A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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CONSTRUCTION OF E–OPTIMAL SPRING BALANCE WEIGHING DESIGNS FOR EVEN NUMBER OF OBJECTS

Abstract. The problem of the construction of spring balance weighing designs satisfying the criterion of E-optimality is discussed. The incidence matrices of partially incomplete block designs are used to construction of the regular E-optimal spring balance weighing design.

Key words: E–optimal design, partially balanced incomplete block design, spring balance weighing design.

I. INTRODUCTION

The study of weighing designs is supposed to be helpful in routine of weighing operations to determine unknown measurements of p objects using n measurement operations. These designs are applicable in a great variety of problems of measurements, for instance in metrology, dynamical system theory, computational mechanics and statistics. Results of experiment can be written as $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$, $\mathbf{\Phi}_{n\times p}(0,1)$ denotes the class of matrices $\mathbf{X} = (x_{ij})$, $i = 1,2,...,n$, $j = 1,2,..., p$, having entries $x_{ij} = 0$ or 1 depending upon whether the j th object is excluded or included in the i th measurement operation, $\mathbf{w} = (w_1, w_2,..., w_p)'$ is a vector representing unknown measurements of objects and $\mathbf{e}$ is an $n \times 1$ vector of random errors. Our basic assumption is the following. There are not systematic errors and the errors are uncorrelated and have different variances, i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is vector of zeros, $\mathbf{G}$ is the $n \times n$ positive definite diagonal matrix of known elements. If the design $\mathbf{X}$ is of full column rank, then all w_j are estimable and the variance matrix of their best linear unbiased estimator is $\sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$.

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In many problems concerning weighing experiments the E–optimal design is considered. For given variance matrix $\sigma^2\mathbf{G}$, the regular E-optimal design there is design for that the maximal eigenvalue of $\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ attains the lower bound. Moreover, in any set of design matrices $\mathbf{\Phi}_{n\times p}(0,1)$, a regular E-optimal design may not exist, whereas E-optimal design exists always. The concept of E–optimality was considered in Raghavarao (1971), Banerjee (1975), Jacroux and Notz (1983), Pukelsheim (1993).

The main purpose of this paper is to obtain a new construction method of regular E–optimal spring balance weighing designs.

II. E–OPTIMAL DESIGN

We consider the experiment in that using $n=\sum_{s=1}^{h} n_s$ measurement operations we determine unknown measurements of p objects. Without loss of generality we can assume that we have at our disposal h different installations with precision factors $g_1, g_2, ..., g_h$ and n_s measurements are taken with the precision g_s, $s=1,2,...,h$, or these n_s measurements are taken in different h conditions at the same installation. Thus

$$\mathbf{G}=\begin{bmatrix} g_1^{-1}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}_{n_2}' & & \mathbf{0}_{n_1}\mathbf{0}_{n_h}' \\ \mathbf{0}_{n_2}\mathbf{0}_{n_1}' & g_2^{-1}\mathbf{I}_{n_2} & & \mathbf{0}_{n_2}\mathbf{0}_{n_h}' \\ & & & \\ \mathbf{0}_{n_h}\mathbf{0}_{n_1}' & \mathbf{0}_{n_h}\mathbf{0}_{n_2}' & & g_h^{-1}\mathbf{I}_{n_h} \end{bmatrix}, \; g_s>0,\, s=1,2,...,h. \tag{1}$$

Now suppose that the design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ is divided into h matrices according to (1). Thus

$$\mathbf{X}=\begin{bmatrix}\mathbf{X}_1\\ \mathbf{X}_2\\ \\ \mathbf{X}_h\end{bmatrix}. \tag{2}$$

The theorem given in Katulska and Rychlińska (2010) presents the necessary conditions determining regular E-optimal design.

Theorem 1. Let p be an even number. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) with the variance matrix of errors $\sigma^2\mathbf{G}$, for $\mathbf{G}$ of (1), is the regular E– optimal if each row of $\mathbf{X}_s$ contains exactly $\frac{p}{2}$ ones and

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p\mathrm{tr}\left(\mathbf{G}^{-1}\right)}{4(p-1)}\mathbf{I}_p + \frac{(p-2)\mathrm{tr}\left(\mathbf{G}^{-1}\right)}{4(p-1)}\mathbf{1}_p\mathbf{1}_p' . \tag{3}$$

In the special case $\mathbf{G} = \mathbf{I}_n$, Theorem 1 was presented in Jacroux and Notz (1983). Method of construction of the regular E–optimal spring balance weighing design for $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) and $\mathbf{G}$ in (1), is given in Katulska and Rychlińska (2010). It is based on the incidence matrices of the balanced incomplete block designs.

For convenience, from now on **G** is the same as defined in (1).

Here, we consider the case p is an even number. Under the assumption that the errors have different variances, we give a new construction method of regular E–optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the form (2). Therefore, we wide the class of possible optimal designs given in Jacroux and Notz (1983) and Katulska and Rychlińska (2010). Presented method is based on incidence matrices of two group divisible designs with the same association scheme.

III. CONSTRUCTION OF THE DESIGN MATRIX

Now, we recall the definition of partially balanced incomplete block design with two associate classes given, for instance, in Clatworthy (1973).

An incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are v $(> k)$ treatments each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments are either first associates or second associates.

(b) Each treatment has exactly α th associates, $\alpha = 1,2$.

(c) Given any two treatments which are α th associates, the number of treatments common to the β th associate of the first and the γ th associate of the second is $p^{\alpha}_{\beta\gamma}$ and is independent of the pair of treatments we start with. Also $p^{\alpha}_{\beta\gamma} = p^{\alpha}_{\gamma\beta}$, $\alpha, \beta, \gamma = 1, 2$.

(d) Two treatments which are α th associates occur together in exactly λ_{α} blocks, $\alpha = 1, 2$.

For a proper partially balanced incomplete block design $\lambda_1 \neq \lambda_2$. The numbers $v, b, r, k, \lambda_1, \lambda_2$ are called parameters of the first kind, whereas the numbers q_{α}, $p^{\alpha}_{\beta\gamma}$, $\alpha, \beta, \gamma = 1, 2$ are called parameters of the second kind.

A group divisible design is a partially balanced incomplete block design with two associate classes for which the treatments may be divided into m groups of t distinct treatments each, such that treatments that belong to the same group are first associates and two treatments belonging to different groups are second associates. For group divisible design it is clear that $v = mt$, $q_1 = t - 1$, $q_2 = t(m-1)$, $(t-1)\lambda_1 + t(m-1)\lambda_2 = r(k-1)$.

Based on incidence matrices of two group divisible designs with the same association scheme, we construct regular E–optimal spring balance weighing design. For this purpose, we consider the design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (2) for

$$\mathbf{X}_s = \begin{bmatrix} \mathbf{N}_{1s} & \mathbf{N}_{2s} \end{bmatrix}', \tag{4}$$

where $\mathbf{N}_{ls}$ is incidence matrix of the group divisible design with the same association scheme with parameters $v, b_{ls}, r_{ls}, k_{ls}, \lambda_{1ls}, \lambda_{2ls}$, $s = 1,2,...,h$, $l = 1,2$.

Furthermore, let the condition

$$\lambda_{11s} + \lambda_{12s} = \lambda_{21s} + \lambda_{22s} = \lambda_s \tag{5}$$

be satisfied for each s. For $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) and (4), we have $p = v$ and $n = \sum_{s=1}^{h}\sum_{l=1}^{2} b_{ls}$.

Theorem 2 Suppose that v is an even number and that's more $\mathbf{N}_{1s}$ and $\mathbf{N}_{2s}$ are incidence matrices of the group divisible designs with the same association scheme with parameters $v,\ b_{ls},\ r_{ls},\ k_{ls},\ \lambda_{1ls},\ \lambda_{2ls}$, $l=1,2$, $s=1,2,...,h$, for that Condition (5) is satisfied. If the conditions

(i) $b_{1s}+b_{2s}=2(r_{1s}+r_{2s})$

(ii) $4\lambda_s(v-1)=(v-2)(b_{1s}+b_{2s})$

are fulfilled simultaneously for each s, then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) and (4) with $\sigma^2\mathbf{G}$, is the regular E–optimal spring balance weighing design.

Proof. For given $\sigma^2\mathbf{G}$, we choose h matrices in (4) and we form $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$. According to above notation, we consider group divisible designs with the same association scheme for that (5) is satisfied. Hence we have

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}=\sum_{s=1}^{h}\sum_{l=1}^{2}g_s^{-1}\mathbf{N}_{ls}\mathbf{N}_{ls}'= \\ =\sum_{s=1}^{h}g_s^{-1}\left(\frac{(b_{1s}+b_{2s})v}{4(v-1)}\mathbf{I}_v+\frac{(b_{1s}+b_{2s})(v-2)}{4(v-1)}\mathbf{1}_v\mathbf{1}_v'\right). \tag{6}$$

Furthermore, $\sum_{s=1}^{h}\sum_{l-1}^{2}\mathbf{N}_{ls}\mathbf{N}_{ls}'=\sum_{s=1}^{h}\left((r_{1s}+r_{2s}-\lambda_s)\mathbf{I}_v+\lambda_s\mathbf{1}_v\mathbf{1}_v'\right)$. Since $\mathbf{N}_{ls}$ are incidence matrices of group divisible design satisfying Conditions (i) and (ii) then it is clear that $k_s=\frac{p}{2}$ for each s. An easy computation shows that (6) is equivalent to (3) if and only if $\lambda_s=\frac{(b_{1s}+b_{2s})(v-2)}{4(v-1)}$ for each s. Thus we have (ii). Taking into consideration Theorem 1 and the equality $\frac{(b_{1s}+b_{2s})v}{4(v-1)}=r_{1s}+r_{2s}-\lambda_s$ we obtain the Condition (i). Hence the result.

In given class $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ we choose h matrices $\mathbf{X}_s$ in such a way that $n=\sum_{s=1}^{h}\sum_{l=1}^{2}b_{ls}$. Let us mention, we don't have to choose different h matrices $\mathbf{X}_s$. The matrices $\mathbf{X}_s$ we form using the incidence matrices of group divisible designs with the same association scheme that parameters are given below. We present series of parameters of group divisible designs based on the book of Clatworthy (1973). The restrictions related to parameters $r,k\le 10$ and u follow from assumption given in Clatworthy (1973).

Theorem 3. Let $v = 4$ and $n = \sum_{s=1}^{h}\sum_{l=1}^{2} b_{ls}$. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times 4}(0,1)$ in (2) and (4), where $\mathbf{N}_{1s}$ and $\mathbf{N}_{2s}$ are incidence matrices of group divisible designs with the same association scheme with parameters

(i) $b_{1s} = 2(3t+1)$, $r_{1s} = 3t+1$, $k_{1s} = 2$, $\lambda_{11s} = t+1$, $\lambda_{21s} = t$ and $b_{2s} = 2(3q+2)$, $r_{2s} = 3q+2$, $k_{2s} = 2$, $\lambda_{12s} = q$, $\lambda_{22s} = q+1$, $t = 1,2,3$, $q = 0,1,2$,

(ii) $b_{1s} = 2(3t+2)$, $r_{1s} = 3t+2$, $k_{1s} = 2$, $\lambda_{11s} = t+2$, $\lambda_{21s} = t$ and $b_{2s} = 2(3q+4)$, $r_{2s} = 3q+4$, $k_{2s} = 2$, $\lambda_{12s} = q$, $\lambda_{22s} = q+2$, $t = 1,2$, $q = 0,1,2$,

(iii) $b_{1s} = 2(u+3)$, $r_{1s} = u+3$, $k_{1s} = 2$, $\lambda_{11s} = u+1$, $\lambda_{21s} = 1$ and $b_{2s} = 4u$, $r_{2s} = 2u$, $k_{2s} = 2$, $\lambda_{12s} = 0$, $\lambda_{22s} = u$, $u = 1,2,3,4,5$,

(iv) $b_{1s} = 16$, $r_{1s} = 8$, $k_{1s} = 2$, $\lambda_{11s} = 0$, $\lambda_{21s} = 4$ and $b_{2s} = 2(3u+4)$, $r_{2s} = 3u+4$, $k_{2s} = 2$, $\lambda_{12s} = u+4$, $\lambda_{22s} = u$, $u = 1,2$,

(v) $b_{1s} = 18$, $r_{1s} = 9$, $k_{1s} = 2$, $\lambda_{11s} = 5$, $\lambda_{21s} = 2$ and $b_{2s} = 6(u+2)$, $r_{2s} = 3(u+2)$, $k_{2s} = 2$, $\lambda_{12s} = u$, $\lambda_{22s} = u+3$, $u = 0,1$,

with $\sigma^2\mathbf{G}$, is the regular E–optimal spring balance weighing design.

Proof. It is easily seen that parameters the group divisible designs satisfy Conditions (i) and (ii) of Theorem 2.

It is convenient to remark that the proofs of Theorems 4-8 are similar to analysis given in the proof of Theorem 3. We leave to the reader to verify relations between the parameters of group divisible designs.

Theorem 4. Let $v = 6$ and $n = \sum_{s=1}^{h}\sum_{l=1}^{2} b_{ls}$. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times 6}(0,1)$ in (2) and (4), where $\mathbf{N}_{1s}$ and $\mathbf{N}_{2s}$ are incidence matrices of group divisible design with the same association scheme with parameters

(i) $b_{1s} = 4t$, $r_{1s} = 2t$, $k_{1s} = 3$, $\lambda_{11s} = 0$, $\lambda_{21s} = t$ and $b_{2s} = 6t$, $r_{2s} = 3t$, $k_{2s} = 3$, $\lambda_{12s} = 2t$, $\lambda_{22s} = t$, $t = 1,2,3$,

(ii) $b_{1s} = 2(2t+5)$, $r_{1s} = 2t+5$, $k_{1s} = 3$, $\lambda_{11s} = t+1$, $\lambda_{21s} = t+2$ and $b_{2s} = 6t$, $r_{2s} = 3t$, $k_{2s} = 3$, $\lambda_{12s} = t+1$, $\lambda_{22s} = t$, $t = 1,2$,

(iii) $b_{1s} = 12$, $r_{1s} = 6$, $k_{1s} = 3$, $\lambda_{11s} = 4$, $\lambda_{21s} = 2$ and $b_{2s} = 2(5t+4)$, $r_{2s} = 5t+4$, $k_{2s} = 3$, $\lambda_{12s} = 2t$, $\lambda_{22s} = 2t+1$, $t = 0,1$,

(iv) $b_{1s} = 16$, $r_{1s} = 8$, $k_{1s} = 3$, $\lambda_{11s} = 4$, $\lambda_{21s} = 3$ and $b_{2s} = 2(5t+2)$, $r_{2s} = 5t+2$, $k_{2s} = 3$, $\lambda_{12s} = t+2$, $\lambda_{22s} = 2t+1$, $t = 0,1$,

with $\sigma^2\mathbf{G}$, is the regular E–optimal spring balance weighing design.

Theorem 5. Let $v = 8$ and $n = \sum_{s=1}^{h}\sum_{l=1}^{2} b_{ls}$. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times 8}(0, 1)$ in (2) and (4), where $\mathbf{N}_{1s}$ and $\mathbf{N}_{2s}$ are incidence matrices of group divisible design with the same association scheme with parameters

(i) $b_{1s} = 4(t+1),$ $r_{1s} = 2(t+1),$ $k_{1s} = 4,$ $\lambda_{11s} = 0,$ $\lambda_{21s} = t+1$ and $b_{2s} = 4(6-t)$, $r_{2s} = 2(6-t)$, $k_{2s} = 4,$ $\lambda_{12s} = 6,$ $\lambda_{22s} = 5-t,$ $t = 1,2,3,$

(ii) $b_{1s} = 2(3t+2),$ $r_{1s} = 3t+2,$ $k_{1s} = 4,$ $\lambda_{11s} = t+1,$ $\lambda_{21s} = t+2$ and $b_{2s} = 6(4-t)$, $r_{2s} = 3(4-t)$, $k_{2s} = 4,$ $\lambda_{12s} = 4-t,$ $\lambda_{22s} = 5-t,$ $t = 1,2,$

with $\sigma^2\mathbf{G},$ is the regular E–optimal spring balance weighing design.

Theorem 6. Let $v = 10$ and $n = \sum_{s=1}^{h}\sum_{l=1}^{2} b_{ls}$. Any nonsingular $\mathbf{X} \in \mathbf{\Phi}_{n\times 10}(0, 1)$ in (2) and (4), where $\mathbf{N}_{1s}$ and $\mathbf{N}_{2s}$ are incidence matrices of group divisible design with the same association scheme with parameters $b_{1s} = 8u,$ $r_{1s} = 4u,$ $k_{1s} = 5,$ $\lambda_{11s} = 0,$ $\lambda_{21s} = 2u$ and $b_{2s} = 10u,$ $r_{2s} = 5u,$ $k_{2s} = 5,$ $\lambda_{12s} = 4u,$ $\lambda_{22s} = 2u,$ $u = 1,2,$ with $\sigma^2\mathbf{G},$ is the regular E–optimal spring balance weighing design.

Theorem 7. If $\mathbf{N}_1$ and $\mathbf{N}_2$ are incidence matrices of group divisible designs with the same association scheme with parameters $v = 2(2u+1),$ $b_1 = 4u,$ $r_1 = 2u,$ $k_1 = 2u+1,$ $\lambda_{11} = 0,$ $\lambda_{21} = u$ and $v = b_2 = 2(2u+1),$ $r_2 = k_2 = 2u+1,$ $\lambda_{12} = 2u,$ $\lambda_{22} = u,$ $u = 1,2,3,4,$ then any spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{2h(4u+1)\times 2(2u+1)}(0, 1)$ in the form $\mathbf{X} = 1_h \otimes \mathbf{X}^*$ for $\mathbf{X}^* = \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\end{bmatrix}'$, with $\sigma^2\mathbf{G},$ is the regular E–optimal.

Theorem 8. If $\mathbf{N}_1$ and $\mathbf{N}_2$ are incidence matrices of group divisible design with the same association scheme with parameters $v = 4(u+1),$ $b_1 = 2(2u+1),$ $r_1 = 2u+1,$ $k_1 = 2(u+1),$ $\lambda_{11} = 2u+1,$ $\lambda_{21} = u$ and $v = b_2 = 4(u+1),$ $r_2 = k_2 = 2(u+1),$ $\lambda_{12} = 0,$ $\lambda_{22} = u+1,$ $u = 1,2,3,4$, then any spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{2h(4u+3)\times 4(u+1)}(0, 1)$ in the form $\mathbf{X} = 1_h \otimes \mathbf{X}^*$ for $\mathbf{X}^* = \begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\end{bmatrix}'$ with $\sigma^2\mathbf{G},$ is the regular E–optimal.

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KONSTRUKCJA REGULARNEGO E–OPTYMALNEGO SPRĘŻYNOWEGO UKŁADU WAGOWEGO DLA PARZYSTEJ LICZBY OBIEKTÓW

W pracy przedstawiono zagadnienie konstrukcji sprężynowego układu wagowego spełniającego kryterium E–optymalności. Do konstrukcji macierzy układu wykorzystano macierze incydencji częściowo zrównoważonych układów bloków.