A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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OPTIMALITY OF SPRING BALANCE WEIGHING DESIGNS

Abstract. In this paper, the problem of existence of optimal spring balance weighing designs is discussed. Two optimality criterions are compared and the appropriate optimality conditions are presented.

Key words: A–optimal design, D–optimal design, spring balance weighing design.

I. INTRODUCTION

We consider the model of spring balance weighing design

$$\mathrm{E}(\mathbf{y}) = \mathbf{Xw}, \ \mathrm{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}, \quad (1)$$

where $\mathbf{y}$ is an $n \times 1$ random vector of the observations. Each observation measures the sum of the measurements of objects taken to the combination. $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$, $\mathbf{\Phi}_{n\times t}(0, 1)$ denotes the class of $n \times t$ matrices $\mathbf{X} = (x_{ij})$, $i = 1,2,...,n$, $j = 1,2,...,t$, having entries $x_{ij} = 1$ or 0, $\mathbf{w} = (w_1, w_2,..., w_t)'$ is a vector representing true measurements of objects and $\mathbf{e}$ is the $n \times 1$ random vector of errors, $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$, $\mathrm{E}(\mathbf{ee}') = \sigma^2\mathbf{G}$, where $\mathbf{G}$ is positive definite known matrix. If the designs matrix $\mathbf{X}$ is of full column rank, then all w_j are estimable and the variance matrix of their best linear unbiased estimator is $\sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$

In the theory of optimal designs, very important role plays the matrix $\mathbf{G}$. It describe the relations between measurement errors. Each form of that matrix requires specific investigations.

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II. OPTIMALITY CRITERIA

Kiefer (1974) considered the φ_p (for $p \geq 0$) optimality criteria in the form

$$\varphi_p = \begin{cases} \det(\mathbf{\Omega})^{-1} & \text{for} \quad p = 0 \\ \operatorname{tr}(\mathbf{\Omega}^{-1}) & \text{for} \quad p = 1 \end{cases}, \tag{2}$$

where $\mathbf{\Omega} = \mathbf{X}'\mathbf{X}$. The D-, A- criteria of optimality are the same as φ_0 and φ_1, respectively. Minimizing φ_p is equivalent to maximizing Pukelsheim's φ_{-p} (see Pukelsheim (1993)), which is also defined for $-1 \leq p < 0$. Here, we generalize the optimum criterion given by Kiefer (1974). For positive definite matrix $\mathbf{G}$ of known elements, the information matrix for estimating $\mathbf{w}$ is $\mathbf{\Omega_G} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. Thus

$$\varphi_p = \begin{cases} \det(\mathbf{\Omega_G})^{-1} & \text{for} \quad p = 0 \\ \operatorname{tr}(\mathbf{\Omega_G})^{-1} & \text{for} \quad p = 1 \end{cases}$$ and we have the following definition.

Definition 2.1. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is said to be

i) A-optimal if and only if $\operatorname{tr}(\mathbf{\Omega_G})^{-1}$ is minimal,

ii) D-optimal if and only if $\det(\mathbf{\Omega_G})^{-1}$ is minimal.

III. THE DESIGN MATRIX

In Jacroux and Notz (1983) the optimal designs are presented.

Theorem 3.1. Let t be odd and let the condition

$$\mathbf{X}'\mathbf{X} = \frac{(t+1)n}{4t}\left(\mathbf{I}_t + \mathbf{1}_t\mathbf{1}_t'\right) \tag{4}$$

be satisfied. Thus $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$ is A-optimal and D-optimal.

It is worth pointing out that in the class $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$ for a given n and t, the number $n(t+1)(4t)^{-1}$ must be integer. This requirement is very restricted as we are not able to construct optimal design for any n and t and it is needed to select an optimal weighing design. Accordingly, let $\mathbf{\Psi}_{(n-1)\times t}(0,1)$ denotes the class of matrices of spring balance weighing designs in that (4) is true, i.e. for any $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-1)\times t}(0,1)$ the condition

$$\mathbf{X}_1^{'}\mathbf{X}_1 = \frac{(t+1)(n-1)}{4t}\left(\mathbf{I}_t + \mathbf{1}_t\mathbf{1}_t^{'}\right) \tag{5}$$

is satisfied.

Special attention in this paper is given to the determining optimal design in the class $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$, in that optimal design satisfying conditions given by Jacroux and Notz (1983) doesn't exist. Because of this the motivation of this paper is to consider the situation in that the measurements are taken in two different conditions or on two different installations, n_1 measurements are taken with variance σ^2 and one, additionally measurement in other conditions with variance $g^{-1}\sigma^2$. Thus

$$\mathbf{G} = \begin{bmatrix} \mathbf{I}_{n_1} & \mathbf{0}_{n_1} \\ \mathbf{0}_{n_1}^{'} & g^{-1} \end{bmatrix}. \tag{6}$$

According to (6), we consider $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}^{'} \end{bmatrix}, \tag{7}$$

where $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-1)\times t}(0,1)$ satisfies (5), $\mathbf{x}$ is $t\times 1$ vector of elements equal to 1 or 0. The form (7) could be interpreted as construction the design matrix $\mathbf{X}$ in the class $\mathbf{\Phi}_{n\times t}(0,1)$ based on matrix $\mathbf{X}_1$ from the class $\mathbf{\Psi}_{(n-1)\times t}(0,1)$, which is A- and D-optimal. In the other words, the problem is how to add to n_1 measurements one additionally measurement to become optimal design. Theorems given by Katulska and Przybył (2007) and Graczyk (2011) will be required to prove the main result of this paper.

Theorem 3.1. Katulska and Przybył (2007). Let t be odd. If $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ then any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is regular D-optimal.

Theorem 3.2. Graczyk (2011). Let t be odd. The design $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is A-optimal if

i) for fixed g, $g \in (0, P_0)$, if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$,

ii) for fixed g, $g \in (L_s, P_s)$, if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$,

iii) for fixed g, $g \in \left(L_{\frac{p-1}{2}}, \infty\right)$, if $\mathbf{x}'\mathbf{1}_t = 1$,

iv) for fixed $g = P_s$, if $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2} - s$ or $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$,

where $P_0 = \frac{n-1}{p(p-1)}$, $L_s = \frac{(p+1)(2p(s-1)+4s-3)(n-1)}{p(p-2s+1)(p+2s+1)}$,

$P_s = \frac{(p+1)(2ps+4s+1)(n-1)}{p(p-2s-1)(p-2s+1)}$, $s = 1,2,\ldots,\frac{p-3}{2}$.

Theorem 3.3. Let $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ be nonsingular spring balance weighing design in the form (7) for that the condition (5) is satisfied.

i) If $g \in (0, P_0)$ and $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$, then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A- and D-optimal.

ii) If $g \in (L_s, P_s)$ and moreover if

a. $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A-optimal,

b. $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is D-optimal.

iii) If $g \in \left(L_{\frac{p-1}{2}}, \infty\right)$ and moreover if

a. $\mathbf{x}'\mathbf{1}_t = 1$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A-optimal,

b. $\mathbf{x}^{'}\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is D-optimal.

iv) If $g = P_s$ and moreover

a. $\mathbf{x}^{'}\mathbf{1}_t = \frac{t-1}{2} - s$ or $\mathbf{x}^{'}\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A-optimal,

b. $\mathbf{x}^{'}\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is D-optimal.

Proof. For positive definite matrix $\mathbf{G}$ in (6) of known elements, let us consider the design $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-1)\times t}(0,1)$ which satisfies Condition (5), i.e. $\mathbf{X}_1$ is A- and D-optimal. Based on that design, we form $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ in (7). According to optimality criterion we consider $\mathbf{\Omega}_{\mathbf{G}}$. We have

$$\mathbf{\Omega}_{\mathbf{G}} = \mathbf{X}^{'}\mathbf{G}^{-1}\mathbf{X} = \mathbf{X}_1^{'}\mathbf{X}_1 + g\mathbf{x}\mathbf{x}^{'} . \tag{8}$$

The proof falls naturally into two parts. First we determine D-optimal design. We count $\det(\mathbf{\Omega}_{\mathbf{G}})^{-1}$. Katulska and Przybył (2007) showed that

$$\det(\mathbf{\Omega}_{\mathbf{G}}) \le (t+1)\left(1 + \frac{gt}{n-1}\right)\left(\frac{(t+1)(n-1)}{4t}\right)^{t} . \tag{9}$$

and, moreover they showed that maximum of (9), i.e. minimum of $\det(\mathbf{\Omega}_{\mathbf{G}})^{-1}$ is attained if and only if $\mathbf{x}^{'}\mathbf{1}_t = \frac{t+1}{2}$. Thus if $\mathbf{x}^{'}\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is D-optimal. Next we check if in $\mathbf{\Phi}_{n\times t}(0,1)$ exists A-optimal design. Thus we count $\mathrm{tr}(\mathbf{\Omega}_{\mathbf{G}})^{-1}$ and we determine $\mathbf{x}$ for which minimum of $\mathrm{tr}(\mathbf{\Omega}_{\mathbf{G}})^{-1}$ is attained. From Graczyk (2011) we have, if $g \in (0, P_0)$ then

$$\mathrm{tr}(\mathbf{\Omega}_{\mathbf{G}})^{-1} \ge \frac{4t^3(n-1+g(t-1))}{(t+1)^2(n+tg-1)(n-1)} \tag{10}$$

and if $g = P_0$ then

$$\operatorname{tr}(\mathbf{\Omega_G})^{-1} \geq \frac{4t(t-1)}{(t+1)(n-1)}. \tag{11}$$

If $g \in (L_s, P_s)$ then

$$\operatorname{tr}(\mathbf{\Omega_G})^{-1} \geq \frac{4t^3}{(t+1)^2(n-1)} - \frac{4t^2 g(t-2s+1)}{(t+1)^2(n-1)} M_s, \tag{12}$$

where $M_s = \dfrac{t(t+1)+2s(t+2)}{(t+1)^2(n-1)+tg(t-2s+1)(t-2s+3)}$ and if $g = P_s$ then

$$\operatorname{tr}(\mathbf{\Omega_G})^{-1} \geq \frac{4t(t(2s+1)-(4s+1))}{(t+1)^2(n-1)(2s+1)}. \tag{13}$$

If $g \in \left(L_{\frac{p-1}{2}}, \infty\right)$ then

$$\operatorname{tr}(\mathbf{\Omega_G})^{-1} \geq \frac{4t^2\left(t(t+1)(n-1)+4g(t-1)^2\right)}{(t+1)^2(n-1)(2s+1)}. \tag{14}$$

For $g \in (0, P_0)$, minimum of (10) is attained if and only if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$. Hence if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A-optimal. If $g = P_0$ then the equality in (11) is fulfilled if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ or $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2}$. For $g \in (L_s, P_s)$ minimum of (12) is attained if and only if $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2}$. If $g = P_s$ then the equality in (13) is true if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ or $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2} - s$. In this case $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$ is A-optimal. We have, if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0,1)$

is A-optimal. If $g = P_s$ then the equality in (13) is true if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ or $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2} - s$. In this case $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$ is A-optimal. Finally, if $g \in \left(L_{\frac{p-1}{2}}, \infty \right)$ then minimum of (14) is attained if and only if $\mathbf{x}'\mathbf{1}_t = 1$. Thus if $\mathbf{x}'\mathbf{1}_t = 1$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times t}(0, 1)$ is A-optimal. Hence the theorem.

Now, we form the design matrix $\mathbf{X}_1 \in \mathbf{\Psi}_{(n-1)\times t}(0, 1)$ based on the incidence matrix of balanced incomplete block design with the parameters v, b, r, k, λ, see Raghavarao and Padgett (2005), as $\mathbf{X}_1 = \mathbf{N}'$.

Lemma 3.1. If $\mathbf{N}$ is the incidence matrix of balanced incomplete block design with the parameters

1) $v = 4q+1$, $b = 2(4q+1)$, $r = 2(2q+1)$, $k = 2q+1$, $\lambda = 2q+1$ or

2) $v = 4q-1$, $b = 4q-1$, $r = 2q$, $k = 2q$, $\lambda = q$, $q = 1,2,\ldots$.

then for $\mathbf{X}_1 = \mathbf{N}'$ the condition (5) is fulfilled.

The proof is left for the reader.

From now on, we consider $\mathbf{X}_1 = \mathbf{N}'$, where $\mathbf{N}$ is the incidence matrix of balanced incomplete block design with the parameters given in Lemma 3.1. The parameter connected with precision of measurements g is given. The values of g determining respectively intervals are the same as in Theorem 3.2. Thus in next corollaries, according to the value of g we give the conditions determining optimal design $\mathbf{X}$ in the class $\mathbf{\Phi}_{n\times t}(0, 1)$.

Corollary 3.1. If $g \in (0, P_0)$ and if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$, then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0, 1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is A- and D-optimal.

Corollary 3.2. If $g \in (L_s, P_s)$ and moreover if

i) $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0, 1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is A-optimal,

ii) $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0, 1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is D-optimal.

Corollary 3.3. If $g \in \left(L_{\frac{p-1}{2}}, \infty \right)$ and moreover if

i) $\mathbf{x}'\mathbf{1}_t = 1$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is A-optimal,

ii)if $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is D-optimal.

Corollary 3.3. If $g = P_s$ and moreover

i) $\mathbf{x}'\mathbf{1}_t = \frac{t-1}{2} - s$ or $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2} - s$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is A-optimal,

ii) $\mathbf{x}'\mathbf{1}_t = \frac{t+1}{2}$ then $\mathbf{X} \in \mathbf{\Phi}_{(b+1)\times t}(0,1)$ in the form (7) with the variance matrix of errors $\sigma^2\mathbf{G}$ in (6) is D-optimal.

IV. EXAMPLE

Let us consider experiment in that using $n = 11$ measurements we determine unknown weights of $t = 5$ objects. For the construction of design matrix we use the incidence matrix $\mathbf{N}$ of balanced incomplete block design with the parameters $v = 5$, $b = 10$, $r = 6$, $k = 3$, $\lambda = 3$ given as

$$\mathbf{N} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Thus if $g \in \left(0, \frac{1}{2}\right)$ then $\mathbf{X}_2$ is A- and D-optimal, if $g \in \left(\frac{1}{2}, \frac{45}{2}\right)$ then $\mathbf{X}_2$ is D-optimal and $\mathbf{X}_3$ is A-optimal, if $g \in \left(\frac{45}{2}, +\infty\right)$ then $\mathbf{X}_2$ is D-optimal and $\mathbf{X}_4$ is A-optimal, moreover if $g = \frac{1}{2}$ then $\mathbf{X}_2$ is A- and D-optimal and $\mathbf{X}_3$ is A-optimal, finally if

$g = \frac{45}{2}$ then $\mathbf{X}_2$ is D-optimal and $\mathbf{X}_3$, $\mathbf{X}_4$ are A-optimal, where $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4 \in \mathbf{\Phi}_{11\times 5}(0,1)$ are given in as

$$\mathbf{X}_2' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{X}_3' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{X}_4' = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

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OPTYMALNOŚĆ W SPRĘŻYNOWYCH UKŁADACH WAGOWYCH

W pracy przedstawiono zagadnienie A- i D-optymalności sprężynowego układu wagowego. Rozważania teoretyczne zostały zobrazowane przykładem konstrukcji macierzy układu w oparciu o macierze incydencji układów zrównoważonych o blokach niekompletnych.