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GOLDFELD-QUANDT TEST: UNBIASEDNESS VS SYMMETRY

ABSTRACT. The Goldfeld-Quandt test for homoscedasticity in a classical linear regression appears to be biased. In the paper an unbiased test is constructed. The result is extended to families of distributions with scale parameter.

Key words: Goldfeld-Quandt test, test for scale, unbiased test, chi-square test for variance.

I. INTRODUCTION

Consider a classical linear regression problem $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $i = 1,\ldots, n$. It is assumed that random errors ε_i's are independent random variables distributed as $N(0,\sigma^2)$. After fitting the model the homoscedasticity of random errors should be checked. One of the commonly applied tests is the Goldfeld-Quandt test (Greene 2000). In the test, the sample is divided into two disjoint subsamples of size n_1 and n_2 respectively, so two models are considered $Y_i = \beta_0^{(1)} + \beta_1^{(1)} x_i + \varepsilon_i^{(1)}$, $i = 1,\ldots, n_1$ and $Y_i^{(2)} = \beta_0^{(2)} + \beta_1^{(2)} x_i + \varepsilon_i^{(2)}$, $i = 1,\ldots, n_2$ with the assumption: $\varepsilon^{(1)} \sim N(0,\sigma_1^2)$ and $\varepsilon^{(2)} \sim N(0,\sigma_2^2)$. Verified hypothesis is $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$. The Goldfeld-Quandt test statistic is S_1^2 / S_2^2, where S_1^2 and S_2^2 are residual variances in the first and the second model respectively. Hypothesis is rejected at the significance level α, if

$$\frac{S_1^2}{S_2^2} < F\left(1-\frac{\alpha}{2}, n_1-2, n_2-2\right) \text{ or } \frac{S_1^2}{S_2^2} > F\left(\frac{\alpha}{2}, n_1-2, n_2-2\right),$$

where $F(\alpha; u, v)$ is the α-critical value of the F distribution with (u, v) degrees of freedom. Unfortunately, it appears that the test is biased one, i.e. the power of the test may be smaller than the significance level. In the Figure 1 the power of the test for $n_1 = 10$, $n_2 = 15$ and $\alpha = 0.05$ is shown.

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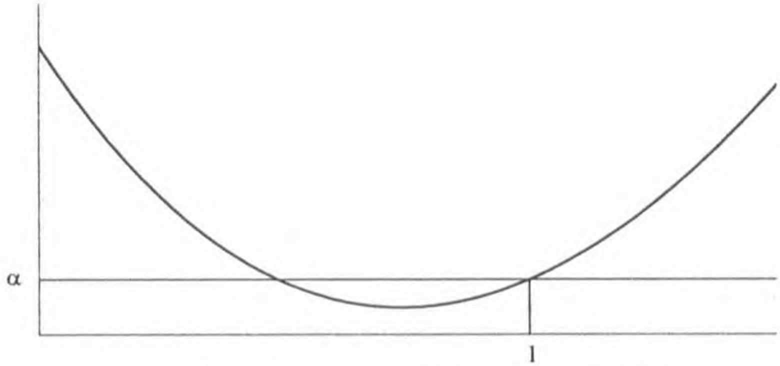

Figure 1. The power of the Goldfeld-Quandt test

Hence it is harder to reject the hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ for some variances σ_1^2 smaller then σ_2^2. In the following table there are given variances from the alternative hypothesis along with appropriate probabilities p of rejecting H_0:

$\frac{\sigma_1^2}{\sigma_2^2}$	0.896	0.912	0.928	0.944	0.960	0.976	0.992	1.000
p	0.05004	0.04947	0.04913	0.04898	0.04904	0.04929	0.04972	0.05000

In practice application of the test may lead to serious misstatement that variances are equal though they are not.

The question is, does there exist an unbiased version of Goldfeld-Quandt test and, if yes, how to construct such a test. The above test will be referred to as a classical one.

The similar situation is met in the problem of testing $H_0 : \sigma^2 = \sigma_0^2$ vs $H_1 : \sigma^2 \neq \sigma_0^2$ in normal distribution $N(\mu, \sigma^2)$. The most commonly used test (e.g. Bickel and Doksum 1980; Bartoszyński and Niewiadomska-Bugaj 1996, Chow 1983, Müller 1991, Storm 1979), based on a sample $X_1, ..., X_n$ with the sample mean $\bar{X} = (X_1 + ... + X_n)/n$, is as follows: reject H_0 at a significance level α if

$$\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma_0^2} < \chi^2\left(1 - \frac{\alpha}{2}; n-1\right) \text{ or } \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\sigma_0^2} > \chi^2\left(\frac{\alpha}{2}; n-1\right).$$

Here $\chi^2(\alpha;\nu)$ denotes the α-critical value of the chi-square distribution with ν degrees of freedom. Some statistical packages (for example Statgraphics and Statistica) implement exactly the above lower and upper critical values of the test. An unbiased test for this hypothesis may be found in Lehmann (1986). A construction of an unbiased test for the scale parameter of the exponential distribution and $n = 1$ one can find in Knight (2000). In what follows we give a general construction of an unbiased test for scale families of distributions. Some important for practical applications examples as well as a remark on the shortest confidence intervals, are also presented.

II. UNBIASED TEST FOR SCALE PARAMETER

Consider a statistical model $\left(R_+,\{F_\lambda(x), \lambda > 0\}\right)$ with scale parameter, i.e. $F_\lambda(x) = F(x/\lambda)$. The problem is to test $H_0 : \lambda = \lambda_0$ vs $H_1 : \lambda \neq \lambda_0$ at a significance level α. If T is a scale equivariant estimator of λ, then critical region is of the form:

$$(*) \qquad \frac{T}{\lambda_0} < G^{-1}(\alpha_2) \text{ or } \frac{T}{\lambda_0} > G^{-1}(\alpha_1),$$

where $\alpha_1 + \alpha_2 = \alpha$ and G^{-1} is the quantile function of the null distribution of T/λ_0.

Theorem. There exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 = \alpha$ and the test with critical region $(*)$ is unbiased.

Proof. The power of the test for a fixed $\alpha_1 \in (0,\alpha)$ is

$$M(\lambda) = P_\lambda\left\{\frac{T}{\lambda_0} < G^{-1}(\alpha - \alpha_1) \quad\text{or}\quad \frac{T}{\lambda_0} > G^{-1}(1-\alpha_1)\right\} =$$

$$= P_\lambda\left\{\frac{T}{\lambda} < \frac{\lambda_0}{\lambda} G^{-1}(\alpha - \alpha_1) \quad\text{or}\quad \frac{T}{\lambda} > \frac{\lambda_0}{\lambda} G^{-1}(1-\alpha_1)\right\} =$$

$$= G\left(\frac{\lambda_0}{\lambda} G^{-1}(\alpha - \alpha_1)\right) + \left(1 - G\left(\frac{\lambda_0}{\lambda} G^{-1}(1-\alpha_1)\right)\right).$$

Then

$$\frac{dM(\lambda)}{d\lambda}=-\frac{\lambda_0 G^{-1}(\alpha-\alpha_1)}{\lambda^2}g\left(\frac{\lambda_0}{\lambda}G^{-1}(\alpha-\alpha_1)\right)+\frac{\lambda_0 G^{-1}(1-\alpha_1)}{\lambda^2}g\left(\frac{\lambda_0}{\lambda}G^{-1}(1-\alpha_1)\right).$$

It follows that $M(\lambda)$ is strictly decreasing to the left of an λ_*, strictly increasing to the right of λ_*, and achieves its minimum at λ_* such that

$$G^{-1}(\alpha-\alpha_1)g\left(\frac{\lambda_0}{\lambda_*}G^{-1}(\alpha-\alpha_1)\right)=G^{-1}(1-\alpha_1)g\left(\frac{\lambda_0}{\lambda_*}G^{-1}(1-\alpha_1)\right).$$

The test is unbiased iff $\lambda_*=\lambda_0$ which holds iff $0<\alpha_1<\alpha$ is a solution of the equation

$$(*)\qquad G^{-1}(\alpha-\alpha_1)g(G^{-1}(\alpha-\alpha_1))=G^{-1}(1-\alpha_1)g(G^{-1}(1-\alpha_1))$$

Note that if $\alpha_1\to 0$ then LHS of (*) tends to $G^{-1}(\alpha)g\left(G^{-1}(\alpha)\right)>0$ and RHS of (*) tends to zero. On the other hand, if $\alpha_1\to 0$ then LHS of (*) tends to zero and RHS of (*) tends to $G^{-1}(1-\alpha)g\left(G^{-1}(1-\alpha)\right)>0$. Hence, there exists α_1 which is a solution of (*). □

Critical values of an unbiased test are illustrated in the Fig. 2.

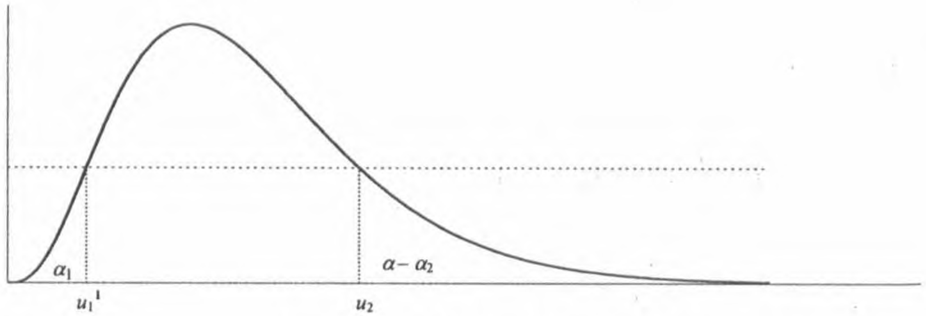

Figure 2. Critical values of the unbiased test

It is obvious that the power of the unbiased test and the power of the symmetric test are not comparable. Those powers are shown in the Fig. 3.

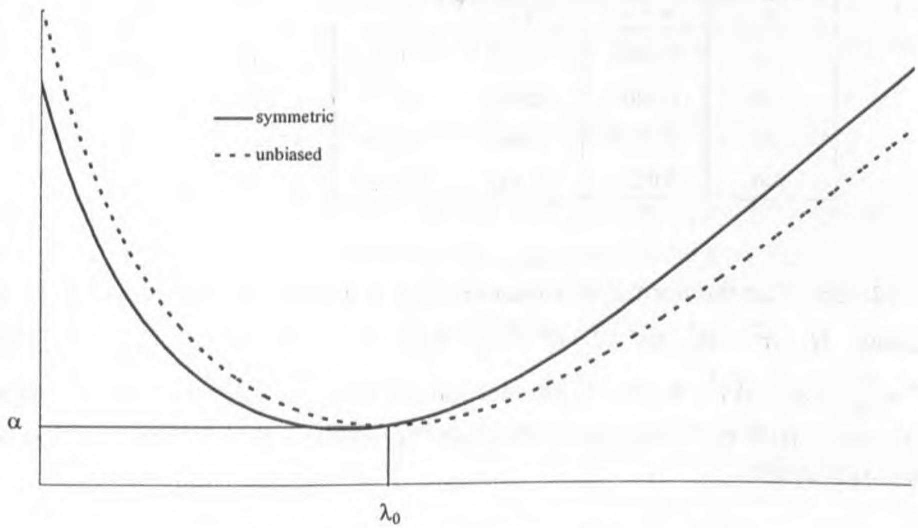

Figure 3. Powers of the unbiased test and the symmetric test

III. THREE IMPORTANT EXAMPLES

1. Let F be the exponential distribution, i.e. $F_\theta(x)=1-\exp\left\{-\frac{x}{\theta}\right\}$ and consider the problem of testing $H_0:\theta=\theta_0$ vs $H_1:\theta\neq\theta_0$. Let $X_1,\ldots,X_n$ be a sample and let $T=\sum_{i=1}^n X_i$. The distribution of T is the gamma one with density function:

$$g_\theta(t)=\frac{1}{\theta^n\Gamma(n)}t^{n-1}\exp\left\{-\frac{t}{\theta}\right\}, \qquad \text{for } t>0.$$

To find the unbiased test one has to find α_1 such that

$$G_1^{-1}(\alpha-\alpha_1)g_1\left(G_1^{-1}(\alpha-\alpha_1)\right)=G_1^{-1}(1-\alpha_1)g_1\left(G_1^{-1}(1-\alpha_1)\right).$$

The solution may be find with the aid of computer (for $n=1$ see Knight 2000). In the Table 1 (for $\alpha=0.05$) there are given critical values for unbiased test (u_1 and u_2) as well as for classical test (c_1 and c_2).

Table 1

Critical values of unbiased and classical test in exponential distribution

n	$\alpha-\alpha_1$	u_1	u_2	c_1	c_2
10	0.0189	4.979	17.613	4.795	17.085
20	0.0207	12.439	30.137	12.217	29.671
50	0.0223	37.372	65.195	37.111	64.781
100	0.0231	81.645	120.919	81.364	120.529

2. Let F be the normal distribution $N(\mu,\sigma^2)$ and consider the problem of testing $H_0:\sigma^2=\sigma_0^2$ vs $H_1:\sigma^2\neq\sigma_0^2$. Let $X_1,\ldots,X_n$ be a sample and let $T=\sum_{i=1}^n(X_i-\overline{X})^2$, where $\overline{X}$ denotes sample mean. The distribution of T is the chi-square with $n-1$ degrees of freedom. To find the unbiased test one has to find α_1 such that

$$G^{-1}(\alpha-\alpha_1)g\left(G^{-1}(\alpha-\alpha_1)\right)=G^{-1}(1-\alpha_1)g\left(G^{-1}(1-\alpha_1)\right),$$

where G and g denotes cdf and pdf of chi-square distribution with $n-1$ degrees of freedom respectively. The solution may be find with the aid of computer. In the Table 2 (for $\alpha=0.05$) there are given critical values for unbiased test (u_1 and u_2) as well as for classical test (c_1 and c_2).

Table 2

Critical values of unbiased and classical test in normal

n	$\alpha-\alpha_1$	u_1	u_2	c_1	c_2
10	0.0161	2.953	20.305	2.700	19.023
20	0.0188	9.267	33.921	8.907	32.852
50	0.0211	32.020	71.128	31.555	70.222
100	0.0222	73.882	129.253	73.361	128.422

3. Consider the problem of testing $H_0:\sigma_1^2=\sigma_2^2$ vs $H_1:\sigma_1^2\neq\sigma_2^2$ in two normal distributions with variances σ_1^2 and σ_2^2. Note that the Goldfeld-Quandt test is a special case of the considered test. Let S_1^2 and S_2^2 be two estimators respectively of σ_1^2 and σ_2^2 such that they are independent random variables distributed as $\chi^2(\nu_1)/\nu_1$ and $\chi^2(\nu_2)/\nu_2$ and let $T = S_1^2 / S_2^2$. The distribution of T is the F (Snedecor) with ν_1 and ν_2 degrees of freedom. In the Goldfeld-Quandt test $\nu_1 = n_1 - 2$ and $\nu_2 = n_2 - 2$. To find the unbiased test one has to find α_1 such that

$$G_{\nu_1,\nu_2}^{-1}(\alpha-\alpha_1)g_{\nu_1,\nu_2}\left(G_{\nu_1,\nu_2}^{-1}(\alpha-\alpha_1)\right)=G_{\nu_1,\nu_2}^{-1}(1-\alpha_1)g_{\nu_1,\nu_2}\left(G_{\nu_1,\nu_2}^{-1}(1-\alpha_1)\right),$$

where G_{ν_1,ν_2} and g_{ν_1,ν_2} denotes cdf and pdf of F (Snedecor) distribution with ν_1 and ν_2 degrees of freedom respectively. The solution may be find with the aid of computer. In the Table 3 (for $\alpha = 0.05$) there are given critical values for unbiased test (u_1 and u_2) as well as for classical test (c_1 and c_2).

Table 3

Critical values of unbiased and classical test in two normal distributions

n	m	$\alpha - \alpha_1$	u_1	u_2	c_1	c_2
10	10	0.0250	0.269	3.717	0.269	3.717
20	10	0.0215	0.348	3.290	0.361	3.419
50	10	0.0188	0.410	3.026	0.432	3.221
100	10	0.0177	0.434	2.936	0.459	3.152

Note that if $\nu_1 = \nu_2$ then $g_{\nu_1,\nu_1}(x) = g_{\nu_1,\nu_1}\left(\frac{1}{x}\right)$. Hence, in the case the unbiased test is the classical one, i.e. $\alpha_1 = \alpha/2$.

IV. UNBIASED TEST AND THE SHORTEST CONFIDENCE INTERVAL

Now consider the problem of constructing the shortest confidence interval for scale parameter λ at the confidence level $1 - \alpha$. Because T is scale equivariant estimator of λ then the confidence interval for λ is of the form:

(**) $$\lambda \in \left(\frac{T}{G^{-1}(1-\alpha_1)};\frac{T}{G^{-1}(\alpha-\alpha_1)}\right).$$

The shortest confidence interval is the solution of the following problem:

$$\frac{1}{G^{-1}(\alpha-\alpha_1)}-\frac{1}{G^{-1}(1-\alpha_1)}=\min,$$

$$\int_{G^{-1}(\alpha-\alpha_1)}^{G^{-1}(1-\alpha_1)} g(x)dx=\alpha.$$

Application of the method of Lagrange multipliers gives the following condition for quantiles of the distribution of T statistics:

$$G^{-1}(\alpha-\alpha_1)g\left(G^{-1}(\alpha-\alpha_1)\right)=G^{-1}(1-\alpha_1)g\left(G^{-1}(1-\alpha_1)\right).$$

This is the same condition as obtained for unbiased test. Hence, the shortest confidence interval for λ is the acceptance region in the unbiased test for $H_0:\lambda=\lambda_0$ vs $H_1:\lambda\neq\lambda_0$.

V. NUMERICAL IMPLEMENTATION

Critical values of an unbiased test may be found numerically. In what follows there is given a short Mathematica program which allows to find critical values of the unbiased version of the Goldfeld-Quandt test. For other problems similar programs may be written. Of course, there is also a possibility to use other mathematical or statistical packages (in a similar way) to find out critical values of an unbiased test.

```
    <<Statistics`ContinuousDistributions`

G[n_,m_]=FRatioDistribution[n,m] (*definition of a F distribution*)

Kw[n_,m_,q_]:=Quantile[G[n, m],q] (*quantile function*)
FF[n_,m_,x_]:=CDF[G[n,m],x] (*cumulative distribution function*)
HH[n_,m_,x_]:=PDF[G[n,m],x] (*probability density function*)
RR[n_,m_,alfa_,beta_]:=Kw[n,m,1-(alfa-beta)]*HH[n,m,Kw[n,m,1-(alfa-beta)]]
 -Kw[n,m,beta]*HH[n,m,Kw[n,m,beta]]
 (*equation to be solved with respect to beta=alfa-alfa1*)

alfa=0.05; n=20; m=10;
b1=beta /. FindRoot[RR[n,m,alfa,beta]==0, beta,0.01]
u1=Kw[n,m,b1]
u2=Kw[n,m,1-(alfa-b1)]
```

VI. CONCLUSIONS

It was shown that classical test for scale parameter is biased. It is recommended to use unbiased test critical values of which are nowadays easy obtainable by standard software. An additional advantage is such that the shortest confidence intervals for scale parameters may be constructed.

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TEST GOLDFELDA-QUANDTA: NIEOBCIĄŻONOŚĆ A SYMETRIA

Test Goldfelda-Quandta homoscedastyczności stosowany w klasycznym modelu regresji liniowej jest testem obciążonym. W pracy skonstruowano odpowiedni test nieobciążony. Wynik został rozszerzony na rodziny rozkładów z parametrem skali.