

A HANDFUL OF THEOREMS ABOUT POINT INCREASE

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Abstract. In classic calculus increasing functions are considered. The idea of increasing is connected with an interval. In this article the idea of increasing function is connected with a point and some neighbourhood of the point. It is defined as an increasing point of the function. This is proved by the theorem: if each point of an interval is a point of increasing a function then the function is increasing at an interval.

Keywords: increasing function, point of increasing, continuous function, decreasing function, point of decreasing, sequence of increasing points.

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In classical calculus, the term “increasing” is used to describe increasing and decreasing functions, thus it applies to a function and an interval. In theory of measure the term “support of measure” is used. The distribution function of a measure is increasing in each interval containing the points of the support. Which point of the support and distribution function is connected with the term “increasing at a point”? In classical calculus there does not exist consideration applicable to the term “increasing at a point”. In this paper we present some properties of the function increasing at a point. It considers the relation between the terms ‘increasing in an interval’ and ‘increasing at a point’.

The increase of function f transforming the set of real numbers into R is usually considered in intervals. It is said that:

Definition 1. A real function f is weakly increasing in an interval $V \subset R$ if

$$f(x) \leq f(z) \tag{1}$$

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whenever x and z are in V with $x < z$. Also, f is strictly increasing if

$$f(x) < f(z) \quad (2)$$

whenever $x < z$ (Janaszak 1987, 2004).

If the inequalities (1) or (2) are oppositely directed it is said that function f is weakly decreasing or strictly decreasing in V .

Usually, if it is clearly defined, instead of the terms *weakly increasing* or *strictly increasing*, simply *increasing* is used. In this article the terms *weakly increasing* or *strictly increasing* are used consistently, the same remark applies to definition 2.

The interval V can be closed, open, or half-closed, bounded or unbounded, included in the domain of function f . By this definition increase is a property of the function on the interval, not on a point situated in the domain of the function. Increase at a point has been defined by only a few authors.

Definition 2. Let x be a point of interval V and its right-hand side neighborhood is included in V . It is said that function f is right-hand weakly increasing at the point x if there exists a number $\delta > 0$ such that $[x, x + \delta) \subset V$ and for each $z \in (x, x + \delta)$ the inequality (1) holds, it is said that function f is right-hand strictly increasing at the point x if the inequality (2) holds. Similarly it is defined as a left-hand weakly increased or left-hand strictly increased function f at the point x , i.e. if for a number $\delta > 0$ the inclusion $(x - \delta, x] \subset V$ is true, and for each $y \in (x - \delta, x)$ the inequality

$$f(y) \leq f(x) \quad (3)$$

or

$$f(y) < f(x) \quad (4)$$

holds. If f is simultaneously left-hand and right-hand weakly or strictly increasing at x it is said that f is weakly or strictly increasing at x . Also, it is said that x is the point of increase, or the increasing point of function f , if it is necessary to use the term *increase* or *increasing* we may add the adverbs *weakly* or *strictly*.

If the inequality (1) or (3) is oppositely directed it is said that function f is right-hand or left-hand weakly decreasing at point x , similarly, if the inequality (2) or (4) is oppositely directed it is said that function f is right-hand and left-hand strictly decreasing at point x .

The examples beneath illustrate the differences between the presented concepts.

Example 1. Let there be the following function

$$f(x) = \begin{cases} a \cdot x & \text{if } x \in \mathcal{Q} \\ b \cdot x & \text{if } x \in \mathbb{R} \setminus \mathcal{Q} \end{cases} \quad (5)$$

where \mathbb{R} is the set of real numbers, \mathcal{Q} the set of rational numbers and the inequality $0 < a < b$ holds:

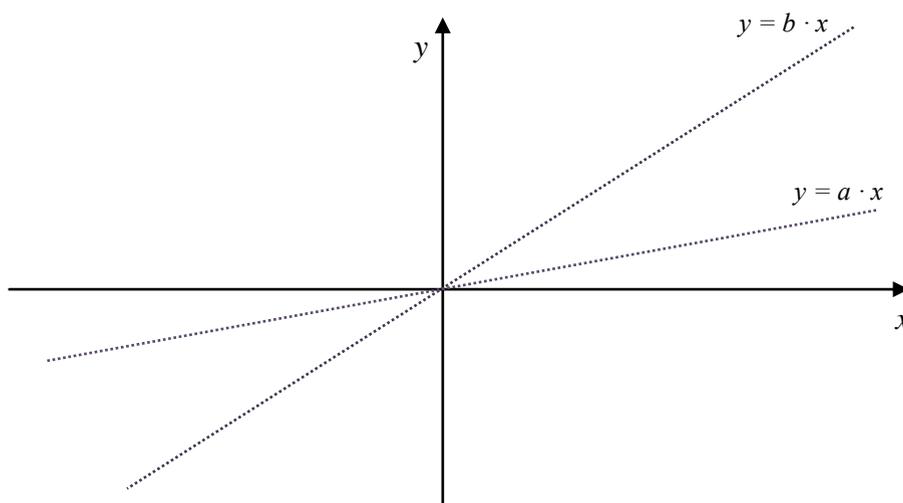


Fig. 1. The graph of the function given by formula 5

Source: own elaboration.

This function is neither weakly and consequently nor strictly increasing – in the sense of definition 1 in any interval on the real line, which is its domain, but it is strictly increasing at the point $x = 0$ – by definition 2.

Example 2. The function is given by the formula

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathcal{Q} \\ b \cdot x & \text{if } x \in \mathbb{R} \setminus \mathcal{Q} \end{cases} \quad (6)$$

where $0 < b$. This function presented in the figure above is neither weakly and consequently nor strictly increasing – in the sense of definition 1, and is not strictly increasing at the point $x=0$, but it is weakly increasing at the point $x=0$, see definition 2.

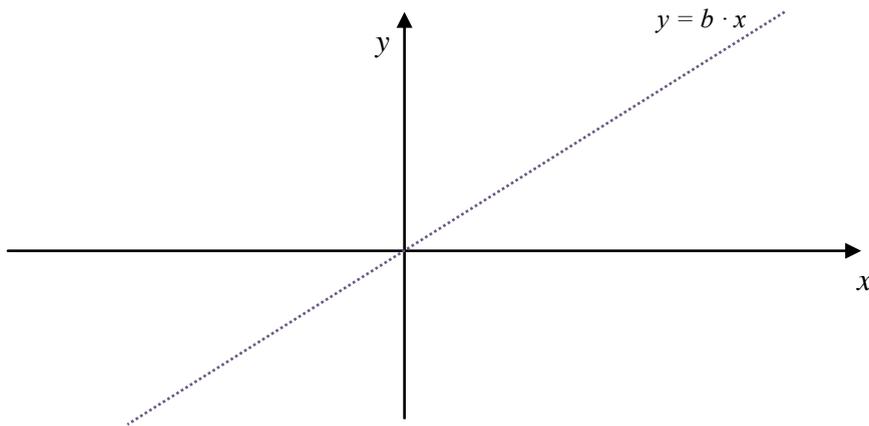


Fig. 1. The graph of the function given by formula 6

Source: own elaboration.

Example 3. Let there be the following function:

$$f(x) = \begin{cases} \left(1 - \sin^2 \frac{\pi}{2 \cdot x}\right) \cdot a \cdot x + \left(\sin^2 \frac{\pi}{2 \cdot x}\right) \cdot b \cdot x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (7)$$

where $0 < a < b$, see Figure 3.

The function f is near the zero point, a denser and denser sinusoid placed between two lines passing through the origin of coordinates with positive direction coefficients, and thus passes through the third and first quarter, hence it is strictly increasing function at the zero point – in the sense of definition 2.

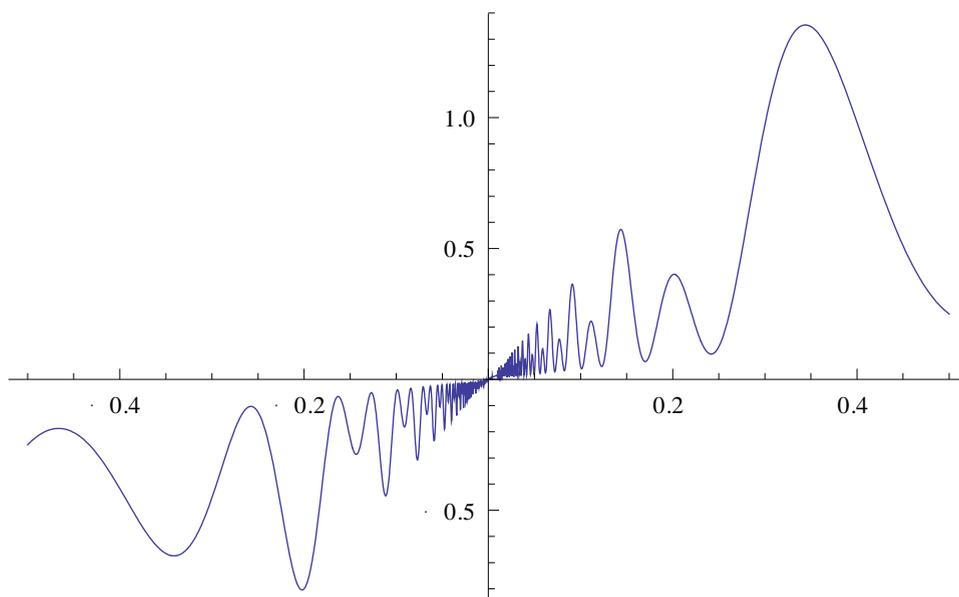


Fig. 3. The graph of the function given by formula 7

Source: own elaboration.

It is shown below that f is not an increasing function on any interval containing the point $x = 0$ – in the sense of definition 1. For each natural number n the inequality

$$\frac{1}{2n+1} < \frac{1}{2n} < \frac{1}{2n-1} \quad (8)$$

holds. At these points function f takes the values:

$$f\left(\frac{1}{2n \pm 1}\right) = \frac{b}{2n \pm 1}$$

and

$$f\left(\frac{1}{2n}\right) = \frac{a}{2n}.$$

The equality

$$\frac{b}{2n+1} - \frac{a}{2n} = \frac{2n \cdot (b-a) - a}{2n \cdot (2n+1)}$$

holds. Since n is a natural number the denominator of the right side of the equality above is positive, and for sufficiently large n the numerator is positive too. So inequality

$$f\left(\frac{1}{2n}\right) < f\left(\frac{1}{2n+1}\right)$$

holds, thus the double inequality

$$f\left(\frac{1}{2n}\right) < f\left(\frac{1}{2n+1}\right) < f\left(\frac{1}{2n-1}\right)$$

holds too, which in comparison with the double inequality (8) proves that the considered function is neither weakly nor strictly increasing on any interval containing the point $x = 0$ – in the sense of definition 1.

Example 4. The function is given by

$$f(x) = \begin{cases} \left(\sin^2 \frac{\pi}{2 \cdot x}\right) \cdot b \cdot x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \quad (9)$$

where $b > 0$.

This function is weakly increasing, but not strictly increasing, at the point $x = 0$ – in the sense of definition 2, it is not increasing (neither weakly nor strictly) increasing in any interval containing the zero point – in the sense of definition 1. In the neighborhood of this point it is a densifying sinusoid placed between the line passing through the origin of the coordinates and x -axis, see Figure 4.

The function of examples 3 or 4 is continuous in the all real axis, but the function of examples 1 or 2 is continuous only at the point $x = 0$.

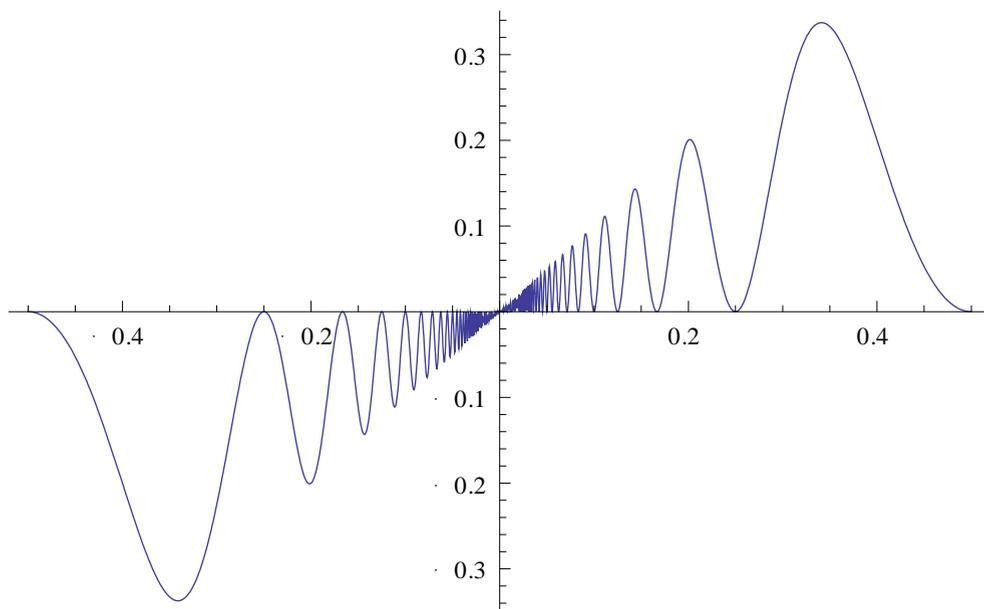


Fig. 4. The function of the formula (9)

Source: own elaboration.

If the function is weakly increasing (or strictly increasing) on a given interval – in the sense of definition 1, then – in the sense of definition 2, it is weakly increasing (or strictly increasing) at each point of the interior of that interval; and it is right-hand or left-hand increasing (or strictly increasing) at each of two endpoints of the interval, if the endpoint belongs to the interval. This result is obvious, and the reverse theorem is also true, as shown below (Billingsley 1987; Kudravcev 1973; Janaszak 2002, 2003):

Theorem 1. Function f maps the open interval V into R , and it is supposed that f is strictly increasing at each point of V – in the sense of definition 2, then f is strictly increasing on V – in the sense of definition 1.

Proof. Let $x_1 < x_2$ be two points belonging to the interval V . It is necessary to show that

$$f(x_1) < f(x_2). \quad (10)$$

Let W be the following set

$$W = \{ x \in (x_1, x_2] : f(x_1) < f(x) \}, \quad (11)$$

W is a non-empty set because f is right-hand strictly increasing at the point x_1 , the set W includes some interval $(x_1, x_1 + \delta)$, as follows from definition 2.

By c there is denoted the least upper bound of the set W , i.e.

$$c = \sup W. \quad (12)$$

Below it is shown that $c \in W$:

for that let it be supposed, on the contrary, that

$$c \notin W. \quad (13)$$

Hence, by assumption, there is a positive number δ such that

$$(c - \delta, c) \subset (x_1, c)$$

and for each argument $y \in (c - \delta, c)$ the inequality

$$f(y) < f(c) \quad (14)$$

holds. From the equality (12) it follows that the set $(c - \delta, c)$ contains at last one point y belonging to the set W . By the formula (11) the inequality

$$f(x_1) < f(y) \quad (15)$$

holds. The formulas (14) and (15), by the transitivity of the inequality relation, entail that

$$f(x_1) < f(c), \quad (16)$$

i.e. the point c belongs to the set W , which is at variance with (13).

To complete the proof it is shown that $c = x_2$. Let it indirectly be assumed that

$$c \neq x_2, \quad (17)$$

which implies that $c < x_2$, as $c \in W$ and $W \subset (x_1, x_2]$. Since function f is strictly increasing at the point c , so there is the positive number δ such that

$$(c, c + \delta) \subset (c, x_2)$$

and for each argument $z \in (c, c + \delta)$ the inequality

$$f(c) < f(z)$$

holds. This means in turn that $z \in W$ and against inequality $c < z$ is a contradiction with the formula (12), i.e. the assumption (17) is false, which completes the proof.

A similar theorem may be proved for the concept *weakly increasing function*.

Accordingly, point x is a local strictly (or weakly) minimum of function f if x is left-hand strictly (or weakly) decreasing and right-hand strictly (or weakly) increasing point of this function. Similarly, x is a local strictly (or weakly) maximum if x is left-hand strictly (or weakly) increasing and right-hand strictly (or weakly) decreasing point.

If it is known which inequality: *less than* or *less than or equal to*, similarly, *more than* or *more than or equal to* is used, it is unnecessary to use the adverb weakly or strictly.

Example 1a. The function $|f(x)|$, where $f(x)$ is given by formula (5) is strictly minimum at the point $x = 0$. This function is not decreasing – in the sense of definition 2, on any interval $(-\delta, 0)$, where $\delta > 0$, and is not increasing on any interval $(0, \delta)$.

Similarly the function $-|f(x)|$, where $f(x)$ is strictly maximum at the point $x = 0$. This function is not increasing – in the sense of definition 2, on any interval $(-\delta, 0)$, where $\delta > 0$, and is not decreasing on any interval $(0, \delta)$.

Example 2a. The function $|f(x)|$, where $f(x)$ is given by formula (6) is weakly minimum at the point $x = 0$. This function is not decreasing – in the sense of definition 2, on any interval $(-\delta, 0)$, where $\delta > 0$, and is not increasing on any interval $(0, \delta)$.

Similarly the function $-|f(x)|$, where $f(x)$ is weakly maximum at the point $x = 0$. This function is not increasing – in the sense of definition 2,

on any interval $(-\delta, 0)$, where $\delta > 0$, and is not decreasing on any interval $(0, \delta)$.

Example 3a. The function $|f(x)|$, where $f(x)$ is given by formula (7) is strictly minimum at the point $x = 0$. This function is not decreasing – in the sense of definition 2, on any interval $(-\delta, 0)$, where $\delta > 0$, and is not increasing on any interval $(0, \delta)$.

Similarly the function $-|f(x)|$, where $f(x)$ is strictly maximum at the point $x = 0$. This function is not increasing – in the sense of definition 2, on any interval $(-\delta, 0)$, where $\delta > 0$, and is not decreasing on any interval $(0, \delta)$.

The functions of the examples 3a is continuous in the all real axis but in the examples 1a or 2a it is continuous only at the point $x = 0$.

Theorem 2. Let us also suppose function f maps the interval $[a, b)$ into the real line, it is right-hand continuous at a and is strictly increasing at every point of interior of the interval. Then function f is right-hand increasing at the point a .

Proof. Let δ be a positive number such that $a + \delta < b$ and x be a point of the interval $(a, a + \delta)$. It is necessary to show that $f(a) < f(x)$. By assumption, according to theorem 1, f is a strictly increasing function on the interval (a, b) , hence for every sequence x_n decreasing to the point a the values $f(x_n)$ form a decreasing sequence. By the continuity of function f the sequence $f(x_n)$ is convergent to the point $f(a)$. Let n be a sufficiently large integer so that $a < x_n < x$, then there are two inequalities: $f(a) < f(x_n)$ and $f(x_n) < f(x)$, hence $f(a) < f(x)$, and the theorem is proved.

Corollary 1. If function f meets the same assumptions as in theorem 2, then f is strictly increasing on the interval $[a, b)$.

Theorem 3. Function f maps the interval $(a, b]$ into the real line, it is left-hand continuous at a , and it is strictly increasing at every point of interior of the interval. Then function f is left-hand increasing at the point b .

Corollary 2. If function f meets the same assumptions as in theorem 3, then f is strictly increasing on the interval $(a, b]$.

Theorem 4. Let function f map the interval V into the real line, it is continuous at the point c located in the interior of V . It is supposed that for every point $x \in \text{int } V$ and $x \neq c$ function f is strictly increasing at x . Then function f is strictly increasing at c .

Proof. Let the points a and b be located in the interior of V . In the interval $[c, b)$ function f satisfies the assumptions of theorem 2, so it is right-hand increasing in the point c . Similarly, in the interval $(a, c]$ function f satisfies the assumptions of theorem 3, so it is left-hand increasing in the point c . Therefore the thesis is realised.

From the theorem, via theorem 1, follows the conclusion.

Theorem 4 can be generalized.

Theorem 5. Let function f map the interval V into the real line. Let the set C be included in the interior of the set V and consist of the isolated points, i.e. for each $c \in C$ there exists $\delta > 0$ such that $(c - \delta, c + \delta) \subset (V \setminus C) \cup \{c\}$, i.e. the interval $(c - \delta, c + \delta)$ is included in V , and does not contain, other than c , points of the set C , function f is continuous at each point of C , and strictly increasing at each point of the set $V \setminus C$. Then f is strictly increasing at every point $c \in C$, and accordingly it strictly increases in the whole interval V .

Proof. For each point $c \in C$ exist two points a and b such that the formula $c \in (a, b) \subset (V \setminus C) \cup \{c\}$ occurred, next, the argument proceeds as in theorem 4.

Corollary 3. The thesis of theorem 5 is true when the set C is finite.

Theorem 5 can still be generalized.

Theorem 6. Let function f map the interval V into the real line. The elements of the sequence c_n belong to the interval V and the sequence to the point c of the set V . It is supposed that function f is continuous at the points c_n , and the point c , in other points of the set V it is strictly increasing. Then it is increasing in the points c_n , and accordingly, it increases in the whole interval V .

Proof. The points of the sequence c_n form a set C , which has the properties occurring in the assumptions of theorem 5. Hence function f strictly increases at every point c_n . Regardless of the position of the point c in the interval V , the theorems 2, 3, 4 ensure that c is the strictly increasing point of f , so by theorem 1 function f increases in V .

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