcal{F}_C(R^n)$ be an integrably bounded fuzzy process and define

$$H_F(t)=\frac{1}{b-a}\int_a^b F\left(t\,x+(1-t)\frac{a+b}{2}\right)dx$$

for $t\in[0,1]$.

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Theorem 4.2. Let F be a h-convex integrably bounded fuzzy process on an interval $[a,b]$. Then H_F is h-convex on $[0,1]$ and

$$H_F(t) \subseteq \frac{1}{2h\left(\frac{1}{2}\right)} H_F(0), \forall t \in [0,1].$$

Proof. The h-convexity of the mapping H_F is a consequence of the h-convexity of the mapping F. Namely, for $t_1, t_2 \in [0,1]$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ we have:

$$H_F(\alpha t_1 + \beta t_2)$$

$$= \frac{1}{b-a}\int_a^b F\left((\alpha t_1 + \beta t_2)x + (1-(\alpha t_1+\beta t_2))\frac{a+b}{2}\right)dx$$

$$= \frac{1}{b-a}\int_a^b F\left(\alpha(t_1 x + (1-t_1)\frac{a+b}{2} + \beta(t_2 x + (1-t_2)\frac{a+b}{2}\right)dx$$

$$\supseteq \frac{1}{b-a}\int_a^b h(\alpha) F\left(t_1\, x + (1-t_1)\frac{a+b}{2}\right)dx$$

$$+ \frac{1}{b-a}\int_a^b h(\beta) F\left(t_2\, x + (1-t_2)\frac{a+b}{2}\right)dx$$

$$= h(\alpha)H_F(t_1) + h(\beta)H_F(t_2)$$

what means that H_F is h-convex.

Now, let $t \in [0,1]$. Taking $y = t\,x + (1-t)\frac{a+b}{2}$ we obtain

$$H_F = \frac{1}{d-c}\int_c^d F(y)dy$$

where $c = ta + (1-t)\frac{a+b}{2}$ and $d = tb + (1-t)\frac{a+b}{2}$. By Theorem 4.1 for $\eta(d,c) = d - c$ we have

$$\frac{1}{d-c}\int_c^d F(y)dy \subseteq \frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{c+d}{2}\right) = \frac{1}{2h\left(\frac{1}{2}\right)} F\left(\frac{a+b}{2}\right)$$

$$= \frac{1}{2h\left(\frac{1}{2}\right)} H_F(0).$$

what completes the proof.

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