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OPTIMAL WEIGHING DESIGNS FOR ESTIMATION OF TOTAL WEIGHT

Abstract. Optimal chemical and spring balance weighing designs for estimation total weight are considered. In this paper we study weighing designs in which the errors are non-positive correlated and have equal variances. A lower bound for the variance of estimated total weight is attained and the necessary and sufficient conditions for the attainability of this lower bound are given. There are given construction methods and a few examples of the design matrices.

Key words: block design, chemical balance weighing design, spring balance weighing design, total weight.

I. INTRODUCTION

We consider the standard Gauss-Markov model $\mathbf{y} = \mathbf{Xw} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ vector of observations, $\mathbf{X}$ is the $n \times p$ design matrix, $\mathbf{w}$ is a $p \times 1$ vector of unknown parameters and $\mathbf{e}$ is an $n \times 1$ vector of random errors with $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is vector of zeros, σ^2 is the constant variance of errors, $\mathbf{G}$ is the $n \times n$ symmetric positive definite matrix of known elements. There are two types of weighing designs, chemical balance weighing design and spring balance weighing design. In a chemical balance weighing design each weighing measures the difference between the total weight of objects put on one of the two pans and the total weight of those on the other pan. In a spring balance weighing design, each observation measures the total weight of the objects put on the balance.

For a chemical balance weighing design, $\mathbf{X}$ is an $n \times p$ matrix with (i, j)-th entry equal to –1, 1 or 0 depending upon whether in the i-th weighing j-th object is put on the left pan, right pan or is not present, while for a spring balance weighing design, each entry of $\mathbf{X}$ is 0 or 1, indicating whether a particular object is absent or present in each weighing.

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Any weighing design is nonsingular if the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular. It is obvious that $\mathbf{G}$ is the symmetric positive definite matrix then any weighing design is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular and then all parameters are estimable. Even if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, there exists a design which estimates the total weight with a smaller variance than the design which is most efficient for the estimation of individual weights. The examples of such designs are available in literature, see Banerjee (1975), Chacko and Dey (1978). When $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular, although unknown measurements off all objects are not estimable, but some linear functions of $\mathbf{w}$ may be estimable. One of the estimable function is the total weight of objects, i.e. $\mathbf{1}'_p\mathbf{w}$, where $\mathbf{1}_p$ denotes the $p\times 1$ vector of ones. Some examples of optimal singular weighing design for estimated total weight are given in Dey and Gupta (1977), Ceranka and Katulska (1986, 1990, 1996), Kageyama (1988, 1990), Katulska (1989). The total weight will be estimable if and only if there exists an $n\times 1$ vector $\mathbf{a}$ such that $\mathbf{a}'\mathbf{X}=\mathbf{1}'_p$. The condition is equivalent to the $\mathbf{1}'_p\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}=\mathbf{1}'_p$, where $\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-}$ denotes a generalized inverse (g-inverse) of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$, i.e. $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}=\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$.

It is therefore assumed that $k\ (<p)$ objects can be weighted simultaneously in each weighing. Under this restriction, a lower bound for the variance of the estimated total weight is obtained using a weighing design permitting the estimation of total weight. Design for which the lower bound is attainable have been called optimum.

In this paper we present the estimation of total weight of objects in the weighing design assuming that the errors have the same variances and they are non-positive correlated, i.e. for the random vector of errors $\mathbf{e}$, $\mathrm{Cov}(\mathbf{e})=\sigma^2\mathbf{G}$, where

$$\mathbf{G}=g\left[(1-\rho)\mathbf{I}_n+\rho\mathbf{1}_n\mathbf{1}'_n\right],\quad g>0,\frac{-1}{n-1}<\rho\leq 0\,, \tag{1}$$

$\mathbf{I}_n$ is $n\times n$ identity matrix. Let note, for $g>0$ and $g>0,\dfrac{-1}{n-1}<\rho\leq 0$, the matrix $\mathrm{Cov}(\mathbf{e})=\sigma^2\mathbf{G}$ is positive definite and $\mathbf{G}^{-1}=\dfrac{1}{g(1-\rho)}\left[\mathbf{I}_n-\dfrac{\rho}{1+\rho(n-1)}\mathbf{1}_n\mathbf{1}'_n\right]$.

For the case $\mathbf{G}=\mathbf{I}_n$, Banerjee (1975) and Dey and Gupta (1977) present the problem related to the spring balance weighing designs. For some patterns of $\mathrm{Cov}(\mathbf{e})=\sigma^2\mathbf{G}$ the conditions determining optimal design for estimating total weight were given in Katulska (1989). Similarly for the case $\mathbf{G}=\mathbf{I}_n$, Banerjee

(1975) and Pukelsheim (1983) produce the issue related to the chemical balance weighing designs. Moreover, for the case $n = p$, Kageyama (1990) gives conditions to determine the optimal design for estimating the total weight. For some special forms of $\text{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}$, the conditions of estimation of the total weight were presented in Ceranka and Katulska (1990, 1996).

II. THE LOWER BOUND OF VARIANCE

Let $\mathbf{D}_{n\times p}$ denote the class of all $n \times p$ (–1,0,1)-matrices such that

1. Each row of matrix $\mathbf{X} \in \mathbf{D}_{n\times p}$ contains at most $k\ (< p)$ elements different from zero, i.e. $a_i + f_i = k_i \le k < p$, $i = 1,2,...,n$, where $a_i \ge 0,\ f_i \ge 0,\ a_i$ denotes the number of $x_{ij} = -1$ and f_i denotes the number of $x_{ij} = 1$.

2. For every $\mathbf{X} \in \mathbf{D}_{n\times p}$ the total weight is estimable.

Let $\mathbf{D}^s_{n\times p} \subset \mathbf{D}_{n\times p}$ denote the subclass of matrices for which $a_i = 0$ and $f_i > 0$ for each $i = 1,2,...,n$ and let $\mathbf{D}^c_{n\times p} \subset \mathbf{D}_{n\times p}$ denote the subclass of matrices for which $a_i \ge 1$ and $f_i \ge 1$ for each $i = 1,2,...,n$. If $\mathbf{X} \in \mathbf{D}^s_{n\times p}$ then $\mathbf{X}$ is the matrix of spring balance weighing design, while $\mathbf{X} \in \mathbf{D}^c_{n\times p}$ then $\mathbf{X}$ is the matrix of chemical balance weighing design.

The following lemma given in Ceranka and Katulska (1996) will be needed to prove the next theorem.

Lemma 1. For any symmetric positive definite matrix $\text{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}$ and any $\mathbf{X} \in \mathbf{D}_{n\times p}$ a necessary condition for the total weight to be estimable is that $a_i \ne f_i$ for at least one i.

Moreover, Katulska (1989) proved the following lemma.

Lemma 2. For any symmetric positive definite $n \times n$ matrix $\text{Cov}(\mathbf{e}) = \sigma^2\mathbf{G}$, $\mathbf{X} \in \mathbf{D}_{n\times p}$ and any vector $\mathbf{c} \ne \mathbf{0}$ satisfying the condition $\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mathbf{c}'$,

$$\mathbf{c}'(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{c} \ge \frac{(\mathbf{c}'\mathbf{c})^2}{\mathbf{c}'\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{c}}. \tag{2}$$

Equality holds in (2) if and only if $\mathbf{c}$ is an eigenvector of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$.

Theorem 1. In any weighing design $\mathbf{X} \in \mathbf{D}_{n \times p}$ with the covariance matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (1) the variance of the estimator of the total weight is given as

$$Var(\widehat{\mathbf{1}'_p \mathbf{w}}) \geq \frac{\sigma^2 p^2 g(1+\varsigma(n-1))}{nd^2} \qquad (3)$$

The equality holds if $d = \begin{cases} k, & \mathbf{X} \in \mathbf{D}^s_{n \times p} \\ k-2m, & \mathbf{X} \in \mathbf{D}^c_{n \times p} \end{cases}$, where $\min_i (a_i, f_i) = m_i$, $\min_i m_i = m \geq 1$.

Proof. In the case considered here, under the above assumptions and considering Lemma 2 we obtain $Var(\widehat{\mathbf{1}'_p \mathbf{w}}) = \mathrm{Var}\left(\mathbf{1}'_p \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-} \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}\right) =$ $= \sigma^2 \mathbf{1}'_p \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)\mathbf{1}_p \geq \dfrac{\sigma^2 p^2}{\mathbf{1}'_p \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)\mathbf{1}_p}$ and the equality holds if $\mathbf{1}_p$ is an eigenvector of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. Furthermore,

$$\mathbf{1}'_p \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)\mathbf{1}_p = \frac{1}{g(1-\rho)}\left(\mathbf{1}'_p \mathbf{X}'\mathbf{X}\mathbf{1}_p - \frac{\rho}{1+\rho(n-1)}\mathbf{1}'_p \mathbf{X}'\mathbf{1}_n\mathbf{1}'_n\mathbf{X}\mathbf{1}_p\right) =$$

$$= \frac{1}{g(1-\rho)}\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{p} x_{ij}\right)^2 - \frac{\rho}{1+\rho(n-1)}\left(\sum_{i=1}^{n}\sum_{j=1}^{p} x_{ij}\right)^2\right) \leq$$

$$\frac{1}{g(1-\rho)}\left(nd^2 - \frac{\rho}{1+\rho(n-1)}n^2 d^2\right) = \frac{nd^2}{\rho(1+\rho(n-1))}.$$

If $\mathbf{X} \in \mathbf{D}^s_{n \times p}$, the equality holds if and only if $k_i = k$ for all $i = 1,2,...,n$. If $\mathbf{X} \in \mathbf{D}^c_{n \times p}$, the equality holds if and only if $k_i - 2m_i = k - 2m$ for all $i = 1,2,...,n$. Thus (3) and the proof is completed.

Remark 1. In the special case $\mathbf{X} \in \mathbf{D}^s_{n \times p}$ and $\mathbf{G} = \mathbf{I}_n$, Theorem 1 was given by Dey and Gupta (1977), whereas in Ceranka and Katulska (1986) was proved under assumption $\mathbf{G} = diag(g_1, g_2, ..., g_n)$, $g_i > 0$ for $i = 1,2,...,n$, and in Ceranka and Graczyk (2011) when $\mathbf{G}$ is of the form (1).

Remark 2. In the special case $\mathbf{X} \in \mathbf{D}_{n \times p}^{c}$ and $\mathbf{G} = \mathbf{I}_n$ and in addition $a_i = 1$ and $f_i = k - 1$ or $a_i = k - 1$ and $f_i = 1$ for $i = 1,2,...,n$, the inequality (3) was proved by Chacko and Dey (1978) and the equality in (3) was given in Kageyama (1988). For $\mathbf{G}$ in the form (1), Theorem 1 was given by Ceranka and Graczyk (2012).

From now on, $\mathbf{G}$ is in the form (1).

Definition 1. Any weighing design $\mathbf{X} \in \mathbf{D}_{n \times p}$ with the covariance matrix $\sigma^2 \mathbf{G}$ is said to be optimal for the estimated total weight if the variance of the estimator of total weight attains the lower bound given in Theorem 1, i.e.

$$Var(\widehat{\mathbf{1}'_p \mathbf{w}}) = \frac{\sigma^2 p^2 g(1 + p(n-1))}{nd^2}. \tag{4}$$

Theorem 2. Any weighing design $\mathbf{X} \in \mathbf{D}_{n \times p}$ with the covariance matrix $\sigma^2 \mathbf{G}$ is said to be optimal for the estimated total weight if and only if

(i) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mu \mathbf{1}_p$ and

(ii) $\begin{cases} k_i = k & \text{for} \quad \mathbf{X} \in \mathbf{D}_{n \times p}^{s} \\ |a_i - f_i| = k - 2m & \text{for} \quad \mathbf{X} \in \mathbf{D}_{n \times p}^{c} \end{cases}$

for each $i = 1,2,...,n$.

Proof. Let us consider $\mathbf{X} \in \mathbf{D}_{n \times p}$ and $\mathbf{G}$. Let denote, $\mathbf{u} = \mathbf{G}^{-\frac{1}{2}}\mathbf{X}\mathbf{1}_p$ and $\mathbf{v} = \mathbf{G}^{-\frac{1}{2}}\mathbf{X}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{1}_p$. Applying the Cauchy-Schwarz inequality on $\mathbf{u}$ and $\mathbf{v}$ we have $(\mathbf{u}'\mathbf{v})^2 \le (\mathbf{u}'\mathbf{u})(\mathbf{v}'\mathbf{v})$. Equality holds if and only if $\mathbf{u} = \mu \mathbf{v}$ for some scalar μ. Substituting for $\mathbf{u}$ and $\mathbf{v}$, the condition $\mathbf{u} = \mu \mathbf{v}$ reduces to $\mathbf{G}^{-\frac{1}{2}}\mathbf{X}\mathbf{1}_p = \mu \mathbf{G}^{-\frac{1}{2}}\mathbf{X}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-}\mathbf{1}_p$ which is equivalent to $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \mu \mathbf{1}_p$ that proves (i). The equality in (3) is attained if and only if $k_i = k,$ when $\mathbf{X} \in \mathbf{D}_{n \times p}^{s}$ or $k_i - 2m_i = k - 2m,$ when $\mathbf{X} \in \mathbf{D}_{n \times p}^{c}$, $i = 1,2,...,n.$ Hence (ii) is true.

Theorem 3. In any weighing design $\mathbf{X} \in \mathbf{D}_{n \times p}$ with the covariance matrix $\sigma^2 \mathbf{G}$ the conditions (i) and (ii) of Theorem 2 are equivalent to

(i) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \vartheta \mathbf{1}_p$ and

(ii) $\mathbf{X}\mathbf{1}_p = d\mathbf{1}_n$, where $\vartheta = \frac{\mu}{d}$.

Proof. To prove the theorem we first observe that from (ii) of Theorem 2 we get $\mathbf{X1}_p = d\mathbf{1}_n$. Under condition (i) of Theorem 2, $\mu\mathbf{1}_p$ implies $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_n = \frac{\mu}{d}\mathbf{1}_p$. On the other hand, we assume that the conditions given in Theorem 3 are true. From $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \vartheta\mathbf{1}_p$ we have $\mathbf{X}'\mathbf{G}^{-1}d\mathbf{1}_n = \vartheta d\mathbf{1}_p$. Taking $\mathbf{X1}_p$ for $d\mathbf{1}_n$ we obtain $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X1}_p = \vartheta d\mathbf{1}_p = \mu\mathbf{1}_p$. Moreover, $\mathbf{X1}_p = d\mathbf{1}_n$ is equivalent to (ii) of Theorem 2 and we get required result.

Note that for $\mathbf{G}$, $\mathbf{G1}_n = \alpha\mathbf{1}_n$ and $\mathbf{G}^{-1}\mathbf{1}_n = \frac{1}{\alpha}\mathbf{1}_n$, where $\alpha = g(1+\rho(n-1))$.

Corollary 1. In any weighing design $\mathbf{X} \in \mathbf{D}_{n\times p}$ with the covariance matrix $\sigma^2\mathbf{G}$, the condition (i) of Theorem 3 is equivalent to $\mathbf{X}'\mathbf{1}_n = \omega\mathbf{1}_p$, where $\omega = \frac{\mu\alpha}{d}$.

Above considerations imply that $\mathbf{X}'\mathbf{1}_n = \omega\mathbf{1}_p$, i.e. the sum of elements in each column of the design matrix $\mathbf{X} \in \mathbf{D}_{n\times p}$ is the same. On the other hand $\mathbf{X}'\mathbf{1}_n = \mu\alpha d^{-1}\mathbf{1}_p$, i.e. the sum of elements in each column of the design matrix $\mathbf{X}$ depends on the matrix $\mathbf{G}$. Comparing equalities $\omega\mathbf{1}_p$ and $\mu\alpha d^{-1}\mathbf{1}_p$ and place forms of α, μ, d and $\mathbf{G}^{-1}$ we obtain identity. Here is why we conclude that the optimal design $\mathbf{X} \in \mathbf{D}_{n\times p}$ is the same for any ρ, $-(n-1)^{-1} < \rho \le 0$, i.e. this design is robust for different ρ. The results given in above theorems imply next corollary.

Corollary 2. Any weighing design $\mathbf{X} \in \mathbf{D}_{n\times p}$ with the covariance matrix $\sigma^2\mathbf{I}_n$ is optimal for the estimated total weight if and only if such design is optimal for the estimated total weight with the covariance matrix $\sigma^2\mathbf{G}$.

III. CONSTRUCTION OF THE DESIGN MATRIX
III.I. CONSTRUCTION FOR $\mathbf{X} \in \mathbf{D}_{n\times p}^{S}$

Let $\mathbf{N}$ denote the usual $v \times b$ binary incidence matrix of block design, where v and b mean the number of treatments and number of blocks, respectively. Let $\mathbf{N1}_b = r\mathbf{1}_v$ and $\mathbf{N}'\mathbf{1}_v = k\mathbf{1}_b$, where r is the number of replications of ith treatment and k is the size of jth block, $i = 1,2,...,n$, $j = 1,2,...,b$.

Theorem 4. Any weighing design $\mathbf{X} \in \mathbf{D}_{n\times p}^{s}$, $\mathbf{X} = \mathbf{N}$ $\left(\text{or } \mathbf{X} = \mathbf{N}'\right)$, with the covariance matrix $\sigma^2\mathbf{G}$ is optimal for estimated total weight of $p = b$ $\left(\text{or } p = v\right)$ objects in $n = v$ $\left(\text{or } n = b\right)$ weighings.

Proof. Let note, if $\mathbf{X} = \mathbf{N}$ then $d = r$, if $\mathbf{X} = \mathbf{N}'$ then $d = k$. Taking $\mathbf{a} = k^{-1}\mathbf{1}_v$ $\left(\text{or } \mathbf{a} = r^{-1}\mathbf{1}_b\right)$ it is clear that condition $\mathbf{a}'\mathbf{N} = \mathbf{1}_p'$ is satisfied for $\mathbf{X} = \mathbf{N}$ $\left(\text{or } \mathbf{X} = \mathbf{N}'\right)$. The condition given in Corollary 1 and condition (ii) of Theorem 3. follow from the equalities $\mathbf{N1}_b = r\mathbf{1}_v$ and $\mathbf{N}'\mathbf{1}_v = k\mathbf{1}_b$.

IV. CONSTRUCTION FOR $\mathbf{X} \in \mathbf{D}_{n\times p}^{c}$

Let $\mathbf{N}_h$ be the incidence matrix of block design with parameters v, b_h, r_h, k_h, $h = 1,2$. Thus

$$\mathbf{X} = \left[s\mathbf{N}_1 \quad 2\mathbf{N}_2 - \mathbf{1}_v\mathbf{1}_{b_2}'\right], \; s = 1 \text{ or } -1. \tag{5}$$

Theorem 5. Any weighing design $\mathbf{X} \in \mathbf{D}_{n\times p}^{c}$, in the form (5) with the covariance matrix $\sigma^2\mathbf{G}$ is optimal for estimated total weight of $p = b_1 + b_2$ objects in $n = v$ weighings if

(i) $sr_1 \neq b_2 - 2r_2$ and

(ii) $sk_1 = 2k_2 - v$.

Proof. According to the condition (ii) of Theorem 3, $\mathbf{X1}_p = \left(sr_1 + 2r_2 - b_2\right)\mathbf{1}_n = d\mathbf{1}_n$. Furthermore, $\mathbf{X}'\mathbf{1}_n = \left[sk_1\mathbf{1}'_{b_1} \quad \left(2k_2 - v\right)\mathbf{1}'_{b_2}\right]'$, hence $sk_1 = 2k_2 - v$. The condition $a_i \neq f_i$ given in Lemma 1 implies that $sr_1 \neq b_2 - 2r_2$. Hence the result.

Now, let $\mathbf{N}_h$ be the incidence matrix of the incomplete block design with parameters v_h, b, r_h, k_h, $h = 1,2$. Thus

$$\mathbf{X} = \left[s\mathbf{N}_1' \quad 2\mathbf{N}_2' - \mathbf{1}_b\mathbf{1}_{b_{v2}}' \right], \; s = 1 \text{ or } -1. \tag{6}$$

Theorem 6. Any weighing design $\mathbf{X} \in \mathbf{D}_{n\times p}^{c}$, in the form (6) with the covariance matrix $\sigma^2\mathbf{G}$ is optimal for estimated total weight of $p = v_1 + v_2$ objects in $n = b$ weighings if

(i) $sk_1 \neq v_2 - 2k_2$ and

(ii) $sr_1 = 2r_2 - b$.

Proof. Condition (ii) of Theorem 3 implies that $\mathbf{X1}_p = (sk_1 + 2k_2 - v_2)\mathbf{1}_n = d\mathbf{1}_n$. Besides $\mathbf{X}'\mathbf{1}_n = \left[sr_1\mathbf{1}'_{v_1} \quad (2r_2 - b)\mathbf{1}'_{v_2} \right]'$, hence $sr_1 = 2r_2 - b.$ The condition $a_i \neq f_i$ given in Lemma 1 implies that $sk_1 \neq v_2 - 2k_2$. Thus we become the result.

IV. EXAMPLES
EXAMPLE 1

Let us consider $\mathbf{X} \in \mathbf{D}_{6\times 15}^{c}$. According to Theorem 5, we construct the incidence matrix $\mathbf{N}_1$ of partially balanced incomplete block design with the parameters v_h, $b_1 = 9$, $r_1 = 3$, $k_1 = 2$, $\lambda_{11} = 0$, $\lambda_{21} = 1$ (design SR6, Clatworthy, 1973) and the incidence matrix $\mathbf{N}_2$ of partially balanced incomplete block design with the parameters $v = 6$, $b_2 = 6$, $r_2 = 4$, $k_2 = 4$, $\lambda_{12} = 3$, $\lambda_{22} = 2$ (design R94, Clatworthy, 1973), where

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \; \mathbf{N}_2 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Thus the design matrix $\mathbf{X} \in \mathbf{D}_{6\times 15}^{c}$ of the form (5) for $s = 1$ is given as

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Let us consider the covariance matrix $\sigma^2\mathbf{G}$ for $g=1$, where

$$\mathbf{G} = \begin{bmatrix} 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 \end{bmatrix}.$$

We have $\alpha = \frac{2}{7}$, $\mathbf{X}\mathbf{1}_{15} = 5\cdot\mathbf{1}_6$. Thus $d=5$, $\mathbf{X}'\mathbf{1}_6 = 2\cdot\mathbf{1}_{15}$. So, we have $\omega = 2$, $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_6 = 7\cdot\mathbf{1}_{15}$ and $\vartheta = 7$. Since $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{1}_{15} = 35\cdot\mathbf{1}_6$, then $\mu = 35$ and $2 = \omega = \frac{\mu\alpha}{d} = \frac{35\cdot\frac{2}{7}}{5}$. Moreover, for the design $\mathbf{X}$ with $\mathbf{G}$, $Var(\widehat{\mathbf{1}'_{15}\mathbf{w}}) = 0{,}43\sigma^2$. It is easy to see that for covariance matrix of errors $\sigma^2\mathbf{G}$, the design $\mathbf{X}\in\mathbf{D}^c_{6\times 15}$ that satisfies Theorem 3 is optimal for estimation of the total weight.

EXAMPLE 2

Let us consider the experiment in which we determine total weight of $p=6$ objects in $n=3$ measurements operations. According to Theorem 4 we consider design matrix $\mathbf{X}\in\mathbf{D}^s_{3\times 6}$ and the covariance matrix $\sigma^2\mathbf{G}$ for $g=1$, where

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 1 \end{bmatrix}.$$

We have $\alpha = \frac{1}{3}$, $\mathbf{X1}_6 = 4 \cdot \mathbf{1}_3$. Hence $d = 4$, $\mathbf{X}'\mathbf{1}_3 = 2 \cdot \mathbf{1}_6$. So, we have $\omega = 2$, $\mathbf{X}'\mathbf{G}^{-1}\mathbf{1}_6 = 6 \cdot \mathbf{1}_6$ and $\vartheta = 6$. Since $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X1}_6 = 24 \cdot \mathbf{1}_3$, then $\mu = 24$ and $2 = \omega = \frac{\mu\alpha}{d} = \frac{24 \cdot \frac{1}{3}}{4}$. Moreover, for the design $\mathbf{X}$ with $\mathbf{G}$, $Var(\widehat{\mathbf{1}_6'\mathbf{w}}) = 0{,}25\sigma^2$. It is easy to see that for covariance matrix of errors $\sigma^2\mathbf{G}$, the design $\mathbf{X} \in \mathbf{D}^s_{3\times 6}$ that satisfies Theorem 3 is optimal for estimation of the total weight.

EXAMPLE 3

Now, we consider $\mathbf{X} \in \mathbf{D}^c_{8\times 12}$. According to the Theorem 6 we construct the incidence matrix $\mathbf{N}_1$ of partially balanced incomplete block design with the parameters $v_1 = 4$, $b = 8$, $r_1 = 4$, $k_1 = 2$, $\lambda_{11} = 2$, $\lambda_{21} = 1$ (design R1, Clatworthy, 1973) and the incidence matrix $\mathbf{N}_2$ of partially balanced incomplete block design with the parameters $v_2 = 8$, $b = 8$, $r_2 = 6$, $k_2 = 6$, $\lambda_{12} = 6$, $\lambda_{22} = 4$ (design S19, Clatworthy, 1973), where

$$\mathbf{N}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \ \mathbf{N}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Thus the design matrix $\mathbf{X} \in \mathbf{D}^c_{8\times 12}$ in the form (6) for $s = 1$ is given as

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & -1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

Condition (i) and (ii) of Theorem 3 are fulfilled and $\mathbf{X}$ given above is optimal for the estimated total weight with $Var(\widehat{\mathbf{1}'_{12}\,\mathbf{w}}) = \frac{\sigma^2 g(1+7p)}{2}$.

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OPTYMALNE UKŁADY WAGOWE DLA ESTYMACJI CAŁKOWITEJ MASY OBIEKTÓW

W pracy przedstawiono teorię estymacji całkowitej masy obiektów zarówno w sprężynowym jak i w chemicznym układzie wagowym przy założeniu, że błędy pomiarów dokonywanych w tych układach są ujemnie skorelowane. Podano dolne ograniczenie wariancji estymatorów oraz warunki konieczne i dostateczne, przy spełnieniu których to dolne ograniczenie jest osiągnięte. Praca jest podsumowaniem i zebraniem wiadomości dotyczących tego zagadnienia poszerzonym o metody konstrukcji i przykłady macierzy odpowiednich układów eksperymentalnych.