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POWER OF TESTS FOR MULTIVARIATE NORMALITY BASED ON SKEWNESS AND FLATNESS COEFFICIENTS

ABSTRACT. There are many methods of construction of multivariate normality tests. The current review of the literature proves that there are at least 60 procedures of verification of the hypothesis about multivariate normality of variable and random distributions. We can indicate a few factors which prove an analysis of this class's tests based on skewness and kurtosis measures. It is easy to notice that these tests application contributes also a better multivariate analysis of the considered variable.

The paper presents results of power tests based on analytic deliberations and Monte Carlo methods.

Key words: tests for multivariate normality, power of tests, quantiles of distributions of tests statistics.

I. INTRODUCTORY REMARKS

Tests which make use of multivariate measures of skewness and flatness constitute an important category of tests for multivariate normality. This approach complements earlier studies based on skewness and flatness coefficients and assessing normality of uniform distributions. A characteristics of these tests together with tables of quantiles of test functions can be found in the study of Snedecor and Cochran (1989).

The fact that other distributions may have the same value is the main disadvantage of testing univariate normality, in order to become convinced whether skewness and flatness are equal to values taken by these parameters for normal distribution.

For instance, every symmetric distribution not only for variable of normal distribution, will have skewness coefficient equal to zero. Therefore, testing of univariate normality is dominated by such tests as Kolmogorov-Smirnov test or, particularly, by Shapiro-Wilk test. This fact has not hampered, however, the development

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of multivariate skewness and flatness measures and their application for constructing tests for multivariate normality.

Mardia (1970) worked out a generalisation of skewness and flatness measures for multivariate distributions. The measures introduced by Mardia are invariant.

A statistic from a sample for multivariate skewness is defined by the following formula:

$$b_{1,p} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\left(y_i - \bar{y}\right)' s^{-1}\left(y_j - \bar{y}\right)\right]^3. \tag{1}$$

An analogous statistic from a sample for flatness takes the form:

$$b_{2,p} = \frac{1}{n}\sum_{j=1}^{n}\left[\left(y_j - \bar{y}\right)' s^{-1}\left(y_j - \bar{y}\right)\right]^2. \tag{2}$$

Let us notice that both Mardia's skewness and flatness coefficients are functions of Mahalanobis squared distances. This fact causes that these measures, and particularly multivariate flatness measure, are useful for detecting "sticking out" values. When compared to expected value of normal distribution, values of this coefficient indicate that one or more observations are characterised by a long Mahalanobis distance, and for that reason they are placed far from the intersection point of geometric solid set of observations.

II. HYPOTHESES ON MULTIVARIATE NORMALITY

Let $\underline{X}_1,...,\underline{X}_n$ be a set of n observable p-variate random vectors being independent realisations of random vectors $\underline{X}$. Distribution of $\mathcal{P}_p$ of random vectors $\underline{X}$ is defined by a distribution function $G_p(\underline{x};\underline{\vartheta}) \equiv G_p(\underline{x}), \underline{x} \in R^p$ where $\underline{\vartheta}$ is a vector of parameters which belong to a given space. The distribution function $G_p(\underline{x})$ may be unknown both as its form and its parameters are concerned. We assume , however, that for every $\underline{x} \in R^p$ it is a continuous function.

Let us denote by $\mathcal{N}_p = \{N_p(\underline{\mu},\underline{\Sigma}) : \underline{\mu} \in R^p, \underline{\Sigma} \in \varphi_p^{>}, \beta_{1p} = 0, \beta_{2p} = p(p+2)\}$ a family of p-variate non-singular normal distributions (normal statistical space), where β_{1p} and β_{2p} are multivariate measures of shape of random vectors $\underline{X}$ of the following form, respectively :

$$\beta_{1p} = E\left\{ [(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}_{*}-\underline{\mu})]^{3}\right\}, \tag{3}$$

$$\beta_{2p} = E\left\{ [(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}_{*}-\underline{\mu})]^{2}\right\} \tag{4}$$

and they express multivariate asymmetrical kurtosis, where $\underline{x}$ and $\underline{x}_{*}$ are independent and have identical distribution . The distrbution function and density of distributions from $\mathcal{N}_p$ are denoted by $F_p(\underline{x};\underline{\mu}.\underline{\Sigma})$ and $f_p(\underline{x};\underline{\mu},\underline{\Sigma}), \underline{x}\in R^p$. If $\mathcal{N}_p$ contains the known $\underline{\mu}$ or $\underline{\Sigma}$ then symbols $\mathcal{N}_p$ $(\underline{\mu}_0)$ or $\mathcal{N}_p$ $(\underline{\Sigma}_0)$ denote that in $\mathcal{N}_p$ respective $\underline{\mu}=\underline{\mu}_0$ or $\underline{\Sigma}=\underline{\Sigma}_0$ are known. We introduce some more denotations. The family of distributions from $\beta_{1p}\neq 0$ and $\beta_{2p}=p(p+2)$ is denoted by A_1, and similarly, the family of distributions from $\beta_{1p}=0$ and $\beta_{2p}\neq p(p+2)$ by A_2 , and finally, from $\beta_{1p}\neq 0$ and $\beta_{2p}\neq p(p+2)$ by A_3. The family of distributions which does not contain normal distributions is denoted by A, i.e. $A=A_1\cup A_2\cup A_3$, in the sense of the criteria studied in the article. Exactly A is the set alternative distributions differed from normal distribution asymetry or flatness.

As we have at our disposal the observed set of vectors $\underline{X}_1,...,\underline{X}_n$ we intend to investigate the consistency of distribution functions $G_p(\underline{x})$ and $F_p(\underline{x};\underline{\mu},\underline{\Sigma})$ i.e. we ask whether distribution function $G_p(\underline{x})$ can be assumed as identical with $F_p(\underline{x};\underline{\mu},\underline{\Sigma})$, or whether it belongs to the $\mathcal{N}_p$ family , which is denoted as $\mathcal{P}_p\in\mathcal{N}_p$ .The assumption whose validity we want to prove on the basis of multivariate sample $\underline{X}_1,...,\underline{X}_n$, is expressed by a complex non-parametric zero hypothesis:

$$H_0:\mathcal{P}_p\in\mathcal{N}_p,$$

against a complex alternative hypothesis:

$$H_1:\mathcal{P}_p\notin\mathcal{N}_p, \text{ or } H_1:\mathcal{P}_p\in A.$$

Hypothesis H_1 can be denoted in the form of the sum of $H_1 = H_{11} \cup H_{12} \cup H_{13}$, and the problem posed above can be formulated in one of the situations listed below:

a) $H_0 : \mathcal{P}_p \in \mathcal{N}_p$, $H_{11} : \mathcal{P}_p \in A_1$ (the family of asymmetric distributions with kurtosis equal to multivariate normal kurtosis);

b) $H_0 : \mathcal{P}_p \in \mathcal{N}_p$, $H_{12} : \mathcal{P}_p \in A_2$ (the family of symmetric distributions with kurtosis different from normal);

c) $H_0 : \mathcal{P}_p \in \mathcal{N}_p$, $H_{13} : \mathcal{P}_p \in A_3$.

III. TESTS OF MULTIVARIATE NORMALITY BASED ON SKEWNESS (b_{1p}) AND FLATNESS (b_{2p}) STATISTICS

Normal p-variate distribution has parameters of distribution shape i.e. skewness $\beta_{1p} = 0$ and flatness $\beta_{2p} = p(p+2)$.

While investigating p-variate empirical distribution with the use of independent observable random vectors $\underline{X}_1, ..., \underline{X}_n$ we ask whether they come from a multivariate population $\beta_{1p} = 0$ or $\beta_{2p} = p(p+2)$,or simultaneously $\beta_{1p} = 0$ and $\beta_{2p} = p(p+2)$.

This leads us to define zero hypotheses which were already given in Section 2. Families of distributions A_1, A_2, A_3, for defining respective alternative distributions , were also given in that section. Numerous statistical tests for verifying hypothesis $H_0 : \mathcal{P}_p \in \mathcal{N}_p$, against alternative hypotheses defined by a class of distributions A_1, A_2 or A_3 are based on statistics b_{1p} and b_{2p}.

Let us distinguish here two types of tests i.e. direction tests and omnibus tests.

DEFINITION 1. Statistical tests for a given class of alternative distributions are called direction tests

DEFINITION 2. Statistical tests which are most powerful in a class of possible alternative distributions are called omnibus tests.

For verifying hypothesis $H_0 : \mathcal{P}_p \in \mathcal{N}_p$ against $H_1 : P_p \in A_1$ or $H_2 : P_p \in A_2$ we apply tests of multivariate normality based on test statistics (test checks) being respective functions b_{1p} lub b_{2p}. The applied direction tests will be most powerful for the class of distributions A_1 lub A_2. However, when we analyse the alternative defined by a family of A_3 distributions we make use of test statistics which are functions of b_{1p} and b_{2p} for omnibus tests. Omnibus tests have such a property that they show simultaneously a departure of p-variate empirical distribution from p-variate distribution $\beta_{1p} = 0$ and $\beta_{2p} = p(p+2)$. Omnibus tests are recommended whenever we do not have any a priori information on distribution specified with the use of alternative hypothesis.

Test statistics based on b_{1p} and b_{2p} have distributions known for large n and based on limit theorems . Detailed distributions for small n are not known.

LEMMA 1.

$nb_{1p}/6 \sim \chi_f^2$ and $n \to \infty$, $f = p(p+1)(p+2)/6$, when $\underline{U} \sim MN$

Proof. Mardia (1970). See also Domanski and Wagner (1984).

LEMMA 2.

$(b_{2p} - E(b_{2p}))/D(b_{2p)} \sim N(0,1)$ and $n \to \infty$, when $\underline{U} \sim MN$

Proof. Mardia (1970, 1974).

The above lemmas are used for constructing tests of multivarate normality. As far as their application is concerned, tests of multivariate normality based on measures b_{1p} and b_{2p} with respect to the above defined classes of alternative distributions, can be divided as follows:

A_1 – tests M_1, C_1, $L(b_{1p})$, $U(b_{1p})$, $W(b_{1p})$, Q_1,

A_2 – tests M_2, C_2, $U(b_{2p})$, $W(b_{2p})$, Q_2,

A_3 – tests M_3, C_3, C_4, S_L^2, S_N^2, S_W^2, C_N^2, C_W^2, C_R^2, Q.

We present tests of multivariate normality and limit ourselves to providing forms of respective test statistics and their distributions. We make an assumption that $\underline{U} \sim MN_p$. First we list:

(1) the author or authors, then, (2) test statistics, and finally (3) distribution of test statistics for $n \to \infty$.

Tests for hypothesis $H_0 : \mathcal{P}_p \in \mathcal{N}_p$ against $H_1 : P \in A_1$

(a) (1) Mardia (1970),

(2) $M_1 = nb_{1p}/6$,

(3) χ_f^2 (lemma 1);

(b) (1) Bera and John (1983)

(2) $C_1 = n\sum_{i=1}^{p} T_i^2/6$,

where

$$T_i = \sum_{j=1}^{n} Y_{ij}^3 / n, \; i = 1, ..., p$$

$$Y_j = (Y_{1j}, ..., Y_{pj})' = \underline{S}^{-1/2}(\underline{x}_j - \bar{\underline{x}}),$$

(3) χ_p^2; if $n \to \infty$

(c)(1) Mardia and Foster (1983),

(2) $L(b_{1p}) = \gamma + \delta \ln(b_{1p} - \xi)$, when $n \to \infty$

where γ, δ, ξ are parameters in the family of log-normal S_L Johnson distributians.

Following Kendall and Stuart (1963) we show the way of determing γ, δ, ξ. We make substitutions: $t = (w^2 - 1)^{1/2}$, $w = \exp(1/\delta^2)$ and $\rho = \exp(-\gamma/\delta)$. Then we determine t from cubic equation using Cordano equations

$$t^3 = 3t - 2f\sqrt{2f} = 0,$$

and finally, we calculate w. From the above given formulas for parameters of b_{1p} distribution we get:

$$E(b_{1p}) = w \; \rho \; = 6f/n \text{ i } D^2(b_{1p}) = \rho^2 w^2 (w^2 - 1) = 72f/n,$$

what allows us, having the known w, to calculate ρ.

Parameter ξ is determined from the formula:

$$\xi = E(b_{1p}) - D(b_{1p})/t = 6[f - 2f/t]/n.$$

With the known w and ρ we calculate δ and γ;

(3) $N(0,1)$;

(d) (1) Mardia and Foster (1983);

(2) $U(b_{1p}) = (b_{1p} - E(b_{1p}))/D(b_{1p}) = [b_{1p} - 6f/n]/[6(2f/n^2)^{1/2}]$,

and $n \to \infty$

(3) $N(0,1)$;

(e)(1) Mardia and Foster (1983),

(2) $W(b_{1p})=[6(4nf^2 b_{1p}/3)^{1/3}-18f+4]/(2f)^{1/2}$ (Wilson–Hilferty approximation of b_{1p} distribution).

(3) $N(0,1)$:

Tests for hypothesis $H_0: \mathcal{P}_p \in \mathcal{N}_p$ against $H_1: \mathcal{P}_p \in A_2$:

(f) (1) Mardia (1970),

(2) $M_2=(b_{2p}-g)^2/(8g/n),\ g=p(p+2)$,

(3) χ_1^2;

(g) (1) Bera and John (1983),

(2) $C_2=n[\sum_{i=1}^{p}(T_{ii}-3)^2/24+\sum_{1\le i<i'\le p}(T_{ii'}-1)^2/4]$,

where

$$T_{ii}=\sum_{j=1}^{n} Y_{ij}^4/n,\ i=1,\dots,p,$$

$$T_{ii'}=\sum_{j=1}^{n}(Y_{ij}Y_{i'j})^2/n,\quad i,i'=1,\dots,p; i\neq i'$$

and Y_{ij} defined in (*b*),

(3) $\chi^2 p(p+1)/2$;

(h) (1) Mardia and Foster (1983),

(2) $U(b_{2p})=[b_{2p}-g(n-1)/(n+1)]/(8g/n)^{1/2}$,

(3) $N(0,1)$;

(i)(1) Mardia and Foster (1983),

(2) $W(b_{2p})=3(f_1/2)^{1/2}\{1-2/gf_1-(1-2/gf_1)/[1+a(2/(f_1-4))^{1/2}]\}^{1/3}$

where

$$f_1=6+4[d+\sqrt{d+d^2}] \text{ and } d=np(p+2)/2(p+8)^2$$
$$\text{and } a=(b_{2p}-E(b_{2p}))/D(b_{2p})),$$

(3) $N(0,1)$;

Tests for hypothesis $H_0: \mathcal{P}_p \in \mathcal{N}_p$, against $H_1: \mathcal{P}_p \in A_3$:

(j) (1) Jarque and McKenzie (1982),

(2) $M_3=M_1+M_2$,

(3) $\chi^2_{[(p/6)(p+1)(p+2)+1]}$

(k)(1) Bera and John (1983),

(2) $C_3 = n\{\sum_{i=1}^{p} T_i^2 / 6 + \sum_{i=1}^{p} (T_{ii} - 3)^2 / 24\}$,

(3) χ^2_{2p};

(l)(1) Bera and John (1983),

(2) $C_4 = C_1 + C_2$,

(3) $\chi^2_{p(p+3)/2}$;

(m) (1) Mardia and Foster (1983)

(2) $S_L^2 = L^2(b_{1p}) = U^2(b_{2p})$,

$S_N^2 = U^2(b_{1p}) + U^2(b_{2p})$,

$S_W^2 = W^2(b_{1p}) + W^2(b_{2p})$,

$C_N^2 = \underline{b}'\underline{v}^{-1}\underline{b}$,

$\underline{b} = (b_{1p} - 6f/n,\ b_{2p} - g(n-1)/(n+1))'$

$$\underline{v} = \begin{bmatrix} 72f/n^2 & 12\text{ph/n}^2 \\ 12ph/n^2 & 8g/n \end{bmatrix}, \qquad h = 8p^2 - 13p + 23,$$

$C_W^2 = \underline{c}'\,\underline{W}^{-1}\underline{c}$,

$\underline{c} = (W(b_{1p}),\ W(b_{2p}))'$,

$$\underline{W} = \begin{bmatrix} 1 & \gamma \\ \gamma & 1 \end{bmatrix},$$

$\gamma = Cov(W(b_{1p})$,

$W(b_{2p})) = 3(f_1/2)^{1/2}(72f_1)^{-1/2} - 40/9(1 - 2/f_1)/(f_1 - 4) + n(1 - 2/f_1)^{1/3}/$

$3D(b_{2p})[2/(f_1 - 4)]^{1/2} Cov(b_{1p}, b_{2p})$

$C_R^2 = d'D^{-1}\underline{d}$

$\underline{d}' = (\sqrt{b_{1p}} - E(\sqrt{b_{1p}}),\ b_{2p} - g(n-1)/(n+1))$,

$$\underline{D} = \begin{bmatrix} D^2\sqrt{b_{1p}} & \text{Cov}(\sqrt{\text{b}_{1\text{p}}}, b_{2p}) \\ Cov(\sqrt{b_{1p}}, b_{2p}) & 8\text{g/n} \end{bmatrix},$$

(3) $S_L^2, S_N^2, S_W^2, C_W^2, C_R^2 \sim \chi^2_2$;

(n)(1) Small (1980),

(2) $Q_1 = \underline{Y}'_{(1)}\underline{U_{(1)}^{-1}}\underline{Y}_1$, with $\mathcal{P}_p \in A_1$,

$$Q_2 = \underline{Y}'_{(2)} \underline{U_{(2)}^{-1}} \underline{Y}_2, \quad \text{with } \mathcal{P}_p \in A_2,$$

$$Q = Q_1 + Q_2, \text{ with } \mathcal{P}_p \in A_3,$$

$$(3)\ Q_1 \sim \chi^2_p, \quad Q_2 \sim \chi^2_p, \quad Q \sim \chi^2_{2p}$$

where

$$\underline{Y}_{(1)} = \delta_1 \sinh^{-1}(\sqrt{b_1(x_1)}/\lambda_1, \ldots, \delta_1 \sinh^{-1}(\sqrt{b_1(x_p)}/\lambda_1))',$$

$$\underline{Y}_{(2)} = \gamma_2 + \delta_2 \sinh^{-1}[(b_2(x_1) - \xi_2^2)/\lambda_2], \ldots, \gamma_2 + \delta_2 \sinh^{-1}[(b_2(x_p) - \xi_2)/\lambda_2])';$$

$$\underline{U}_{(1)} = (r^3_{ii'}), \quad \underline{U}_{(2)} = (r^4_{ii'}), \quad i, i' = 1, \ldots, p; \quad i \neq i', \quad r_{ii} = 1,$$

where $r_{ii'}$ are rectilinear correlation coefficients from $\underline{U}$. $\delta_i, \lambda_i, \gamma_i, \xi_i$ are the transformation parameters of Johnson's system $z = \gamma + \delta g\left(\frac{x-\xi}{\nu}\right)$ where δ_i, γ_i are shape parameters, ξ_i is the location parameter and ν is the range parameter

IV. EXAMINATION OF POWER OF TESTS

There are numerous tests for multivariate normality and as many rules of constructing test stastistics for them . Having such a variety of tests to choose from it seems worthwhile to ask a few questions about them.

Which of them are best in the sense of power? Which of them have the properties of omnibus tests? Which are direction tests? And finally, which of them can be recommended for practical use?.

While seeking answers to all these questions it is best to refer to Monte Carlo simulation experiments. It is a well known fact that examinations of power of multivariate normality tests based on skewness and flatness measures have been conducted for almost forty years now.

The development of tests for multivariate normality dates back to the year 1968 when Wagle's work entitled "Multivatiate Beta Distribution and a Test for Multivariate Normality" was published.

The next stage in the development of tests for multivariate normality is closely connected with Mardia and his works (1970, 1974, 1975, 1980). As it was mentioned above he introduced the measure of multivariate asymmetry and kurtosis being the generalised measures of Pearson shape. Taking these measures as the basis, several tests for multivariate normality were constructed. Omnibus and direction tests using the above measures were developed in the works of: Mardia and Foster (1983), Bera and John (1983).

A different approach towards constructing tests for multivariate normality was adopted by Malkovich and Afifi (1973) who, making the use of Roy's union and intersection principle, gave measures of shape for multivariate distributions, taking as a basis Cramer-Wold theorem. The mentioned above principle enabled to generalise tests for multivariate normality of the class of tests such as: Shapiro-Wilk, Kolmogorov-Smirnow, and Cramer –von Mises, as well as standardised third and fourth central moment from a sample in a multivariate case (cf Domanski and Wagner, 1984).

Although several general examinations of power of tests for mutivariate normality can be found, none of them is fully universal in character. This is due to the fact that it would be pointless to examine every existing method and impossible to test every departure from normality. The majority of more universal examinations limit the scope of their analyses to selected categories of tests, or to most popular or most promising ones. Unfortunately, none of the tests for multivarate normality can be described as the one which has been fully examined.

Ward (1988) compared the power of Mardia's skewness and flatness tests, Shapiro-Wilk test generalised by Malkovich-Afifi, Anderson-Darling test modified by Hawkins, Mardia-Foster test, and two of his own propositions which developed Kolmogorov-Smirnow and Anderson-Darling tests.

In most cases Mardia tests seemed to be most powerful, yet none of them was considered to be the best. Multivariate measures of skewness and flatness prove useful both as statistics characterising multivariate sample, and as the basis for normality tests. For that reason examinations of multivariate tests based on measures of skewness and flatness were carried out.

In the conducted experiment simulating Monte-Carlo method the power of the following ten tests was investigated: $M_1, C_1, U(b_{1p}), M_2, C_2, U(b_{2p})$, M_3, C_4, S_{2n}, C_{2n}.

The experiment involved 50,000 repetitions for both multivariate normal and alternative distributions for n=20, 30, 40, 50, 100; p=2, 3, 4, 5.

The obtained results are presented in tables 1.1., 2.1 and 2.2 in figures 1.1–1.4, 2.1–2.4, 3.1–3.4.

The conducted examination allows us to form the following conclusions

1. Mardia and Foster test based on statistics $W(b_{1p}), W(b_{2p})\, i\, S_n^2$ and Bera and John test based on statistic C_3 for p$\geq$2, with the assumption of the truthfulness of the hypothesis that multivariate distribution is normal, exceed the ac-

cepted significance levels. In such cases it is recommended to take quantiles of statistical distributions, obtained with the use of Monte-Carlo method, as the basis for analysis. In the further stages of the analysis these tests were disregarded. Figures 1.1–1.4 present power of tests M_1, M_2 i $M_3 = MSK$ for zero multivariate normal distribution for p=2, 3, 4 and 5.

2. Mardia test (M_1), Mardia and Foster $(U(b_{1p})\, i\, (S_n^2))$ and Jargun and McKenzie (M_3) tests are most powerful (cf table 2.1and fig. 3.1–3.4). These tests proved to be better for symmetric distributions (cf table 2.2 and 2.1–2.4).

3. Power of tests for multivariate normality based on measures of shape for $n \leq 30$ decreases according to the increase in p.

4. The considered tests are to be applied for samples $n \geq 30$.

V. FINAL REMARKS

The assumption that a sample comes from a multivariate normal distribution is a fundamental one for many commonly used multivariate statistical techniques. If this assumption does not hold good, then the results of statistical analysis become dubious.Even now power of numerous multivariate analyses is hardly acceptable due to the fact that researchers are frequently forced to use samples which are far from perfect ; either because of the sample size or because of the applied methodology.

The first attempts to test multivariate normality were undertaken almost forty years ago. Healy (1968) developed Q–Q diagram to chi-square which is fequently used for graphic evaluation of multivariate normality. Mardia proposed multivariate measures of skewness and flatness. These measures prove helpful both as a descriptive statistic for a multivariate sample and as the basis of many useful tests for multivariate normality. Mardia tests are apparently the most often used consistency procedures for multivariate normal distribution.

Some other comparisons of power of tests for multivariate normality were also made (cf e.g. Meklin and Mundfrom, 2004). However, no uniformity in the analysed tests or in alternative distributions was observed. The only type of tests which were taken into consideration in every examination of power, are Mardia skewness and flatness tests. All in all, Mardia tests are thought to be generally effective, although their use as diagnostic tests which allow to find the reason for the lack of normality was questioned by Horswell and Looney (1992). Other tests which are potentially useful include: Koziol test (1986) and Royston test (1983), and particularly, Henze and Zinkler test, (1990).

As it was shown by the earlier research none of the methods is good enough when multivariate normality is taken into account. The graphic approach alone e.g. visual examination of chi-square or beta diagram, will signal considerable departures from normality. Multivariate measures of skewness and flatness are useful both as discriptive statistics of multivariate set of data, and as a basis of tests for normality.

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Monovariate case $p = 1$ is well known in the literature. Determination of empirical critical values requires complex experiment by Monte Carlo method.

Table 1.1

Empirical power of selected tests for multivariate normal distribution for $\alpha = 0.05$, $p = 2, 3, 4, 5$ and $n = 20, 30, 40, 50, 100$

Test statistics	Sample size (n)				
	20	30	40	500	100
1	2	3	4	5	6
			$p = 2$		
M_1	.0137	.0266	.0309	.0382	.0419
C_1	.0157	.0261	.0324	.0971	.0399
$U(b_{1p})$	.0133	.0258	.0306	.0372	.0418
M_2	.0022	.0093	.0150	.0177	.0298
C_2	.0074	.0166	.0212	.0269	.0357
$U(b_{2p})$	.0032	.0086	.0131	.0167	.0249
M_3	.0098	.0205	.0258	.0317	.0383
C_4	.0141	.0250	.0321	.0376	.0450
S_N^2	.0118	.0229	.0273	.0341	.0400
C_N^2	.0003	.1753	.0635	.0530	.0462
			$p = 3$		
M_1	.0080	.0210	.0283	.0311	.0435
C_1	.0141	.0245	.0326	.0320	.0413
$U(b_{1p})$	.0104	.0210	.0264	.0277	.0392
M_2	.0316	.0359	.0350	.0387	.0462
C_2	.0086	.0214	.0276	.0320	.0414
$U(b_{2p})$	.0009	.0051	.0120	.0140	.0284
M_3	.0067	.0167	.0238	.0363	.0557
C_4	.0151	.0285	.0358	.0411	.0503
S_N^2	.0078	.0186	.0261	.0271	.0397
C_N^2	.0001	.0007	.0007	.4367	.0671

Table 1.1 (cont.)

1	2	3	4	5	6
			$p=4$		
M_1	.0039	.0154	.0203	.0275	.0406
C_1	.0119	.0255	.0285	.0321	.0443
$U(b_{1p})$	.0559	.0421	.0400	.0399	.0449
M_2	.1410	.1039	.0869	.0770	.0612
C_2	.0077	.0222	.0282	.0324	.0478
$U(b_{2p})$	.0003	.0035	.0102	.0135	.0312
M_3	.0026	.0128	.0167	.0231	.0391
C_4	.0127	.0305	.0358	.0419	.0556
S_N^2	.0374	.0361	.0376	.0382	.0460
C_N^2	.0001	.0005	.0013	.0010	.2773
			$p=5$		
M_1	.0011	.0102	.0163	.0208	.0393
C_1	.0127	.0246	.0264	.0298	.0420
$U(b_{1p})$	.1449	.0891	.0705	.0597	.0540
M_2	.3349	.2120	.1571	.1318	.0892
C_2	.0048	.0213	.0255	.0323	.0463
$U(b_{2p})$	.0008	.0053	.0117	.0189	.0335
M_3	.0009	.0089	.0144	.0182	.0356
C_4	.0107	.0269	.0334	.0400	.0526
S_N^2	.1111	.0783	.0635	.0623	.0558
C_N^2	.0000	.0002	.0005	.0008	.0018

Source: Author's own calculations.

Table 1.2

Empirical power of tests for alternative multivariate gamma distribution
($\alpha = 2; \theta = 2$) for p =2, 3, 4, 5 and $\alpha = 0{,}05$ (in ‰)

Test statistics	Sample size (n)				
	20	30	40	500	100
1	2	3	4	5	6
$p = 2$					
M_1	331	659	858	941	1000
C_1	129	272	409	525	881
$U(b_{1p})$	327	655	855	940	1000
M_2	83	223	351	458	785
C_2	127	281	412	510	817
$U(b_{2p})$	119	269	401	503	809
M_3	344	675	874	954	1000
C_4	167	339	488	607	914
S_N^2	286	593	808	916	1000
C_N^2	2	892	933	974	1000
$p = 3$					
M_1	290	662	877	962	1000
C_1	98	220	350	471	860
$U(b_{1p})$	273	644	867	960	1000
M_2	44	181	333	467	840
C_2	104	261	408	521	839
$U(b_{2p})$	75	250	408	539	872
M_3	296	674	890	972	1000
C_4	133	307	478	602	922
S_N^2	231	575	816	938	1000
C_N^2	27	23	1	1000	1000
$p = 4$					
M_1	211	632	889	934	1000
C_1	102	260	439	590	945
$U(b_{1p})$	186	601	870	969	1000
M_2	45	145	300	461	872
C_2	81	259	413	545	866
$U(b_{2p})$	37	221	395	555	905

Table 1.2 (cont.)

1	2	3	4	5	6
M_3	214	642	900	979	1000
C_4	118	319	499	655	953
S_N^2	152	531	818	950	1000
C_N^2	14	124	138	35	1000
			$p=5$		
M_1	132	588	871	970	1000
C_1	121	336	534	721	867
$U(b_{1p})$	116	545	848	962	1000
M_2	94	104	250	423	704
C_2	72	265	423	561	767
$U(b_{2p})$	13	174	371	543	765
M_3	133	598	880	976	1000
C_4	114	345	544	701	914
S_N^2	92	474	794	935	1000
C_N^2	4	105	233	278	.

Source: Author's own calculations.

Tablica 1.3

Empirical power of tests for alternative empirical gamma distribution ($\alpha=10\,;\theta=2$) for p=2, 3, 4, 5 and $\alpha=0{,}05$ (in ‰)

Test statistics	Sample size (n)				
	20	30	40	500	100
1	2	3	4	5	6
			$p=2$		
M_1	67	153	242	333	729
C_1	35	70	100	137	296
$U(b_{1p})$	67	150	239	330	726
M_2	13	38	67	94	200
C_2	29	64	91	121	239
$U(b_{2p})$	20	49	80	110	223
M_3	76	168	261	356	762
C_4	41	86	118	157	318
S_N^2	57	130	203	283	672
C_N^2	1	125	306	359	725

Table 1.3 (cont.)

1	2	3	4	5	6
			$p=3$		
M_1	41	124	229	32	759
C_1	25	50	85	104	246
$U(b_{1p})$	38	113	214	308	742
M_2	28	38	60	76	182
C_2	20	58	94	122	237
$U(b_{2p})$	7	32	64	90	215
M_3	42	13	241	340	783
C_4	31	74	116	149	300
S_N^2	30	97	181	263	679
C_N^2	3	6	3	730	801
			$p=4$		
M_1	19	99	199	303	773
C_1	24	59	94	130	318
$U(b_{1p})$	43	95	178	277	744
M_2	108	71	63	73	162
C_2	17	59	99	129	248
$U(b_{2p})$	2	20	49	75	203
M_3	20	102	207	315	793
C_4	25	77	123	164	338
S_N^2	30	86	148	230	672
C_N^2	1	8	18	19	918
			$p=5$		
M_1	7	70	160	274	757
C_1	24	68	120	168	422
$U(b_{1p})$	86	80	144	245	721
M_2	264	128	85	86	141
C_2	12	54	101	134	249
$U(b_{2p})$	1	13	39	66	189
M_3	7	72	164	284	773
C_4	21	76	131	178	376
S_N^2	66	70	123	203	643
C_N^2	0	4	13	50	11

Source: Author's own calculations.

Fig. 1.1 Empirical power of tests M1, M2, MSK for multivariate normal distribution Np(0,1) for p=2 and alpha=0.05 depending on the sample size

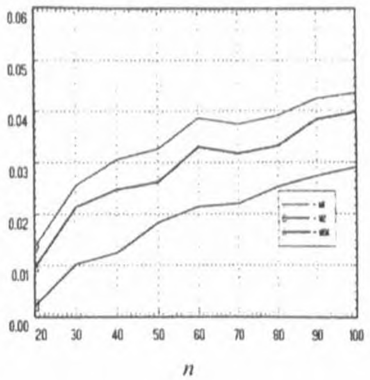


Fig. 1.2 Empirical power of tests M1, M2, MSK for multivariate normal distribution Np(0,1) for p=3 and alpha=0.05 depending on the sample size

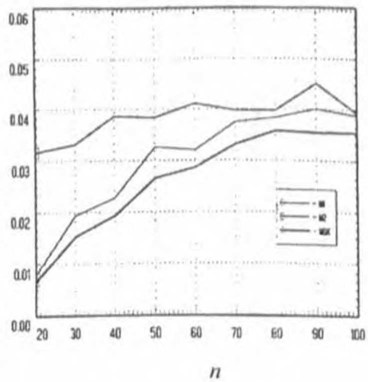


Fig. 1.3 Empirical power of tests M1, M2, MSK for multivariate normal distribution Np(0,1) for p=4 and alpha=0.05 depending on the sample size

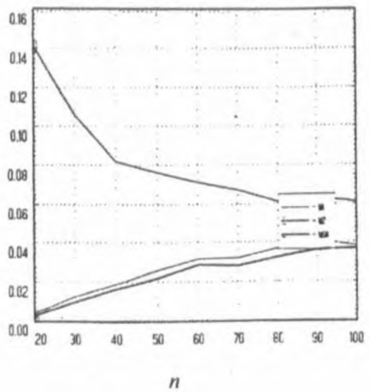


Fig. 1.4 Empirical power of tests M1, M2, MSK for multivariate normal distribution Np(0,1) for p=5 and alpha=0.05 depending on the sample size

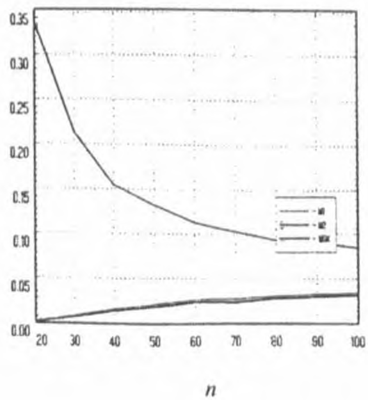


Fig. 2.1 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=50; theta=2) for p=2 and alpha=0.05 depending on the sample size

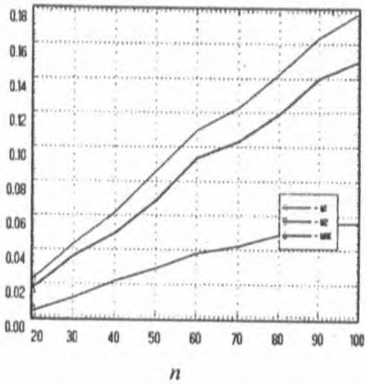


Fig. 2.2 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=50; theta=2) for p=3 and alpha=0.05 depending on the sample size

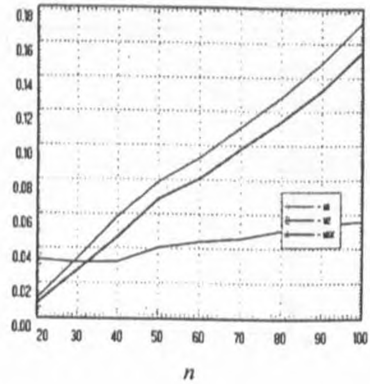


Fig. 2.3 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=50; theta=2) for p=4 and alpha=0.05 depending on the sample size

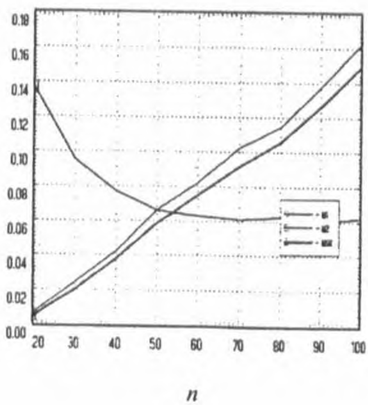


Fig. 2.4 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=50; theta=2) for p=5 and alpha=0.05 depending on the sample size

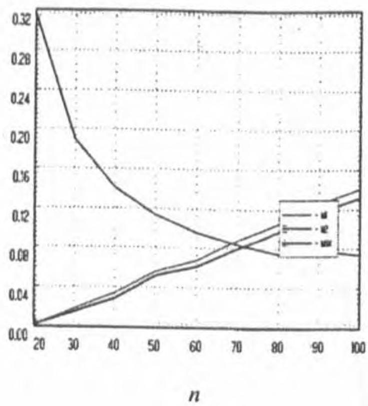


Fig. 3.1 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=2; theta=2) for p=2 and alpha=0.05 depending on the sample size

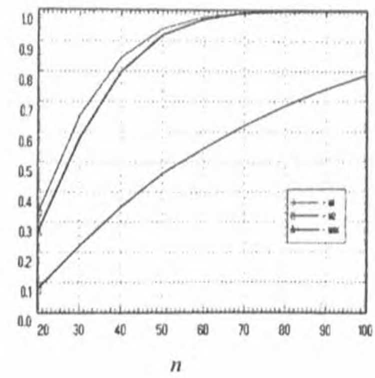


Fig. 3.2 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=2; theta=2) for p=3 and alpha=0.05 depending on the sample size

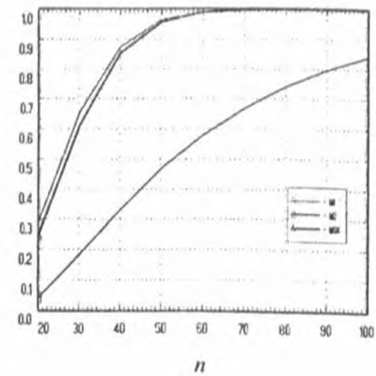


Fig. 3.3 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=2; theta=2) for p=4 and alpha=0.05 depending on the sample size

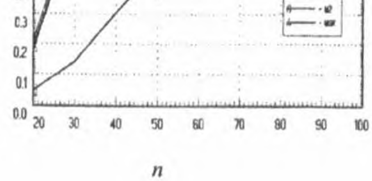
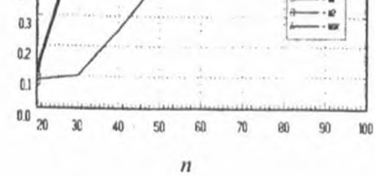


Fig. 3.4 Empirical power of tests M1, M2, MSK for alternative multivariate gamma distribution (alpha=2; theta=2) for p=5 and alpha=0.05 depending on the sample size



Czesław Domański

MOC TESTÓW WIELOWYMIAROWEJ NORMALNOŚCI OPARTYCH NA MIARACH SKOŚNOŚCI I SPŁASZCZENIA

Istnieje wiele zaproponowanych metod konstrukcji testów wielowymiarowej normalności. Aktualny przegląd literatury dowodzi, że istnieje przynajmniej 60 procedur weryfikacji hipotezy o wielowymiarowej normalności rozkładów zmiennych losowych. Kilka przesłanek uzasadnia analizę testów tej klasy, które oparte są na miarach skośności i spłaszczenia. Łatwo można spostrzec, że zastosowanie tych testów wpływa także na lepszą ogólną analizę wielowymiarową rozważanej zmiennej.

W opracowaniu prezentowane są wyniki badania mocy wielu autorów i własne oparte na rozważaniach analitycznych i metodach Monte Carlo.