

*Wiesław Pasewicz**, *Wiesław Wagner***

PROBABILITY MODEL OF TIES IN MULTISTAGE DECISION PROCESS

Abstract. Probability model on multistage decision process is discussed with particular emphasis on special case using the rule $R(4, 2)$. An idea of importance graph ties is presented. Possibility recording probability of success in multistage decision process as linear combination others probabilities of the decision process is presented as well.

Key words: Multistage decision process, importance of ties, rule $R(k, \lambda)$.

1. INTRODUCTION

Theory of games developed essentially in two directions, containing so called extended games and matrix games. The first case consists of graphs with ties and edges. Walking from one tie of the graph to the next one is a decision process because the graph has a hierarchical structure of multistage game. It is important to know probability of ending the decision process for each tie of the graph. Such permanent monitoring permits us to undertake rational decisions.

In the paper we present a probability model for a particular scheme of multistage decision process. Besides, we give an a measure of importance measure for ties of the graph.

2. A RULE OF REACHING A SUCCESS ON THE GRAPH

Let us assume that the decision process is leaded on graph $G(T, E)$. The graph consists of ties T and edges E . From each of the tie there are going out two edges named strategies L (left) and R (right). Each of these

* PhD, Agricultural University, Szczecin.

** Prof., University School of Physical Education, Poznań.

two edges is going to the one of two different ties with probabilities p and $q = 1 - p$, respectively.

The player taking decision (strategies L or R) is random walking on the graph. The game is finished when the player reached a success according to the following rule $R(k, \lambda)$:

The player ought to obtain at least k strategies L . The number of these strategies has to exceed the number m strategies R with advantage at least λ , that is $k - m \geq \lambda$, $\lambda = 1, 2, 3, \dots$

We note that the graph $G(T, E)$ depends of parameters k and λ . Taking $k = 4$ and $\lambda = 2$ we have the rule $R(4, 2)$, which is used in tennis. The rule $R(4, 2)$ will be considered in the next point of the paper.

3. PROBABILITY OF SUCCESS IN MULTISTAGE DECISION PROCESS USING THE RULE $R(4, 2)$

Let us consider the rule $R(k, \lambda)$, where $k = 4$ and $\lambda = 2$. Figure 1 shows graph $G(T, E)$ in the case.

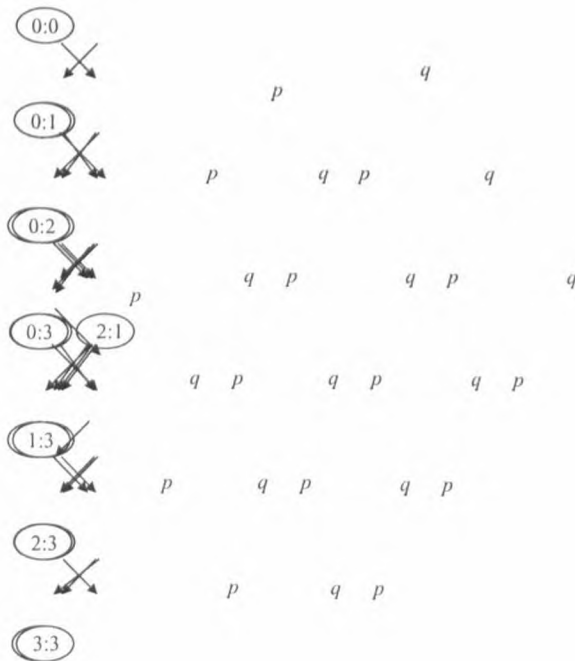


Fig. 1. Graph $G(T, E)$ of multistage decision process using in the rule $R(4, 2)$

These ties denote states $a:b$, where $a(b)$ is the number of strategies $L(R)$ reached by the player on the graph. There are three states finishing the game with victory the player, that is 4:0, 4:1 and 4:2. After state 3:3 the game is continued until $(a+2):a$, for $a = 3, 4, \dots$ with the player as a winner.

The main idea considered problem is calculate probabilities $P(a:b)$ that the player reach a success in decision process depending on states, illustrated in Fig. 1. In recent papers e.g. B. P. Hsi and D. M. Burych (1971), L. H. Riddle (1988), W. Pasewicz and W. Wagner (2000) we can see the following formulas connected with tennis:

$P(0:0) = p^4(1 + 4q + 10q^2) + \frac{20p^5q^3}{1 - 2pq}$ - probability of success the player in state 0:0,

$20p^2q^3$ - probability that the process starts in state 0:0 and ends in state 3:3,

$v = \frac{p^2}{1 - 2pq}$ - probability of success the player after state 3:3.

Analysis of states for every stage decision process from graph $G(T, E)$ gives the general rule for the probability success ending the game of the form

$$P(a:b) = p^{4-a} \sum_{i=b}^2 \binom{3-a-b+i}{i-b} q^{i-b} + \binom{b-a-b}{3-b} p^{3-a} q^{3-b} v \quad (1)$$

For instance if $a = 0$ and $b = 2$, we have $P(0:2) = p^4 + 4p^3qv$, because there is only one path 0:2|1:2|2:2|3:2|4:2 from state to 0:2 state 4:2 and four paths: 0:2|0:3|1:3|2:3|3:3; 0:2|1:2|1:3|2:3|3:3; 0:2|1:2|2:2|2:3|3:3 and 0:2|1:2|2:2|3:2|3:3 from state 0:2 to state 3:3.

4. DEFINITION OF THE IMPORTANCE TIES IN MULTISTAGE DECISION PROCESS

The player beings in one of ties can choose the following ties choosing strategies L or R . He can be interested in so called importance of the ties, as well. C. Morris (1977) has introduced the following definition of importance tie I as a difference between two conditional probabilities that is (see Croucher 1998):

$I = P(S(GD) | \text{the player chooses strategy } L) - P(S(GD) | \text{the player chooses strategy } R)$, where $S(GD)$ denotes "the player reaches a success on the graph decision". Importance measure of tie $a:b$ we calculate using the following formula

$$I_{ab} = P(a+1:b) - P(a:b+1) \quad (2)$$

where $a, b = 0, 1, 2, 3$ or $a = 4, 5, \dots$, and $b = a-1, a, a+1$ and $P(a:b)$ is given by (1).

At present, we will show three examples application of formula (2). Let $I(a:b) I(c:d)$ denotes that tie $a:b$ is "more important" than tie $c:d$.

The following inequalities: (a) $I(2:3) > I(1:2)$, (b) $I(3:2) > I(2:1)$ and (c') $I(2:3) \geq I(2:2)$, if $p \geq 1/2$, (c'') $I(2:3) < I(2:2)$, if $p < 1/2$ are true, because of:

$$I(2:3) = P(3:3) - P(2:4) = v - 0 = v,$$

$$I(1:2) = P(2:2) - P(1:3) = p^2 + 2pqv - p^2v = (1 - 2pq)v = (1 - p^2)v,$$

$$I(3:2) = P(4:2) = P(4:2) - P(3:3) = 1 - v = \frac{q^2}{1 - 2pq},$$

$$I(2:1) = P(3:1) - P(2:2) = p + pq + q^2v - p^2 - 2pq = \frac{pq^2(1+q)}{1-2pq},$$

$$I(2:2) = P(3:2) - P(2:3) = p + qv - pv,$$

and

$$(a) \quad I(2:3) - I(1:2) = p^2v > 0,$$

$$(b) \quad I(3:2) - I(2:1) = \frac{q^4}{1+2pq} > 0,$$

$$(c') \quad I(2:3) - I(2:2) = \frac{p(2p-1)}{1-2pq} \geq 0, \text{ if } p \geq \frac{1}{2},$$

$$(c'') \quad I(2:3) - I(2:2) = \frac{p(2p-1)}{1-2pq} < 0, \text{ if } p < \frac{1}{2}.$$

5. PROBABILITY OF SUCCESS IN DECISION PROCESS USING THE RULE $R(K, 2)$

Natural generalization of the rule $R(4, 2)$ is $R(k, 2)$, where $k = 2, 3, 4, \dots$. Then the equality (1) will be form

$$P_k(a:b) = p^{k-a} \sum_{i=b}^{k-2} \binom{k-a-b-1+i}{i+b} q^{i-b} + \binom{2K-a-b-2}{k-b-1} p^{k-a-1} q^{k-b-1} v \quad (3)$$

Let us write down a few particular cases of equality (3) for $a, b = 0, 1$ and $k = 2, 3, 4, 5$. We have

$$P_k(0:0) = p^k \sum_{i=0}^{k-2} \binom{k-1+i}{i} q^i + \binom{2k-2}{k-1} p^{k-1} q^{k-1} v,$$

$$P_2(0:0) = p^2 + 2pqv,$$

$$P_3(0:0) = p^3(1 + 3q) + 6p^2q^2v,$$

$$P_4(0:0) = p^4(1 + 4q + 10q^2) + 20p^3q^3v,$$

$$P_5(0:0) = p^5(1 + 5q + 15q^2 + 35q^3) + 70p^4q^4v.$$

Similarly,

$$P_k(1:0) = p^{k-1} \sum_{i=0}^{k-2} \binom{k-2+i}{i} q^i + \binom{2k-3}{k-1} p^{k-2} q^{k-1} v,$$

$$P_2(1:0) = p + qv,$$

$$P_3(1:0) = p2(1 + 2q) + 3pq^2v,$$

$$P_4(1:0) = p^2(1 + 3q + 6q^2) + 10p^2q^3v,$$

$$P_5(1:0) = p^4(1 + 4q + 10q^2 + 20q^3) + 35p^3q^4v;$$

and

$$P(0:1) = p^k \sum_{i=1}^{k-2} \binom{k-2+i}{i-1} q^{i-1} + \binom{2k-3}{k-2} p^{k-1} q^{k-2} v,$$

$$P_2(0:1) = pv,$$

$$P_3(0:1) = p^3 + 3p^2qv,$$

$$P_4(0:1) = p^4(1 + 4q) + 10p^3q^2v,$$

$$P_5(0:1) = p^5(1 + 5q + 15q^2) + 35p^4q^3v.$$

Of course the probabilities $P_4(0:0)$, $P_4(1:0)$ and $P_4(0:1)$ we could obtain using to equality (1).

Now, we prove the following relation:

$$P_k(0:0) = pP_k(1:0) + qP(0:1) \quad \text{for } k = 2, 3, 4, \dots \quad (4)$$

Indeed. Using the identity $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$ we obtain

$$\begin{aligned} P_k = (0:0) &= p^k + \sum_{i=1}^{k-2} \binom{k-1+i}{i} p^k q^i + \binom{2k-2}{k-1} p^{k-1} q^{k-1} v = p^k + \sum_{i=1}^{k-2} \left[\binom{k-2+i}{i} + \right. \\ &\quad \left. + \binom{k-2+i}{i-1} \right] p^k q^i + \left[\binom{2k-3}{k-1} + \binom{2k-3}{k-2} \right] p^{k-1} q^{k-1} v = \\ &= p \left[p^{k-1} + \sum_{i=1}^{k-2} \binom{k-2+i}{i} p^{k-1} q^i \right] + p \binom{2k-3}{k-1} p^{k-2} q^{k-1} v + \\ &\quad + q \sum_{i=1}^{k-2} \binom{k-2+i}{i-1} p^k q^{i-1} + q \binom{2k-3}{k-2} p^{k-1} q^{k-2} v = p P_k(1:0) + q P_k(0:1). \end{aligned}$$

In the same manner we can show that:

$$P_k(0:0) = p^2 P_k(2:0) + 2pq P_k(1:1) + q^2 P_k(0:2),$$

$$P_k(0:0) = p^3 P_k(3:0) + 3p^2 q P_k(2:1) + 3pq^2 P_k(1:2) + q^3 P_k(0:3),$$

and generally

$$P_k(0:0) = \sum_{m=0}^n \binom{n}{m} p^{n-m} q^m P_k((n-m):m) \quad (5)$$

for $n = 1, 2, \dots$ and $n \geq m$.

Thus $P_k(0:0)$ is a linear combination of probabilities reaching a success in the decision process when the player is in the tie with state $(n-m):m$ and $P_k((n-m):m)$ we calculate according to formula (3).

6. CONCLUSION

In this paper was considered probability model of multistage decision process using the rule $R(k, \lambda)$ with special case for $k = 4$ and $\lambda = 2$. The player walking on graph $G(T, E)$ (Fig. 1) is taking decisions (strategies L or R) which are dependent on λ, k and ties of the graph. H reach a success (for $\lambda = 2$) if number k of strategies L will be at least by two more than number m of strategies R .

Case for $\lambda = 1$ is a simple case and formula (3) reduces to the form

$$P_k(a:b) = p^{k-a} \sum_{i=b}^{k-1} \binom{k-a-b-1+i}{i-b} q^{i-b}, \quad \text{for } k = 1, 2, 3, \dots$$

Especially interesting case is for $\lambda = 3$. Then the probability model of multistage decision process is more complicated. Problem $R(k, 3)$ authors will be present in the next paper.

REFERENCES

- Croucher J. S. (1998), *Developing Strategies in Tennis*, [in:] J. Bennett (ed.), *Statistics in Sport*, Arnold, New York, 157–171.
- Hsi B. P., Burych D. M. (1971), *Games of Two Players*, J. R. Statist. Soc., Series C, **20**, 86–92.
- Morris C. (1977), *The Most Important Point in Tennis*, [in:] S. P. Ladany, R. E. Machol (eds.), *Optimal Strategies in Sports*, North-Holland, New York, 131–140.
- Pasewicz W., Wagner W. (2000), *Charakterystyka modeli probabilistycznych w opisie gry i seta w tenisie ziemnym* (english: *Description of the Probability Models of Game and Set in Tennis*), „Wyzwania i Dylematy Statystyki XXI wieku”, Akademia Ekonomiczna, Wrocław, 140–147.
- Riddle L. H. (1988), *Probability models for tennis scoring systems*, Appl. Statist., **37**, 1, 63–75.

Wiesław Pasewicz, Wiesław Wagner

PROBABILISTYCZNY MODEL WIĄZAŃ W WIELOETAPOWYM PROCESIE DECYZYJNYM

W artykule rozważany jest model probabilistyczny wielostopniowego procesu decyzyjnego ze specjalnym uwzględnieniem przypadku użycia reguły $R(4, 2)$. Zaprezentowano ideę wiązań w grafach oraz możliwość przedstawienia prawdopodobieństwa sukcesu w wielostopniowym procesie decyzyjnym jako liniową kombinację innych prawdopodobieństw w procesie decyzyjnym.