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A TWO-PARAMETER LINDLEY DISTRIBUTION

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ABSTRACT

A two-parameter Lindley distribution, of which the Lindley distribution (LD) is a particular case, has been introduced. Its moments, failure rate function, mean residual life function and stochastic orderings have been discussed. The maximum likelihood method and the method of moments have been discussed for estimating its parameters. The distribution has been fitted to some data-sets to test its goodness of fit.

Key words: Lindley distribution, moments, failure rate function, mean residual life function, stochastic ordering, estimation of parameters, goodness of fit.

1. Introduction

Lindley (1958) introduced a one-parameter distribution, known as Lindley distribution, given by its probability density function

$$f\left(x;\theta\right)=\frac{\theta^2}{\theta+1}\left(1+x\right)e^{-\theta x};\qquad x>0,\quad \theta>0 \tag{1.1}$$

It can be seen that this distribution is a mixture of exponential$\left(\theta\right)$ and gamma $\left(2,\theta\right)$ distributions. Its cumulative distribution function has been obtained as

$$F\left(x\right)=1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x};\; x>0,\; \theta>0 \tag{1.2}$$

Ghitany et al (2008) have discussed various properties of this distribution and showed that in many ways (1.1) it provides a better model for some applications than the exponential distribution. The first four moments about origin of the Lindley distribution have been obtained as

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$$\mu_1' = \frac{\theta+2}{\theta(\theta+1)}, \quad \mu_2' = \frac{2(\theta+3)}{\theta^2(\theta+1)}, \quad \mu_3' = \frac{6(\theta+4)}{\theta^3(\theta+1)}, \quad \mu_4' = \frac{24(\theta+5)}{\theta^4(\theta+1)} \tag{1.3}$$

and its central moments have been obtained as

$$\mu_2 = \frac{\theta^2+4\theta+2}{\theta^2(\theta+1)^2}, \quad \mu_3 = \frac{2(\theta^3+6\theta^2+6\theta+2)}{\theta^3(\theta+1)^3}, \quad \mu_4 = \frac{3(3\theta^4+24\theta^3+44\theta^2+32\theta+8)}{\theta^4(\theta+1)^4} \tag{1.4}$$

Ghitany et al (2008) studied various properties of this distribution. A discrete version of this distribution has been suggested by Deniz and Ojeda (2011) having its applications in count data related to insurance. Sankaran (1970) obtained the Lindley mixture of Poisson distribution. Mazucheli and Achcar (2011), Ghitany et al (2009, 2011) and Bakouchi et al (2012) are some among others who discussed its various applications. Zakerzadah and Dolati (2009) obtained a generalized Lindley distribution and discussed its various properties and applications.

In this paper, a two parameter Lindley distribution, of which the Lindley distribution (1.1) is a particular case, has been suggested. Its first four moments and some of the related measures have been obtained. Its failure rate, mean residual rate and stochastic ordering have also been studied. Estimation of its parameters has been discussed and the distribution has been fitted to some of those data sets where the Lindley distribution has earlier been fitted by others, to test its goodness of fit.

2. A Two-parameter Lindley distribution

A two-parameter Lindley distribution with parameters α and θ is defined by its probability density function (p.d.f)

$$f(x;\alpha,\theta) = \frac{\theta^2}{\alpha\theta+1}(\alpha+x)\,e^{-\theta x}; \; x>0,\; \theta>0,\; \alpha\theta>-1 \tag{2.1}$$

It can easily be seen that at $\alpha = 1$, the distribution (2.1) reduces to the Lindley distribution (1.1) and at $\alpha = 0$, it reduces to the gamma distribution with parameters $(2,\theta)$.The p.d.f. (2.1) can be shown as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions as follows:

$$f(x;\alpha,\theta) = pf_1(x) + (1-p)\, f_2(x) \tag{2.2}$$

where $p = \dfrac{\alpha\theta}{\alpha\theta+1}$, $f_1(x) = \theta e^{-\theta x}$ and $f_2(x) = \theta^2 x e^{-\theta x}$.

The first derivative of (2.1) is obtained as

$$f'(x) = \frac{\theta^2}{\alpha\theta+1}(1-\alpha\theta - x\theta)\, e^{-\theta x}$$

and so $f'(x) = 0$ gives $x = \dfrac{1-\alpha\theta}{\theta}$. From this it follows that

(i) for $|\alpha\theta| < 1$, $x_0 = \dfrac{1-\alpha\theta}{\theta}$ is the unique critical point at which $f(x)$ is maximum.

(ii) for $\alpha \geq 1$, $f'(x) \leq 0$ i.e. $f(x)$ is decreasing in x.

Therefore, the mode of the distribution is given by

$$\text{Mode} = \begin{cases} \dfrac{1-\alpha\theta}{\theta}, & |\alpha\theta| < 1 \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

The cumulative distribution function of the distribution is given by

$$F(x) = 1 - \frac{1+\alpha\theta+\theta x}{\alpha\theta+1} e^{-\theta x};\ x > 0, \theta > 0, \alpha\theta > -1 \tag{2.4}$$

3. Moments and related measures

The *r*th moment about origin of the two-parameter Lindley distribution has been obtained as

$$\mu_r' = \frac{\Gamma(r+1)(\alpha\theta+r+1)}{\theta^r(\alpha\theta+1)};\ r = 1,2,... \tag{3.1}$$

Taking $r = 1,2,3$ and 4 in (3.1), the first four moments about origin are obtained as

$$\mu_1' = \frac{\alpha\theta+2}{\theta(\alpha\theta+1)},\quad \mu_2' = \frac{2(\alpha\theta+3)}{\theta^2(\alpha\theta+1)},\quad \mu_3' = \frac{6(\alpha\theta+4)}{\theta^3(\alpha\theta+1)},\quad \mu_4' = \frac{24(\alpha\theta+5)}{\theta^4(\alpha\theta+1)} \tag{3.2}$$

It can be easily verified that for $\alpha = 1$, the moments about origin of the distribution reduce to the respective moments of the Lindley distribution. Further, the mean of the distribution is always greater than the mode, the distribution is positively skewed. The central moments of this distribution have thus been obtained as

$$\mu_2 = \frac{\alpha^2\theta^2 + 4\alpha\theta + 2}{\theta^2(\alpha\theta+1)^2}, \tag{3.3}$$

$$\mu_3 = \frac{2\left(\alpha^3\theta^3 + 6\alpha^2\theta^2 + 6\alpha\theta + 2\right)}{\theta^3(\alpha\theta+1)^3}, \tag{3.4}$$

$$\mu_4 = \frac{3\left(3\alpha^4\theta^4 + 24\alpha^3\theta^3 + 44\alpha^2\theta^2 + 32\alpha\theta + 8\right)}{\theta^4(\alpha\theta+1)^4} \tag{3.5}$$

It can be easily verified that for $\alpha = 1$, the central moments of the distribution reduce to the respective moments of the Lindley distribution.

The coefficients of variation (γ), skewness $\left(\sqrt{\beta_1}\right)$ and the kurtosis (β_2) of the two-parameter LD are given by

$$\gamma = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\alpha^2\theta^2 + 4\alpha\theta + 2}}{\alpha\theta + 2} \tag{3.6}$$

$$\sqrt{\beta_1} = \frac{2\left(\alpha^3\theta^3 + 6\alpha^2\theta^2 + 6\alpha\theta + 2\right)}{\left(\alpha^2\theta^2 + 4\alpha\theta + 2\right)^{3/2}} \tag{3.7}$$

$$\beta_2 = \frac{3\left(3\alpha^4\theta^4 + 24\alpha^3\theta^3 + 44\alpha^2\theta^2 + 32\alpha\theta + 8\right)}{\left(\alpha^2\theta^2 + 4\alpha\theta + 2\right)^2} \tag{3.8}$$

4. Failure rate and mean residual life

For a continuous distribution with p.d.f. $f(x)$ and c.d.f. $F(x)$, the failure rate function (also known as the hazard rate function) and the mean residual life function are respectively defined as

$$h(x) = \lim_{\Delta x \to 0} \frac{P\left(X < x + \Delta x \mid X > x\right)}{\Delta x} = \frac{f(x)}{1 - F(x)} \tag{4.1}$$

and $m(x) = E\left[X - x \middle| X > x\right] = \frac{1}{1 - F(x)} \int_x^\infty \left[1 - F(t)\right] dt$ (4.2)

The corresponding failure rate function $h(x)$ and the mean residual life function $m(x)$ of the distribution are thus given by

$$h(x) = \frac{\theta^2 (\alpha + x)}{1 + \alpha\theta + x\theta} \tag{4.3}$$

and $m(x) = \frac{1}{(1 + \alpha\theta + \theta x)e^{-\theta x}} \int_x^\infty (1 + \alpha\theta + \theta t)e^{-\theta t} dt = \frac{2 + \alpha\theta + \theta x}{\theta(1 + \alpha\theta + \theta x)}$ (4.4)

It can be easily verified that $h(0) = \frac{\theta^2 \alpha}{\alpha\theta + 1} = f(0)$ and $m(0) = \frac{\alpha\theta + 2}{\theta(\alpha\theta + 1)} = \mu_1'$. It is also obvious that $h(x)$ is an increasing function of x, α and θ, whereas $m(x)$ is a decreasing function of x, α and increasing function of θ. For $\alpha = 1$, (4.3) and (4.4) reduce to the corresponding measures of the Lindley distribution. The failure rate function and the mean residual life function of the distribution show its flexibility over Lindley distribution and exponential distribution.

5. Stochastic orderings

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behaviour. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $(X \le_{st} Y)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $(X \le_{hr} Y)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $(X \le_{mrl} Y)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $(X \le_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \tag{5.1}$$

$$\Downarrow$$

$$X \le_{st} Y$$

The two-parameter LD is ordered with respect to the strongest ‘likelihood ratio’ ordering as shown in the following theorem:

Theorem: Let $X \sim$ two-parameter LD$\left(\alpha_1,\theta_1\right)$ and $Y \sim$ two-parameter LD$\left(\alpha_2,\theta_2\right)$. If $\alpha_1=\alpha_2$ and $\theta_1 \ge \theta_2$ (or if $\theta_1=\theta_2$ and $\alpha_1 \ge \alpha_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)}=\left(\frac{\theta_1}{\theta_2}\right)^2\left(\frac{\alpha_2\theta_2+1}{\alpha_1\theta_1+1}\right)\left(\frac{\alpha_1+x}{\alpha_2+x}\right)e^{-(\theta_1-\theta_2)x} \quad ; \; x>0$$

Now

$$\log\frac{f_X(x)}{f_Y(x)}=2\log\left(\frac{\theta_1}{\theta_2}\right)+\log\left(\frac{\alpha_2\theta_2+1}{\alpha_1\theta_1+1}\right)+\log\left(\alpha_1+x\right)-\log\left(\alpha_2+x\right)-\left(\theta_1-\theta_2\right)x.$$

Thus

$$\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}=\frac{1}{\alpha_1+x}-\frac{1}{\alpha_2+x}+\left(\theta_2-\theta_1\right)$$

$$=\frac{\alpha_2-\alpha_1}{\left(\alpha_1+x\right)\left(\alpha_2+x\right)}+\left(\theta_2-\theta_1\right) \tag{5.2}$$

Case (i). If $\alpha_1=\alpha_2$ and $\theta_1 \ge \theta_2$, then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}<0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Case (ii). If $\theta_1=\theta_2$ and $\alpha_1 \ge \alpha_2$, then $\frac{d}{dx}\log\frac{f_X(x)}{f_Y(x)}<0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

This theorem shows the flexibility of two-parameter LD over Lindley and exponential distributions.

6. Estimation of parameters

6.1. Maximum likelihood estimates

Let $x_1,x_2,\ldots,x_n$ be a random sample of size *n* from a two-parameter Lindley distribution (2.1) and let f_x be the observed frequency in the sample corresponding to $X=x$ $\left(x=1,2,...,k\right)$ such that $\sum_{x=1}^{k} f_x=n$, where k is the largest observed value having non-zero frequency. The likelihood function, L of the two-parameter Lindley distribution (2.1) is given by

$$L = \left(\frac{\theta^2}{\alpha\theta + 1} \right)^n \prod_{x=1}^{k} \left(\alpha + x \right)^{f_x} e^{-n\theta\overline{X}} \tag{6.1.1}$$

and so the log likelihood function is obtained as

$$\log L = n \log \theta^2 - n \log \left(\alpha\,\theta + 1 \right) + \sum_{i=1}^{k} f_x \log \left(\alpha + x \right) - n\theta\overline{X} \tag{6.1.2}$$

The two log likelihood equations are thus obtained as

$$\frac{\partial \log L}{\partial \theta} = \frac{2n}{\theta} - \frac{n\alpha}{\alpha\theta + 1} - n\overline{X} = 0 \tag{6.1.3}$$

$$\frac{\partial \log L}{\partial \alpha} = - \frac{n\theta}{\alpha\theta + 1} + \sum_{x=1}^{k} \frac{f_x}{\alpha + x} = 0 \tag{6.1.4}$$

Equation (6.1.3) gives $\overline{X} = \dfrac{\alpha\theta + 2}{\theta\left(\alpha\theta + 1 \right)}$, which is the mean of the two-parameter Lindley distribution. The two equations (6.1.3) and (6.1.4) do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = - \frac{2n}{\theta^2} + \frac{n\alpha^2}{\left(\alpha\theta + 1 \right)^2} \tag{6.1.5}$$

$$\frac{\partial^2 \log L}{\partial \theta\, \partial \alpha} = - \frac{n}{\left(\alpha\theta + 1 \right)^2} \tag{6.1.6}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n\theta^2}{\left(\alpha\theta + 1 \right)^2} - \sum_{x=1}^{k} \frac{f_x}{\left(\alpha + x \right)^2} \tag{6.1.7}$$

The following equations for $\hat{\theta}$ and $\hat{\alpha}$ can be solved

$$\begin{bmatrix} \dfrac{\partial^2 \log L}{\partial \theta^2} & \dfrac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \dfrac{\partial^2 \log L}{\partial \theta \partial \alpha} & \dfrac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta} = \theta_0 \\ \hat{\alpha} = \alpha_0}} \begin{bmatrix} \hat{\theta} - \theta_0 \\ \hat{\alpha} - \alpha_0 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \log L}{\partial \theta} \\ \dfrac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta} = \theta_0 \\ \hat{\alpha} = \alpha_0}} \tag{6.1.8}$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close estimates of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

6.2. Estimates from moments

Using the first two moments about origin, we have

$$\frac{\mu_2'}{\mu_1'^2} = k\,(\text{say}) = \frac{2(\alpha\theta+3)(\alpha\theta+1)}{(\alpha\theta+2)^2} \tag{6.2.1}$$

Taking $b = \alpha\,\theta$, we get

$$\frac{\mu_2'}{\mu_1'^2} = \frac{2(b+3)(b+1)}{(b+2)^2} = \frac{2b^2+8b+6}{b^2+4b+4} = k\ .$$

This gives $(2-k)b^2 + 4(2-k)b + 2(3-2k) = 0$ (6.2.2)

which is a quadratic equation in b .Replacing the first and the second moments μ_1' and μ_2' by the respective sample moments $\overline{X}$ and m_2' an estimate of k can be obtained, using which , the equation (6.2.2) can be solved and an estimate of b obtained. Substituting this estimate of b in the expression for the mean of the two-parameter LD, an estimate of θ can be obtained as

$$\hat{\theta} = \left(\frac{b+2}{b+1}\right)\frac{1}{\overline{X}} \tag{6.2.3}$$

Finally to get an estimate of α , we substitute the value b and estimate of θ in the expression $b = \alpha\theta$, which gives an estimate of α as

$$\hat{\alpha} = \frac{b}{\hat{\theta}} \tag{6.2.4}$$

7. Goodness of fit

The two-parameter Lindley distribution has been fitted to a number of data-sets to which earlier the Lindley distribution has been fitted by others and to almost all these data-sets the two-parameter Lindley distribution provides closer fits than the one parameter Lindley distribution.

The fittings of the two-parameter Lindley distribution to three such data-sets have been presented in the following tables. The data sets given in tables-1, 2 and 3 are the data sets reported by Ghitany et al (2008), Bzerkedal (1960) and Paranjpe and Rajarshi (1986) respectively. The expected frequencies according to the one parameter Lindley distribution have also been given for ready comparison with those obtained by the two-parameter Lindley distribution. The estimates of the parameters have been obtained by the method of moments.

Table 1. Waiting times (in minutes) of 100 bank customers

Waiting Time (In minutes)	Observed frequency	Expected frequency One-parameter LD	Expected frequency Two-parameter LD
0 – 5	30	30.4	30.2
5 – 10	32	30.7	30.9
10 – 15	19	19.2	19.3
15 – 20	10	10.3	10.3
20 – 25	5	5.1	5.0
25 – 30	1	2.4	2.4
30 – 35	2	1.1	1.1
35 – 40	1	0.8	0.8
Total	100	100.0	100.0
Estimates of parameters		$\hat{\theta} = 0.187$	$\hat{\theta} = 0.191139$ $\alpha = 0.894052$
χ^2		0.09402	0.07481
d.f.		4	3

Table 2. Data of survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli

Survival Time (In days)	Observed frequency	Expected frequency One-parameter LD	Expected frequency Two-parameter LD
0 – 80	8	16.1	10.7
80 – 160	30	21.9	26.9
160 – 240	18	15.4	17.7
240 – 320	8	9.0	9.2
320 – 400	4	5.5	4.3
400 – 480	3	1.8	1.9
480 – 560	1	2.3	1.3
Total	72	72.0	72.0
Estimates of parameters		$\hat{\theta} = 0.011$	$\hat{\theta} = 0.012992$ $\hat{\alpha} = -20.08359$
χ^2		7.7712	1.2335
d.f.		3	2

Table 3. Mortality grouped data for blackbird species

Survival Time (In days)	Observed frequency	Expected frequency One-parameter LD	Expected frequency Two-parameter LD
0 – 1	192	173.5	168.0
1 – 2	60	98.6	88.4
2 – 3	50	46.5	46.2
3 – 4	20	20.1	24.0
4 – 5	12	8.1	12.4
5 – 6	7	3.2	6.4
6 – 7	6	1.4	3.3
7 – 8	3	0.3	1.7
≥ 8	2	0.3	1.6
Total	352	352.0	352.0
Estimates of parameters		$\hat{\theta} = 0.984$	$\hat{\theta} = 0.731104$ $\hat{\alpha} = 10.266582$
χ^2		49.846	16.5342
d.f.		4	4

It can be seen that the two-parameter LD gives much closer fits than the one parameter Lindley distribution and thus provides a better alternative to the Lindley distribution.

8. Conclusions

In this paper, a two-parameter Lindley distribution (LD), of which the one-parameter LD is a particular case, has been proposed. Several properties of the two-parameter LD such as moments, failure rate function, mean residual life function, stochastic orderings, estimation of parameters by the method of maximum likelihood and the method of moments have been discussed. Finally, the proposed distribution has been fitted to a number of data sets relating to waiting and survival times to test its goodness of fit to which earlier the one-parameter LD has been fitted, and it is found that two-parameter LD provides better fits than those by the one-parameter LD.

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