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THE NUMBER OF STABLE MATCHINGS IN MODELS OF THE GALE–SHAPLEY TYPE WITH PREFERENCES GIVEN BY PARTIAL ORDERS

From the famous Gale–Shapley theorem we know that each classical marriage problem admits at least one stable matching. This fact has inspired researchers to search for the maximum number of possible stable matchings, which is equivalent to finding the minimum number of unstable matchings among all such problems of size *n*. In this paper, we deal with this issue for the Gale–Shapley model with preferences represented by arbitrary partial orders. Also, we discuss this model in the context of the classical Gale–Shapley model.

Key words: *combinatorial problems, stable matching, Gale–Shapley model*

1. Preliminaries

The problem of determining the maximum number of possible stable matchings among all the classical marriage problems of size n was posed by Knuth [6] and still remains an open question. Knuth established that this number exceeds $2^{n/2}$ for $n \geq 2$ and Gusfield and Irving showed [3] that when n is a power of 2 it is at least 2^{n-1}, which can be improved to $(2.28)^n/(1+\sqrt{3})$ based on the construction given by Irving and Leather [4]. Some properties of this number analyzed as a function of the problem size were considered in [10]. The first upper bound on this number was derived in [1]. It approximately equals three quarters of all $n!$ possible matchings and is still far from the expected number of stable matchings, which is asymptotic to $e^{-1}n\ln n$ for $n \to \infty$ [8].

In many practical applications of the marriage problem, the lists of preferences for prospective partners allow incomparable elements. In this context, consider the fol-

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lowing example. Let us assume that a company begins work on n different computer programming projects simultaneously, employing at the same time n people as project managers. These people possess such skills as: competence in human resource management, mobility, knowledge of spoken and programming languages, knowledge of the topic sets connected to the projects, access to different data bases. The task of the company management is to assign managers to the projects, so that each of them is responsible for exactly one project. Such an assignment is considered stable if there exists no unassociated pair (manager i and project j) such that manager i would prefer managing project j rather than the one assigned to him and, moreover, manager i would be better for project j than the manager who is presently assigned. Note that it may be hard to decide who is a better candidate for being the manager of a given project, because some of the managers' attributes are not comparable. It can happen that project i needs both the knowledge of a particular programming language and the ability to negotiate in Finnish. Assume that no manager can do both things, but there are two, let us say m_l, m_k who each possesses one of these skills. This fact shows the incomparability of m_l and m_k with respect to project i. Similarly, specific features of the projects can make two of them incomparable for a given manager. Usually, many attributes are needed to manage a given project and people can possess some of them. This means that the preferences of each manager are described by a partial order over the set of projects. Similarly, the appropriateness of a manager for a given project is given by a partial order over the set of all managers. In this situation, a problem of size n involves finding a matching between n managers and n projects with $2n$ indexed partially ordered sets, where n are partial orders described over the set of projects and n over the set of managers.

In this paper, we investigate the minimum number of unstable matchings in models of the Gale–Shapley type with preferences that are not necessarily linear. Among other things, we explain the difference between the classical model and the model considered in this paper. We show that if preferences are given by arbitrary partial orders, then, unlike the classical model, there is no upper bound on the number of stable matchings concerning all the problems of size n. On the other hand, if all $2n$ Hasse diagrams representing the preferences (for managers and for projects) have pairwise the same shape for two different problems of size n (they are pairwise isomorphic), then the number of stable matchings for both these problems can be bounded from above by the same expression (Theorem 1). In particular, we give a simpler form of this expression in the case when all $4n$ Hasse diagrams for two different problems of order n are isomorphic (Corollary 3). Note that the classical Gale–Shapley model is of this last kind, because all $4n$ Hasse diagrams for any two problems of size n are isomorphic (in fact they are represented by linear orders over an n-element set).

Since people have different and incomparable attributes, the example mentioned above can be written in the language of a marriage problem. We use this classical description in the remaining part of the paper.

2. Main results

In general, we follow the notation and terminology of [9]. Let W be a set of n women $w_1, ..., w_n$ and M be a set of n men $m_1, ..., m_n$ and for a natural number n, let the notation $[n]$ denote the set $\{1, ..., n\}$. Next, suppose that for a fixed $j \in [n]$, the partial order $p(m_j)$ represents the preferences of the man m_j over the set of women. This means that for given $s, k, j \in [n]$ we have $(w_s, w_k) \in P(m_j)$ if and only if either $s = k$ or m_j prefers w_s to w_k. It follows that $(w_s, w_k) \notin P(m_j)$ if and only if $k \neq s$ and either $(w_k, w_s) \in P(m_j)$ or w_s and w_k are incomparable for m_j. Similarly, for fixed $i \in [n]$, a partial order $P(w_i)$ represents the preferences of the woman w_i over the set of men. Thus preferences for prospective partners are represented by the set $\mathbf{P} = \{P(m_1), ..., P(m_n), P(w_1), ..., P(w_n)\}$. Next, a triple $(M, W; \mathbf{P})$ denotes a specific marriage market (of size n). It should be mentioned here that, according to this model, we do not allow any individual to remain single and assume that all prospective partners are preferred to the option of being single. Moreover, all preferences are transitive, but two alternatives can be incomparable. This means that individuals are not necessarily rational.

A matching μ is an arbitrary bijection of W onto M. For simplicity, we write $\mu(i) = j$ instead of $\mu(w_i) = m_j$. By $\Omega(i_1, j_1, i_2, j_2)$ we denote the set of all matchings μ satisfying $\mu(i_1) = j_1$ and $\mu(i_2) = j_2$. In this description, we implicitly assume that $i_1 \neq i_2$ and $j_1 \neq j_2$. A pair w_i, m_j is said to block a matching $\mu \in \Omega(i, j_1, i_2, j)$ in $(M, W; \mathbf{P})$ if $(m_j, m_{j_1}) \in P(w_i)$ and $(w_i, w_{i_2}) \in P(m_j)$. A matching μ is unstable for $(M, W; \mathbf{P})$ if there exists at least one pair blocking μ in $(M, W; \mathbf{P})$, otherwise μ is stable for $(M, W; \mathbf{P})$.

For a given marriage market $(M, W; \mathbf{P})$ of size n and for $i, j \in [n]$, let $a_{i,j} = |\{s: (m_j, m_s) \in P(w_i) \text{ and } s \neq j\}|$ and $b_{i,j} = |\{s: (w_j, w_s) \in P(m_i) \text{ and } s \neq j\}|$. The number $a_{i,j}$ gives us information on how many of the men are worse than m_j according to w_i, and $b_{i,j}$ tells how many of the women are worse than w_j according to m_i. Next, for fixed $i \in [n]$ and $P(m_i)$, by $\mathbf{M}_i$ we denote the set of all linear orders over W, say $(w_{s_1^i}, ..., w_{s_n^i})$, such that $b_{i,s_1^i} \geq ... \geq b_{i,s_n^i}$. Similarly, for fixed $i \in [n]$ and $P(w_i)$, by $\mathbf{W}_i$ we denote the set of all linear orders over M, say $(m_{s_1^i}, ..., m_{s_n^i})$, such that $a_{i,s_1^i} \geq ... \geq a_{i,s_n^i}$.

Proposition 1. Let $(M, W; \mathbf{P})$ be a marriage market of size n and let $\mathbf{P}^*$ be an arbitrary set $\{P^*(m_1), ..., P^*(m_n), P^*(w_1), ..., P^*(w_n)\}$ such that for all $i \in [n]$ the conditions

$P^*(m_i) \in \mathbf{M}_i$ and $P^*(w_i) \in \mathbf{W}_i$ are satisfied. If a matching μ is stable for $(M, W; \mathbf{P})$, then μ is stable for $(M, W; \mathbf{P})$.

Proof. Suppose that Proposition 1 does not hold, i.e. μ is stable for $(M, W; \mathbf{P}^*)$ and unstable for $(M, W; \mathbf{P})$. It follows that there exists a pair w_s, m_k blocking μ when preferences are given by $\mathbf{P}$, which consequently means that there exists a pair (j_1, j_2) such that $\mu \in \Omega(s, j_1, i_2, k)$ and $(m_k, m_{j_1}) \in P(m_s)$ and $(w_s, w_{i_2}) \in P(w_k)$ and also $k \neq j_1$ and $s \neq \imath_2$. From the definition of the numbers $a_{i,j}$, $b_{i,j}$, we obtain $a_{s,k} > a_{s,j_1}$ and $b_{k,s} > b_{k,i_2}$. This yields $(m_k, m_{j_1}) \in P^*(w_s)$ and $(w_s, w_{i_2}) \in P^*(m_k)$. Using the definition of a blocking pair once again, we obtain that w_s, m_k blocks μ when the preferences are given by $\mathbf{P}^*$, contrary to the initial assumption.

Let $\sigma(M, W; \mathbf{P})$ denote the set of all stable matchings for the marriage market $(M, W; \mathbf{P})$. Taking into account Proposition 1 we immediately obtain the following result:

Corollary 1. If $(M, W; \mathbf{P})$ is a marriage market of size n, then $\cup\sigma(M, W; \mathbf{P}^*) \subseteq \sigma(M, W; \mathbf{P})$, where the union is taken over all $\mathbf{P}^* = \{P^*(m_1), ..., P^*(m_n), P^*(w_1), ..., P^*(w_n)\}$ such that $P^*(m_i) \in \mathrm{M}_i$ and $P^*(w_i) \in \mathbf{W}_i$ for each $i \in [n]$.

From Gale and Shapley [2] we know that if all the partial orders in $\mathbf{P}$ are linear, then there exists at least one stable matching for $(M, W; \mathbf{P})$. In the light of the above corollary, this fact implies the next result.

Corollary 2. If $(M, W; \mathbf{P})$ is a marriage market of size n, then there exists at least one stable matching for $(M, W; \mathbf{P})$.

Unfortunately, the conversion to Proposition 1 is not true. For instance in Fig. 1 we present a set $\mathbf{P} = \{P(m_1), P(m_2), P(m_3), P(w_1), P(w_2), P(w_3)\}$ of partial orders, such that the matching μ defined by $\mu(i) = i$ for each permissible i, is stable for $(M, W; \mathbf{P})$. Moreover, μ is unstable for any marriage market $(M, W; \mathbf{P}^*)$ such that $\mathbf{P}^* = \{P^*(m_1), P^*(m_2), P^*(m_3), P^*(w_1), P^*(w_2), P^*(w_3)\}$, where $P^*(m_i) \in \mathbf{M}_i$, $P^*(w_i) \in \mathbf{W}_i$ with $i \in \{1, 2, 3\}$. Indeed, for each $P^*(m_1) \in \mathbf{M}_1$ we have $(w_2, \mathrm{w}_1) \in P^*(m_1)$ and for each $P^*(w_2) \in w_2$ we have $(m_1, m_2) \in P^*(w_2)$. Thus the pair w_2, m_1 blocks μ in the marriage market $(M, W; \mathbf{P}^*)$.

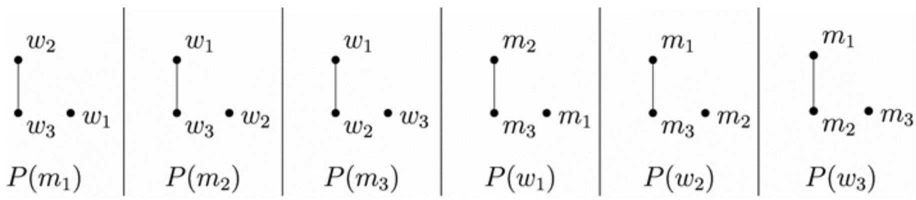

Fig. 1. An illustration of the fact that the containment in Proposition 1 can be strict

Of course there are many examples for which the inclusion in Corollary 1 can be substituted by an equality. Analysis of particular marriage markets having preferences with ties gives us infinitely many such examples. Markets of this type have been considered in many papers, especially from the aspect of the existence of stable matchings [5, 7].

A partial order γ over $[n]$ is called simple if there exists a partition of $[n]$ into k parts $X_1, ..., X_k$, such that $(a, b) \in \gamma$ if and only if $a = b$ or $a \in X_i$, $b \in X_j$ and $i < j$.

Proposition 2. Let $(M, W; \mathbf{P})$ be a marriage market of order n such that all the partial orders from $\mathbf{P}$ are simple. If μ is a stable matching for $(M, W; \mathbf{P})$, then there exists a set $\mathbf{P}^*$ consisting of linear orders $P^*(m_1), ..., P^*(m_n), P^*(w_1), ..., P^*(w_n)$ such that $P^*(m_i) \in \mathbf{M}_i$ and $P^*(w_i) \in \mathbf{W}_i$ for $i \in [n]$ and μ is stable for $(M, W; \mathbf{P}^*)$.

Proof. Let $i, p \in [n]$ satisfy $\mu(i) = p$. Because $P(m_p)$, $P(w_i)$ are simple, there exist partitions $X_1, ..., X_k$ and $Y_1, ..., Y_q$ of W and M, respectively, and indices $s \in [k]$, $j \in [q]$ such that $w_i \in X_s$ and $m_p \in Y_j$. In addition, we know that all the elements of X_s are incomparable based on $P(m_p)$ and all the elements of Y_j are incomparable based on $P(w_i)$. We construct $P^*(m_p) \in \mathbf{M}_p$ and $P^*(w_i) \in \mathbf{W}_i$ in an arbitrary way that satisfies $(w_i, w_l) \in P^*(m_p)$ for each $w_l \in X_s$ and $(m_p, m_l) \in P^*(w_i)$ for each $m_l \in Y_q$. For every other pair of indices $i_1, p_1 \in [n]$ satisfying the equality $\mu(i_1) = p_1$, we use a similar argument to produce a set $\mathbf{P}^* = \{P^*(m_1), ..., P^*(m_n), ..., P^*(w_1), ..., P^*(w_n)\}$ that satisfies the theorem.

It should be mentioned that marriage markets with all preferences given by simple orders are sometimes considered in the literature as markets with linear preferences and ties. Individuals in such markets are indifferent between some elements.

Above we indicated a connection between any marriage market with preferences given by partial orders and some set of markets with preferences given by linear orders. It was shown in [1] that the linearity of preferences guarantees the existence of a non-trivial lower bound on the number of unstable matchings. The ratio of this bound to the number $n!$ of all possible matchings tends to 1/4 when n tends to infinity. We have no such conclusion if for at least one man or at least one woman the preferences for potential spouses are given by a non-linear order. One specific situation occurs when for each woman any two men are incomparable and for each man any two women are incomparable. Namely, in this case, any matching is stable. Thus we cannot guarantee the existence of unstable matchings at all but if they exist, then their number can be bounded from below by parameters defined here. The corresponding result will be preceded by some helpful definitions.

For a marriage market $(M, W; \mathbf{P})$ of size n and for $i, j \in [n]$, let $A_{i,j}(M, W; \mathbf{P})$ denote the set of all tuples (i, j_1, i_1, j) such that w_i, m_j is a blocking pair for each matching $\mu \in \Omega(i, j_1, i_1, j)$. Evidently, $A_{i,j}(M, W; \mathbf{P}) = \{(i, j_1, i_1, j)$: $(m_j, m_{j_1}) \in P(w_i)$ and (w_i, w_{i_1})

$\in P(m_j)$ with $j \neq j_1$ and $i \neq i_1\}$, which means that $|A_{i,j}(M, W; \mathbf{P})| = a_{i,j}b_{j,i}$. Since each tuple (i, j_1, i_1, j) belonging to $A_{i,j}(M, W; \mathbf{P})$ corresponds to $(n-2)!$ unstable matchings from $\Omega(i, j_1, i_1, j)$ and because for any two tuples from $A_{i,j}(M, W; \mathbf{P})$ these sets of unstable matchings are disjoint (each of cardinality $(n-2)!$) we obtain the following fact.

Remark 1. Let $(M, W; \mathbf{P})$ be a marriage market of size n. If $i, j \in [n]$, then $|\sigma(M, W; \mathbf{P})| \leq n! - (n-2)!a_{i,j}b_{j,i}$.

Recall that two partial orders ρ_1, ρ_2 over the sets X, Y, respectively, are isomorphic if there exists a bijection $\psi: X \to Y$ such that for any $v, w \in X$ we have $(v, w) \in \rho_1$ if and only if $(\psi(v), \psi(w)) \in \rho_2$.

In the next theorem, we use Remark 1 to obtain an upper bound on the number $\max|\sigma(M, W; \mathbf{P})|$, where the maximum is taken over all marriage markets $(M, W; \mathbf{P})$ of size n, in which the partial orders that represent preferences for corresponding women and men are pairwise isomorphic. First, we focus on the general case, next we turn our attention to a special case, thus improving the upper bound obtained using the first approach.

Theorem 1. $(M, W; \mathbf{P})$ be a marriage market of size n with $n \geq 2$ and let for $i \in [n]$ the orderings $(s_1^i, \ldots, s_n^i)$, $(k_1^i, \ldots, k_n^i)$ of $[n]$ satisfy $a_{i,s_1^i} \geq \ldots \geq a_{i,s_n^i}$ and $b_{i,k_1^i} \geq \ldots \geq b_{i,k_n^i}$. If

$$c_1 = \begin{cases} \min\left\{a_{i,s^i_{(n/2)+1}} : i \in [n]\right\} \text{when } n \text{ is even} \\ \min\left\{a_{i,s^i_{(n+1)/2}} : i \in [n]\right\} \text{when } n \text{ is odd} \end{cases}, \quad d_1 = \begin{cases} \min\left\{b_{i,k^i_{n/2}} : i \in n\right\} \text{when } [n] \text{ is even} \\ \min\left\{b_{i,k^i_{(n+1)/2}} : i \in n\right\} \text{when } [n] \text{ is odd} \end{cases}$$

and

$$c_2 = \begin{cases} \min\left\{a_{i,s^i_{n/2}} : i \in [n]\right\} \text{when } n \text{ is even} \\ c_1 \text{ when } n \text{ is odd} \end{cases}, \quad d_2 = \begin{cases} \min\left\{b_{i,k^i_{(n/2)+1}} : i \in [n]\right\} \text{when } n \text{ is even} \\ d_1 \text{ when } n \text{ is odd} \end{cases}$$

then $|\sigma(M, W; \mathbf{P})| \leq n! - (n-2)! \max\{c_1d_1, c_2d_2\}$.

Moreover, $n! - (n-2)! \max\{c_1d_1, c_2d_2\}$ is an upper bound on $|\sigma(M, W; \mathbf{P}')|$ for any other marriage market $(M, W; \mathbf{P}')$ of size n such that for each $i \in [n]$ the partial orders $P(m_i)$ and $P'(m_i)$ are isomorphic and the partial orders $P(w_i)$ and $P'(w_i)$ are isomorphic.

Proof. First we shall show that the theorem holds for $(M, W; \mathbf{P})$. It is sufficient to prove that there exist indices $i_1, j_1, i_2, j_2 \in [n]$ such that $a_{i_1, j_1} \geq c_1$ and $b_{j_1, i_1} \geq d_1$ while

$a_{i_2, j_2} \geq c_2$ and $b_{j_2, i_2} \geq d_2$, before using Remark 1. By symmetry, it is enough to show the existence of indices i_1, j_1. To obtain a contradiction, assume that such indices i_1, j_1 do not exist. This means that for any pair of indices (i, j) either $a_{i,j} \leq c_1 - 1$ or $b_{j,i} \leq d_1 - 1$. We define two sets $A = \{(i, j, a_{i,j}): i, j \in [n]\}$ and $B = \{(i, j, b_{i,j}): i, j \in [n]\}$ and a mapping ψ that assigns the element $(j, i, b_{j,i})$ from B to the element $(i, j, a_{i,j})$ from A. Evidently, $|A| = |B| = n^2$ and ψ is a bijection.

In the remaining part of the proof we shall construct two sets $E \subseteq A$ and $F \subseteq B$ such that ψ maps E to a subset of F and the cardinality of F is less than the cardinality of E. Obviously, this contradicts the fact that ψ is a bijection.

First consider the case of even n. From the definition of c_1 it follows that for each $i \in [n]$, the inequality $a_{i,s^i_{(n/2)+1}} \geq c_1$ holds. Moreover, $a_{i,s^i_1} \geq ... \geq a_{i,s^i_{(n/2)+1}}$. Hence for each $i \in [n]$, the numbers $b_{s^i_1, j}, ..., b_{s^i_{(n/2)+1}, j}$ do not exceed $d_1 - 1$. Thus the values j in $b_{s^i_1, j}, ..., b_{s^i_{(n/2)+1}, j}$ satisfy $j \in D$, where $D = \{k^p_l : p \in [n], l \in \{n/2+1, ..., n\}\}$. Now the image under ψ of the set $E = \{(i, j, a_{i,j}): i \in \{n], j \in \{s^i_1, ..., s^i_{(n/2)+1}\}\}$ of cardinality $(n^2/2) + n$ is a subset of $F = \{(i, j, b_{i,j}) : i \in [n], j \in D\}$. Note that the cardinality of F is equal to $n^2/2$, which proves the theorem when n is even.

We use the same reasoning in the case of an odd n by constructing the same mapping ψ, sets A, B and their subsets $E = \{(i, j, a_{i,j}): i \in [n], j \in \{s^i_1, ..., s^i_{(n+1)/2}\}\}$ and $F = \{(i, j, b_{i,j}) : i \in [n], j \in D\}$, where $D = \{k^p_s : p \in [n], s \in \{(n+3)/2, ..., n\}\}$. By the definitions of c_1, d_1, we know that $\psi(E) \subseteq F$. Similarly to the previous case, the cardinality of E equals $n(n + 1)/2$ and the cardinality of F equals $n(n - 1)/2$, which contradicts the fact that ψ is a bijection.

Thus $n! - (n - 2)!\max\{c_1d_1, c_2d_2\}$ is an upper bound on $|\sigma(M, W; \mathbf{P})|$. The last statement of the theorem follows since all the numbers c_1, c_2, d_1, d_2 depend only on the shape of the Hasse diagrams and are independent of the assignments of women and men to them (isomorphic partial orders produce the same numbers c_1, c_2, d_1, d_2).

We illustrate Theorem 1 using a marriage market $(M, W; \mathbf{P})$ of size 5, where $\mathbf{P} = \{P(m_1), ..., P(m_5), P(w_1), ..., P(w_5)\}$ (Fig. 2). In this case, the orderings whose existence is assumed in Theorem 1 could be $(s^1_1, ..., s^1_5) = (1, 2, 3, 4, 5)$, $(s^2_1, ..., s^2_5) = (5, 4, 1, 2, 3)$, $(s^3_1, ..., s^3_5) = (2, 3, 4, 5, 1)$, $(s^4_1, ..., s^4_5) = (1, 2, 3, 4, 5)$, $(s^5_1, ..., s^5_5) = (4, 3, 2, 5, 1)$, $(k^1_1, ..., k^1_5) = (3, 1, 5, 2, 4)$, $(k^2_1, ..., k^2_5) = (1, 5, 4, 3, 2)$, $(k^3_1, ..., k^3_5) = (5, 1, 4, 2, 3)$, $(k^4_1, ..., k^4_5) = (2, 4, 3, 5, 1)$, $(k^5_1, ..., k^5_5) = (1, 5, 2, 3, 4)$. Next, we have $a_{1,s^1_3} = a_{2,s^2_3} = a_{3,s^3_3} = a_{5,s^5_3} = 2$ and $a_{4,s^4_3} = 1$, which gives $c_1 = 1$ (5 is an odd number). Moreover, $b_{1,k^1_3} = b_{2,k^2_3} = b_{3,k^3_3} = b_{4,k^4_3} = b_{5,k^5_3} = 2$, which implies that $d_1 = 2$. Thus, from

Theorem 1, we can find at least 12 unstable matchings for (m, W; P) and it does not depend on the assignment of the elements from the sets W, M to the vertices of Hasse diagrams presented in Fig. 2.

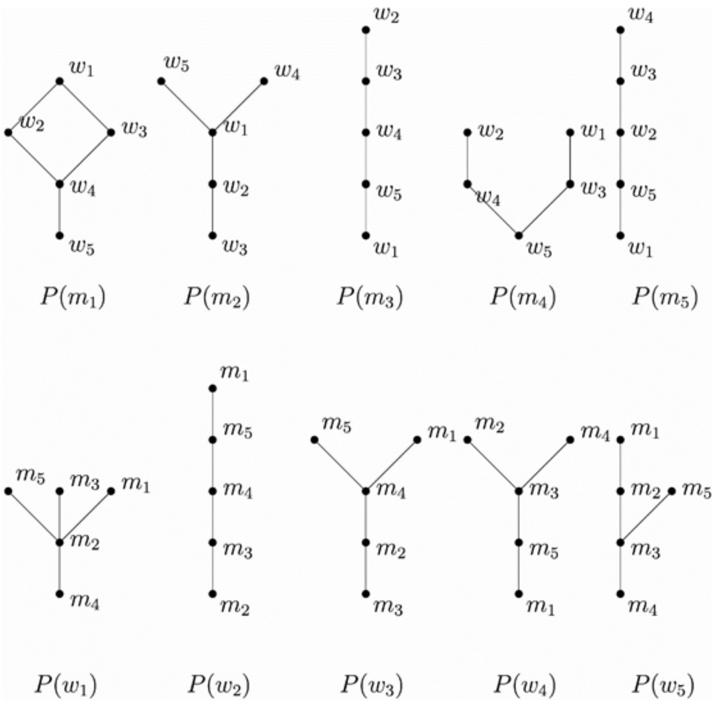

Fig. 2. The Hasse diagrams that represent preferences

For example, the same number of 12 unstable matchings is guaranteed in the marriage market (M, W; **P**) of size 5 whose set of preferences $\mathbf{P}' = \{P'(m_1), ..., P'(m_n), P'(w_1), ..., P'(w_n)\}$ is given in Fig. 3.

Corollary 3. Let (M, W; **P**) be a marriage market of size n such that $\mathbf{P} = \{P(m_1), ..., P(m_n), P(w_1), ..., P(w_n)\}$ and let all the partial orders from **P** be isomorphic to γ. Next, let γ be defined over $[n]$ such that for $e_i = |\{j \in [n]: j \neq i \text{ and } (i, j) \in \gamma\}|$. It follows that $e_1 \geq ... \geq e_n$. If

$$c = \begin{cases} e_{(n/2)+1}, \text{when } n \text{ is even} \\ e_{(n+1)/2}, \text{ when } n \text{ is odd} \end{cases} \qquad \text{and} \qquad d = \begin{cases} e_{n/2}, \text{when } n \text{ is even} \\ e_{(n+1)/2}, \text{ when } n \text{ is odd} \end{cases}$$

then $|\sigma(M, W; \mathbf{P})| \leq n! - cd(n-2)!$.

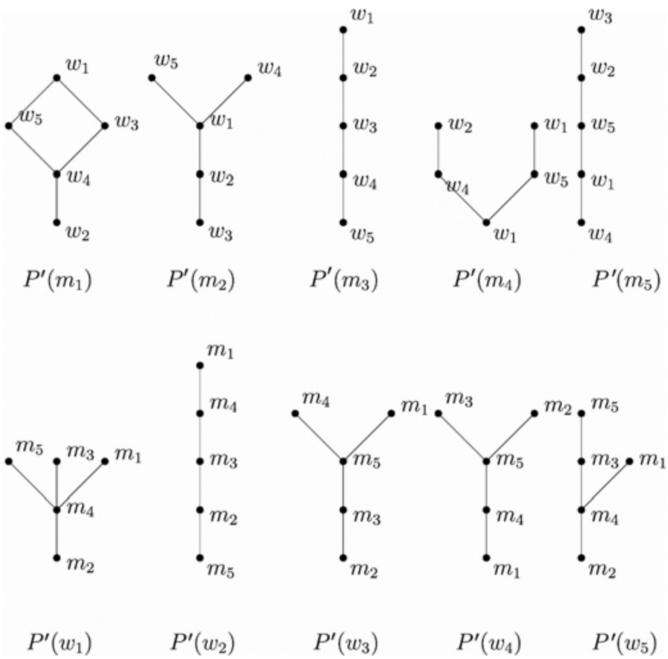

Fig. 3. The Hasse diagrams that represent preferences

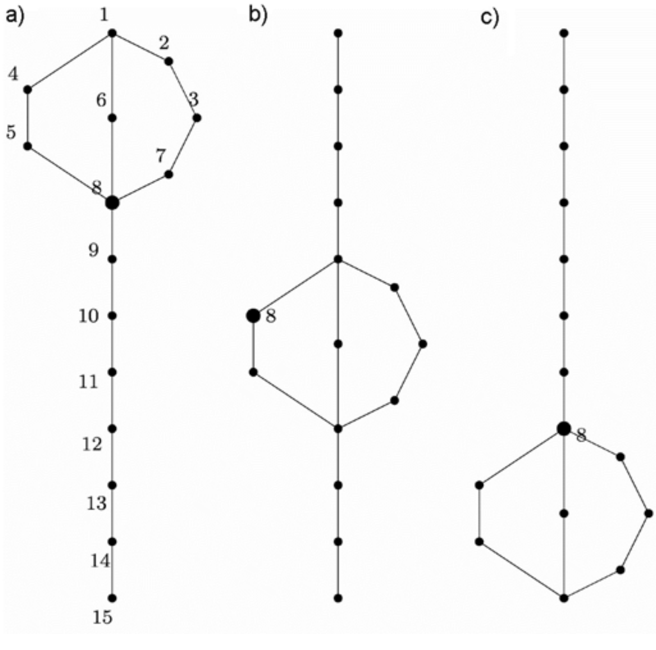

Fig. 4. Hasse diagrams

Note that Corollary 3 can be applied if all the partial orders from **P** are isomorphic. One such possibility occurs when the preferences for all women and all men are linear. In this case, Corollary 3 implies one of the main results given in [1]. To illustrate this corollary using another example, consider an arbitrary marriage market $(M, W; \mathbf{P})$ of size n for which all $P(m_i)$ and all $P(w_i)$ are isomorphic to the one given in Fig. 4a.

Note that the numbers e_i defined in Corollary 3 (the indexes i correspond to the vertices of Hasse diagram from Fig. 4a satisfy $e_1 = 14$, $e_2 = 10$, $e_3 = e_4 = 9$, $e_5 = e_6 = e_7 = 8$, $e_8 = 7$, $e_9 = 6$, $e_{10} = 5$, $e_{11} = 4$, $e_{12} = 3$, $e_{13} = 2$, $e_{14} = 1$, $e_{15} = 0$. This means that $c = e_{(15+1)/2} = e_8 = 7$. Hence, there exist at least $13! \times 49$ unstable matchings for the market $(M, W; \mathbf{P})$ and this does not depend on the correspondence between the individuals and the elements from the set [15].

Concluding remarks

It is worth noting that the upper bound on the number of stable matchings derived in this paper depends on the number of individuals that are worse than given participants of the marriage market. A detailed analysis of the numbers c_1, c_2, d_1, d_2 or c, d, respectively, leads to the conclusion that the value of this bound is very dependent on the participants who occupy central positions in the preference orderings. For a better understanding of this fact, let us consider four classes of marriage markets of size 15. The first, where all of the preferences are represented by linear orders, the second, where all the partial orders describing preferences are isomorphic to the one from Fig. 4a, the third and the fourth where all the partial orders representing preferences are isomorphic to those from Figs. 4b and 4c, respectively.

In the first and second cases, Corollary 3 demonstrates that a randomly selected matching is unstable with the probability of at least $49/(14 \times 15) = 7/30$, in the third one with the probability of at least $25/(14 \times 15) = 5/42$, and in the last one with the probability of at least $4/(14 \times 15) = 2/105$. Thus, the best possible upper bounds correspond to marriage markets in which the worst elements are ordered linearly and any incomparable elements are relatively good.

In [1] the well-known principle of inclusion and exclusion was used to derive an exact expression for the number of stable matchings in a marriage market with preferences given by linear orders. A similar expression can be derived for marriage markets with preferences given by arbitrary partial orders. Indeed, in both cases, we count all the matchings which do not lie in any of the sets $A_{i,j}$. Unfortunately, in the case of arbitrary partial orders, it is harder to determine the sets $A_{i,j}$, whose cardinalities determine the number of stable matchings. For completeness, we give the desired expression, together with the required notions but we omit the proof which is exactly analogous to the one presented for the simpler (linear) case.

Let $(M, W; \mathbf{P})$ be a marriage market of size n and $A \subseteq \bigcup_{(i,j)\in[n]\times[n]} A_{i,j}(M, W; \mathbf{P})$. Next, let G_A be a bipartite graph $(X, Y; E_A)$ such that $X = \{x_1, ..., x_n\}$, $Y = \{y_1, ..., y_n\}$ and $E_A = \{x_p y_q\colon (p, q, i_2, j_2) \in A \text{ or } (i_1, j_1, p, q) \in A\}$. By $F(M, W; \mathbf{P})$ we denote a set $\{A \subseteq \bigcup_{(i,j)\in[n]\times[n]} A_{i,j}(M, W; \mathbf{P})\colon \forall_{(i,j)\in[n]\times[n]} \left|A \cap A_{i,j}(M, W; \mathbf{P})\right| \le 1 \text{ and } \Delta(G_A) = 1\}$, where $\Delta(G)$ is the maximum degree of a vertex in the graph G. Finally, let $F_s^k(M, W, \mathbf{P}) = \{A \in F(M, W; \mathbf{P})\colon |A| = s \text{ and } |E_A| = k\}$.

Theorem 2. Let $n \ge 2$. If $(M, W; \mathbf{P})$ is a marriage market of size n, then

$$\sigma(M, W; \mathbf{P}) = n! + \sum_{s=1}^{n(n-1)} (-1)^s \sum_{k=2}^{n} (n-k)! \left|F_s^k(M, W; \mathbf{P})\right|$$

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