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EVALUATION OF POWER OF SOME LINEARITY TESTS FOR ECONOMETRIC MODEL WITH TWO EXPLANATORY VARIABLES

1. FORMULATION OF THE PROBLEM

Consider an econometric model

(1) $$Y = \alpha_o + \alpha_1 X_1 + \alpha_2 X_2 + \varepsilon,$$

where the error term $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^T$ fulfils usually formulated assumptions and X_1, X_2 are nonstochastic real variables, i.e. $X_1, X_2 \in R^1$.

On the basis of the sample

(2) $$(x_{11}, x_{12}, y_1), (x_{21}, x_{22}, y_2), \ldots, (x_{n1}, x_{n2}, y_n),$$

composed of n independent observations, the hypothesis

(3) $$H_o: E(Y|X_1, X_2) = \alpha_o + \alpha_1 X_1 + \alpha_2 X_2$$

is to be verified. The use of three kinds of tests will be considered in this article: a run test, F test and Theil's test.

For a run test a test statistic is the number of runs in the sequence of residuals

(4) $$e_i = y_i - a_o - a_1 x_{i1} - a_2 x_{i2},$$

where a_o, a_1, a_2 are the OLS estimates of the parameters α_o, α_1, α_2 respectively.

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In the case of a model with one explanatory variable runs are calculated in the sequence $\{e_i\}$ whose order is determined by increasing order of the explained variable. Much more criteria of residuals' ordering may be given when a model with more explanatory variables is considered. In this article a model with two explanatory variables is analysed.

Let f be a certain function in two variables, and u - a permutation ordering numbers $f_i = f(x_{i1}, x_{i2})$ in such a way that

$$f_{u(1)} \leqslant f_{u(2)} \leqslant \dots \leqslant f_{u(n)}. \tag{5}$$

Let f be defined by one of the following patterns:

(a) $f_i = \alpha_o + \alpha_1 x_{i1} + \alpha_2 x_{i2} + \varepsilon_i = y_i$,

(b) $f_i = x_{i1}$,

(c) $f_i = x_{i2}$,

(d) $f_i = x_{i1} + x_{i2}$,

(e) $f_i = x_{i1}^2 + x_{i2}^2$.

The increasingly ordered sequence of numbers f_i determines the permutation u, which orders the residuals into the sequence $\{e_i\}$.

The explanatory variables are assumed to be standardized, what makes the criteria (a)-(e) independent of linear transformations of the variables [2].

For the test based on the number of runs a left-hand-side critical region is constructed (a small number of runs testifies against H_o), using the tables of critical values [1]. In the presented investigation different variants of run tests correspond with different criteria (a)-(e) of residuals' ordering.

F test does not depend on the residuals' order, so the mentioned criteria do not concern it [7].

To verify the linearity hypothesis for a model with two explanatory variables by means of Theil's test [3, 5] the differences

(6) $$R_{s(i)} = y_{s(i)} - a_1 x_{s(i),1} - a_2 x_{s(i),2}$$

should be found. Here s is a permutation that orders the values y_i in the following way

(7) $$y_{s(1)} \leqslant y_{s(3)} \leqslant y_{s(5)} \leqslant \dots \leqslant y_{s(4)} \leqslant y_{s(2)}.$$

The test statistic for Theil's test is given by the pattern [4, 6]

(8) $$T_n = \sum_{i=1}^{n} \sum_{j=i+1}^{n} \operatorname{sign}(R_{s(i)} - R_{s(j)}).$$

The critical values for the tests mentioned above, used in the investigation are presented in Tab. 1 below.

Table 1

Critical values of run, F and Theil's tests

Number of runs

n	α = 0.05				α = 0.10			
	k_1	$P\{S \leqslant k_1\}$	k_2	$P\{S \leqslant k_2\}$	k_1	$P\{S \leqslant k_1\}$	k_2	$P\{S \leqslant k_2\}$
10	2	0.0195	3	0.0898	3	0.0898	4	0.2530
20	6	0.0318	7	0.0553	7	0.0835	8	0.1796
40	14	0.0266	15	0.0541	16	0.0998	17	0.1684
60	23	0.0337	24	0.0587	25	0.0963	26	0.1488
100	41	0.0350	42	0.0537	43	0.0796	44	0.1138

F test

n	α = 0.05	α = 0.10
10	6.940	4.320
20	3.740	2.730
40	3.280	2.470
60	3.168	2.402
100	3.096	2.363

Table 1 (condt.)

Theil's test

n	α = 0.05				α = 0.10			
	k_1	$P(T_n \geq k_1)$	k_2	$P(T_n \geq k_2)$	k_1	$P(T_n \geq k_1)$	k_2	$P(T_n \geq k_2)$
10	19	0.0463	17	0.0642	15	0.0779	13	0.1082
20	50	0.0493	48	0.0563	40	0.0930	38	0.1043
40	144	0.0479	142	0.0503	110	0.0986	108	0.1028
60	268	0.0494	266	0.0507	202	0.0982	200	0.1004
100	552	0.0498	550	0.0505	430	0.0999	428	0.1010

It should be noticed now that run tests require the residuals to be independent. Because for the residuals (4) this is only approximately true, the utilization of the tables of the test statistic distribution is not fully justifiable. However, in the authors' opinion, the magnitude of the error made when choosing the appropriate quantile of the variable in question, is supposed not to affect significantly the result of the mentioncd tests' power comparison.

2. DESCRIPTION OF THE METHOD

To evaluate the power of the proposed tests a Monte-Carlo experiment was utilized. Its procedure was following.

For the fixed sample sizes n = 10, 20, 40, 60, 100 the sequence $\{x_{i1}^*\}$ (i = 1, 2, ..., n) is generated in two variants (w = 1, 2): from uniform distribution (w = 1) and from normal distribution (w = 2). After standardizing the sequences $\{x_{i1}^*\}$ the sequences $\{x_{i1}\}$ (also two variants) are obtained. In the investigation the later are concerned to be the realization of variable X_1.

The sequence $\{x_{i2}\}$ (i = 1, 2, ..., n) for both variants (w = 1, 2) is generated from normal distribution $N(\rho x_{i1}, 1)$, where ρ is the predetermined correlation coefficient between

$\{x_{i1}\}$ and $\{x_{i2}\}$. In the investigation two values of ρ were assumed $\rho = 0.5,\ 0.9$.

For each assumed sample sizes n, both variants of the sequence of pairs $\{(x_{i1}, x_{i2})\}$ and nonlinear function g, determining the alternative hypothesis, the sequence $\{y_i\}$ is generated, where

$$(9)\qquad y_i = g(x_{i1}, x_{i2}) + \xi_i, \quad i = 1, 2, \ldots, n,$$

where:

ξ_i - independent random variables normally distributed $N(0, \sigma_\xi)$.

(Note that σ_ξ reflects the dispersion of the empirical points around the surface described by function g).

In the investigation function g is defined as

$$(10)\qquad g(x_1, x_2) = c_0 + c_1x_1 + c_2x_2 + c_3x_1^2 + c_4x_2^2 + c_5x_1x_2.$$

Assuming $c_0 = 0$ (a shift along a vertical axis does not change the order and magnitude of residuals) and $c_5 = 0$ (the axis Oy constitutes then the axis of paraboloid (8) what significantly simplifies the experiment and does not influence remarkably the generality of conclusions), function g takes on the form

$$(11)\qquad g(x_1, x_2) = c_4(2v^2u_1x_1 - 2u_2x_2 + v^2x_1^2 + x_2^2),$$

where:

$v = \sqrt{c_3/c_4}$ and (u_1, u_2) are the coordinates (on the Ox_1 and Ox_2 axes) of the paraboloid's top. In the future we shall try alternative compositions of c's.

Thus generated triples

$$(12)\qquad (x_{11}, x_{12}, y_1), (x_{21}, x_{22}, y_2), \ldots, (x_{n1}, x_{n2}, y_n)$$

constitute now the sample, on the basis of which the model

$$(13)\qquad y_i = \alpha_0 + \alpha_1x_{i1} + \alpha_2x_{i2} + \varepsilon_i, \quad i = 1, 2, \ldots, n$$

is estimated. The estimates a_o, a_1, a_2 of the parameters α_o, α_1, α_2 obtained by the classical OLS allow to compute the residuals (4).

For each way of ordering (according to the criteria (a)-(e)) the number of runs in the sequence $\{e_i\}$ is computed. Comparing this number with a critical value taken from tables of number of runs' distribution (see Tab. 1) a result of the test is obtained: rejection or acceptance of H_o.

Repeating this process many times (in the described investigation 200 times) a relatively exact frequency of rejecting H_o (empirical power) for all variants of the run test may be calculated.

Notice, that in the presented experiment the sample consists of the points generated from the paraboloid, so, that variant of the test will be better which more frequently allows to reject the linearity hypothesis of the relation between Y and X_1, X_2.

The considerations were not restricted only to one specification of the alternative hypothesis, determined by the form of function g. Some of the parameters of that function were modified. Parameter v was considered in five variants v = = 0.0, 0.2, 0.5, 1.0, 3.0 and the position of the paraboloid's top - in seven variants $(u_1, u_2) = (0, 0)$, (1, 1), (1, 3), (1, 10), (3, 3), (3, 10), (10, 10). However, due to similarity of the results and to the lack of place only one of the mentioned variants is presented v = 1.0, $(u_1, u_2) = (0, 0)$.

The shape of the paraboloid, and also the dispersion of the empirical (generated) points around it vary according to the value of the coefficient

$$\psi^2 = \frac{\sigma_\xi^2}{S_\theta^2 + \sigma_\xi^2}, \tag{14}$$

where:

$$S_\Theta^2 = \frac{1}{n} \sum_{i=1}^{n} \theta_i^2,$$

$\theta_i = g(x_{i1}, x_{i2}) - \hat{g}(x_{i1}, x_{i2})$,

$\hat{g}$ is the OLS linear approximation of g on the set $\{(g(x_{i1}, x_{i2}), x_{i1}, x_{i2})\}$, $i = 1, 2, \ldots, n$.

The higher value ψ^2 takes on the more the paraboloid (11) is similar to a plane, and the more the empirical points are dispersed around it (for lower ψ^2 higher test powers are expected). In the experiment ψ^2 takes on the following values $\psi^2 = 0.01, 0.05, 0.10, 0.25, 0.50, 0.90$ (however more variants were investigated).

In this article only some results are presented, namely those concerning $n = 10, 20, 40, 60, 100$, $w = 1, 2$, $\varrho = 0.5, 0.9$, $(u_1, u_2) = (0,0)$, $v = 1.0$ and $\psi^2 = 0.01, 0.05, 0,10, 0.25, 0.50, 0.75, 0.90$. The results are presented in Tab. 2, 3, 4 below. They give the empirical power of the analysed tests.

Table 2

Empirical power of run, F and Theil's tests
$w = 1$, $\varrho = 0.5$, $v = 1.00$, $\alpha = 0.05$

n	ψ^2	Run test: ordering according to					F test	Theil's test
		y_i	x_{i1}	x_{i2}	$x_{i1}+x_{i2}$	$x_{i1}^2+x_{i2}^2$		
1	2	3	4	5	6	7	8	9
10	0.01	100.00	82.67	100.00	75.30	77.33	100.00	15.78
	0.05	97.62	85.05	98.48	80.93	79.60	100.00	20.17
	0.10	95.02	86.78	97.40	82.23	80.52	100,00	21.52
	0.25	91.55	90.25	95.02	86.57	82.75	80.50	16.17
	0.50	94.37	63.28	95.88	92.85	90.75	30.50	11.48
	0.75	96.75	96.97	96.75	95.45	94.72	13.50	6.02
	0.90	97.62	90.70	98.05	96.97	96.35	8.50	5.83
20	0.01	80.64	96.30	99.12	76.41	94.17	100.00	100.00
	0.05	84.86	87.85	92.61	80.29	82.18	100.00	100.00
	0.10	88.91	88.23	90.32	84.69	75.38	100.00	99.70
	0.25	93.14	88.81	92.43	90.49	78.68	100.00	86.31
	0.50	94.90	93.30	95.25	95.95	89.02	91.50	54.82
	0.75	97.01	95.24	96.51	96.30	93.41	38.00	27.21
	0.90	98.42	97.30	97.54	97.89	95.00	10.50	13.90

1	2	3	4	5	6	7	8	9
40	0.01	88.50	83.33	99.15	100.00	100.00	100.00	100.00
	0.05	80.41	64.22	85.09	99.15	100.00	100.00	100.00
	0.10	80.41	72.24	82.11	97.87	100.00	100.00	100.00
	0.25	82.96	81.67	81.68	76.57	98.59	100.00	89.27
	0.50	90.13	92.11	88.50	79.56	95.48	100.00	43.10
	0.75	92.61	95.87	93.19	92.33	94.98	82.50	19.27
	0.90	96.17	97.72	96.06	95.52	96.43	33.50	10.50
60	0.01	97.03	100.00	98.25	100.00	100.00	100.00	100.00
	0.05	95.98	100.00	97.21	100.00	100.00	100.00	100.00
	0.10	95.28	99.48	96.33	99.65	100.00	100.00	100.00
	0.25	93.36	92.96	92.49	94.93	99.50	100.00	100.00
	0.50	96.03	94.78	95.63	94.76	89.19	100.00	96.00
	0.75	97.28	96.67	97.78	96.36	91.62	98.50	71.52
	0.90	97.00	98.15	97.63	99.00	98.17	54.50	27.02
100	0.01	96.86	100.00	99.61	100.00	100.00	100.00	100.00
	0.05	97.24	100.00	97.55	100.00	100.00	100.00	100.00
	0.10	97.53	99.41	97.45	100.00	100.00	100.00	100.00
	0.25	96.92	95.92	96.47	99.61	100.00	100.00	100.00
	0.50	97.03	91.29	61.64	98.04	93.37	100.00	100.00
	0.75	97.81	97.21	97.82	97.64	92.08	100.00	74.34
	0.90	98.61	98.29	99.01	98.40	96.39	77.00	31.50

Table 3

Empirical power of run, F and Theil's tests
$w = 2,\ \rho = 0.5,\ v = 1.0,\ \alpha = 0.05$

n	ψ^2	Run test: ordering according to					F test	Theil's test
		y_i	x_{i1}	x_{i2}	$x_{i1}+x_{i2}$	$x_{i1}^2+x_{i2}^2$		
1	2	3	4	5	6	7	8	9
10	0.01	100.00	100.00	100.00	59.70	47.30	100.00	0.00
	0.05	100.00	97.83	100.00	69.88	60.62	100.00	0.00
	0.10	98.48	97.18	100.00	73.78	66.00	100.00	0.00
	0.25	96.53	96.97	99.35	80.93	75.55	84.50	0.00
	0.50	96.32	96.10	98.48	86.13	83.97	35.50	0.07
	0.75	96.75	97.62	99.78	93.07	90.10	10.50	0.13
	0.90	98.27	97.83	99.35	96.65	92.83	6.10	1.76

Table 3 (contd)

1	2	3	4	5	6	7	8	9
20	0.01	99.65	64.80	100.00	100.00	100.00	100.00	0.00
	0.05	96.30	68.49	100.00	99.12	100.00	100.00	0.00
	0.10	93.31	73.95	100.00	97.01	100.00	100.00	0.00
	0.25	92.69	84.24	98.59	89.99	91.72	100.00	0.00
	0.50	93.25	92.04	96.48	97.82	79.68	91.50	0.00
	0.75	96.01	95.54	97.36	93.54	92.41	43.50	1.20
	0.90	97.42	97.54	98.42	97.44	96.53	15.50	3.50
40	0.01	98.97	94.89	84.67	100.00	100.00	100.00	0.00
	0.05	85.57	83.81	75.30	100.00	100.00	100.00	0.00
	0.10	86.65	81.26	70.61	99.57	100.00	100.00	0.00
	0.25	88.50	73.09	76.57	93.19	99.48	100.00	0.50
	0.50	91.96	83.94	82.89	75.93	93.13	100.00	2.77
	0.75	93.46	94.74	93.81	89.70	92.26	85.50	5.50
	0.90	95.52	97.65	94.87	96.30	95.00	52.00	6.00
60	0.01	91.62	100.00	96.33	100.00	100.00	100.00	24.61
	0.05	92.84	99.48	94.59	100.00	100.00	100.00	24.02
	0.10	93.54	97.21	91.97	100.00	100.00	100.00	25.52
	0.25	96.51	92.66	92.66	96.33	99.35	100.00	23.00
	0.50	95.31	94.16	95.28	93.21	89.52	100.00	17.50
	0.75	96.60	97.48	97.90	95.46	87.02	94.50	11.02
	0.90	98.50	98.80	98.67	97.98	93.72	57.00	7.00
100	0.01	96.76	100.00	100.00	100.00	100.00	100.00	0.00
	0.05	97.55	100.00	99.80	100.00	100.00	100.00	0.00
	0.10	97.06	99.80	99.31	100.00	100.00	100.00	0.50
	0.25	97.55	98.33	98.33	99.80	100.00	100.00	2.00
	0.50	97.14	96.96	97.15	96.96	93.28	100.00	2.00
	0.75	97.62	98.01	97.82	97.94	90.78	100.00	1.50
	0.90	98.51	98.10	99.51	99.12	94.89	79.50	1.50

Table 4

Empirical power of run, F and Theil's tests
$w = 1$, $\rho = 0.9$, $v = 1.0$, $\alpha = 0.05$

n	ψ^2	Run test: ordering according to					F test	Theil's test
		y_i	x_{i1}	x_{i2}	$x_{i1}+x_{i2}$	$x_{i1}^2+x_{i2}^2$		
10	0.01	100.00	56.67	91.33	56.67	43.33	100.00	99.13
	0.05	100.00	56.67	85.92	56.67	43.90	100.00	86.80
	0.10	98.92	59.27	83.10	58.83	47.72	100.00	77.87
	0.25	94.58	70.52	81.15	67.28	59.92	82.50	65.74
	0.50	95.45	84.62	87.43	84.62	78.28	29.50	40.63
	0.75	97.40	94.58	93.72	.94.37	90.00	12.50	18.54
	0.90	98.70	96.75	95.08	96.53	93.28	8.00	8.52
20	0.01	100.00	100.00	94.54	100.00	100.00	100.00	100.00
	0.05	97.71	100.00	88.03	100.00	99.35	100.00	100.00
	0.10	98.59	98.24	86.45	98.70	95.14	100.00	100.00
	0.25	95.63	89.67	87.15	90.67	77.79	100.00	99.70
	0.50	95.45	88.67	92.43	89.82	79.57	92.50	89.01
	0.75	97.18	93.57	94.54	93.52	90.61	42.00	49.42
	0.90	97.12	97.09	97.56	96.51	95.00	13.50	21.31
40	0.01	100.00	100.00	100.00	100.00	100.00	100.00	100.00
	0.05	97.02	100.00	100.00	100.00	100.00	100.00	100.00
	0.10	96.59	99.57	99.57	100.00	100.00	100.00	100.00
	0.25	93.19	90.63	87.22	97.44	99.56	100.00	100.00
	0.50	93.19	82.17	82.54	84.09	95.61	100.00	99.27
	0.75	95.31	90.91	92.54	87.48	94.11	77.00	82.77
	0.90	97.30	95.02	95.59	95.94	97.13	35.50	34.77
60	0.01	97.36	100.00	100.00	100.00	100.00	100.00	100.00
	0.05	94.17	100.00	100.00	100.00	100.00	100.00	100.00
	0.10	92.85	100.00	100.00	100.00	100.00	100.00	100.00
	0.25	95.75	99.65	99.48	99.83	100.00	100.00	100.00
	0.50	96.58	91.91	94.24	94.41	93.32	100.00	100.00
	0.75	96.58	95.25	96.71	94.28	89.19	97.00	93.50
	0.90	97.33	97.30	97.00	97.28	94.72	51.50	54.50

3. CONCLUSIONS

On the basis of the described experiment some general conclusions concerning the powers of the run, F and Theil's tests may be drawn.

1. The power of all three considered tests increases along with sample size n.

2. As it was expected, the power of the run test is, in general, lower than the power of F test, although for great ψ^2 this relation becomes opposite (for $\psi^2 > 0.5$). However, it should be noticed that not for all specifications of the alternative hypothesis there exists a possibility to apply F test. In the presented research the polynomial function of the degree two was assumed in the alternative hypothesis, what made it possible to verify significarce of parameters at squares and at product of the variables X_1 and X_2 with F test. For more complicated alternative hypotheses the F test application may be inconvenient or even impossible. Theil's test is more sensitive to changes in ψ^2 than F test but not enough to be applicable in practical researches. The comparison of run test power with that of Theil's and F tests with respect to changes in ψ^2 is hindered due to difficulties in distinguishing the effects of changes in the "shape" of the paraboloid from those in the magnitude of the empirical points' dispersion around it.

3. The distribution of variable X_1 does not significantly influence the powers of run and F tests, but strongly affects that of Theil's test. For variant $w = 2$ (normal distribution of variable X_1) Theil's test is very weak, while for variant $w = 1$ (uniform distribution of variable X_1) - relatively strong. In the authors' opinion this may result from better "mixture" of positive and negative residuals in the case of variant $w = 1$ (the problem of constructing an appropriate criterion of residuals' ordering will be a subject of further investigations).

4. The increase in correlation coefficient ρ between variables X_1 and X_2 results, in general, in the increase of run

test power (although the influence is different for the particular criteria) and in the increase in that of Theil's test. On the other hand, test F responds to these changes inversely. One can notice that the increase in ρ causes the greatest reactions in Theil's test, and the smallest - in F test.

5. The obtained in most cases high powers of the considered run tests (e.g. for n = 40 the lowest power exceeds 80%) in comparison with F and Theil's tests confirm practical usefulness of the run tests.

In the light of the obtained results the possibility of generalization of the discussed methods to the case of a model with more than two explanatory variables seems to be quite realistic. In each of the proposed methods the most important is the problem of choosing the appropriate criterion of the observations' ordering. This problem in the case of a model with more explanatory variables can be solved, in the authors' opinion, by natural generalization of criteria (a)-(e). However, the answer to the question concerning powers of such tests seems to be quite difficult at the present stage of the research.

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OCENA MOCY NIEKTÓRYCH TESTÓW LINIOWOŚCI
MODELU EKONOMETRYCZNEGO Z DWIEMA ZMIENNYMI OBJAŚNIAJĄCYMI

W artykule przedstawiona została propozycja metody testowania liniowości modelu ekonometrycznego z dwiema zmiennymi objaśniającymi $Y = \alpha_0 + \alpha_1 X_1 + \alpha_2 X_2 + \varepsilon$.

Rozważa się pięć wariantów testów serii odpowiadających pięciu różnym kryteriom porządkowania reszt empirycznych, modyfikację testu Theila oraz test F. Wyniki przeprowadzonego przez autorów eksperymentu Monte-Carlo pozwalają ocenić i porównać moc badanych testów.