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OPTIMAL DESIGNS FOR $p+1$ OBJECTS BASED ON DESIGNS FOR p OBJECTS

ABSTRACT. The problem of optimizing the estimation of the weights of p objects in n weighing operations using a chemical balance is considered. Conditions under which the existence of an optimum chemical balance weighing design for p objects implies the existence of an optimum chemical balance weighing design for $p+1$ objects are given. We assume that variance matrix of errors is diagonal. We want all variances of estimated measurements to be equal and attaining their lower bound. To construct the design matrix of considered optimum chemical balance weighing design we use the incidence matrices of balanced bipartite weighing designs.

Key words: balanced bipartite weighing design, chemical balance weighing design.

I. INTRODUCTION

Let us suppose we want to estimate the weights of p objects by weighing them n times using a chemical balance, $p \leq n$. The manner of allocation of objects on the pans is described through columns of the $n \times p$ matrix $\mathbf{X}$. Its elements are equal to -1, 1 or 0 if the object is kept on the left pan, right pan or is not included in the particular measurement operation, respectively. It is assumed that $n \times 1$ random column vector of errors $\mathbf{e}$ is such that $E(\mathbf{e}) = \mathbf{0}_n$ and $E(\mathbf{ee}') = \sigma^2\mathbf{G}$ where $\mathbf{0}_n$ is an $n \times 1$ column vector of zeros, $\mathbf{G}$ is an $n \times n$ positive definite diagonal matrix of known elements, $E(\cdot)$ stands for the expectation of $(\cdot)$ and $(\cdot)'$ is used for the transpose of $(\cdot)$. For the estimation of

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the unknown weights of objects we used the weighed least squares method and we get

$$\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$$

and the dispersion matrix of $\hat{\mathbf{w}}$ is

$$\mathrm{V}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1},$$

provided $\mathbf{X}$ is full column rank, $\mathrm{r}(\mathbf{X}) = p$, where $\mathbf{w}$ and $\mathbf{y}$ are column vectors of unknown weights of p objects and of the recorded results in n weighings, respectively.

The problem connected with the optimality of chemical balance weighing design is the choosing of a design matrix $\mathbf{X}$ which minimizes $\phi\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)$ over $\mathbf{D}(n, p)$ for some real-valued function ϕ, where $\mathbf{D}(n, p)$ denotes the class of matrices of n rows, p columns and elements equal to -1, 0 or 1. ϕ is called an optimality criterion. In this paper we consider the optimality criterion as minimum variance for each of the estimated weights.

II. SOME RESULTS ON VARIANCE LIMIT OF ESTIMATED WEIGHTS

We assume that matrix $\mathbf{G}$ is given in the form

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \mathbf{I}_{n_2} \end{bmatrix}, \tag{1}$$

where $n = n_1 + n_2$, $a > 0$ and $\mathbf{I}_{n_h}$ is the $n_h \times n_h$ identity matrix, $h = 1,2$. This structure of the dispersion matrix of errors may be useful in the following situation.

Suppose that are two kinds of chemical balances of different precision. Let n_1 and n_2 be the numbers of times in which the respectively balances are used.

Suppose further that the matrix $\mathbf{X}$ is partitioned correspondingly to the matrix $\mathbf{G}$, i.e.

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}. \tag{2}$$

Ceranka and Graczyk (2004) showed that the minimum attainable variance for each of the estimated weights for a chemical balance weighing design with the design matrix $\mathbf{X}$ given by (2) and the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given in (1), is

$$\mathrm{V}(\hat{w}_j) \geq \frac{\sigma^2}{am_1 + m_2}, \qquad j = 1,2,\ldots,p,$$

where m_1 and m_2 is the number of elements equal to -1 and 1 in the j th column of the matrix $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively.

Definition 1 Any nonsingular chemical balance weighing design with the design matrix $\mathbf{X}$ given in (2) and with the dispersion matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1), is called optimal for the estimated individual weights if

$$\mathrm{V}(\hat{w}_j) = \frac{\sigma^2}{am_1 + m_2}, \qquad j = 1,2,\ldots,p.$$

Theorem 1 Any nonsingular chemical balance weighing design with the design matrix $\mathbf{X}$ given in (2) and with the dispersion matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1), is optimal for the estimated individual weights if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (am_1 + m_2)\mathbf{I}_p.$$

In particular case when $m_1 = n_1$ and $m_2 = n_2$ the Theorem 2.1 was given by Katulska (1989) and if additionally $\mathbf{G} = \mathbf{I}_n$ then it was given in Hotelling (1944).

III. OPTIMUM CHEMICAL BALANCE WEIGHING DESIGN FOR p+1 OBJECTS

Let $\mathbf{X}$ given in (2) be the $n \times p$ matrix of the chemical balance weighing design. Based on this matrix we want to construct matrix $\mathbf{T}$ of the chemical balance weighing design for $p+1$ objects in the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{1}_{n_1} \\ \mathbf{X}_2 & \mathbf{0}_{n_2} \end{bmatrix}, \tag{3}$$

where $\mathbf{1}_{n_1}$ is the $n_1 \times 1$ vector of units.

Theorem 2 If $\mathbf{X}$ given in (2) is the $n \times p$ matrix of the chemical balance weighing design with the dispersion matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is given by (1), then the $\mathbf{T}$ given by (3) is the $n \times (p+1)$ matrix of the optimum chemical balance weighing design with the same dispersion matrix $\sigma^2\mathbf{G}$ if and only if

$$\mathbf{X}_1'\mathbf{X}_1 + \mathbf{X}_2'\mathbf{X}_2 = an_1\mathbf{I}_p \tag{4}$$

and

$$\mathbf{X}_1'\mathbf{1}_{n_1} = \mathbf{0}_p. \tag{5}$$

Proof. The proof is straightforward using the Theorem 2.1.

In the present paper we study some methods of construction the matrix $\mathbf{T}$ of an optimum chemical balance weighing design for $p+1$ objects. The method utilizes the incidence matrices of the balanced bipartite weighing designs for $p = v$ treatments.

IV. BALANCED BIPARTITE WEIGHING DESIGN

A balanced bipartite weighing design (See Huang (1976) and Swamy (1982)) with the parameters v, k_1, k_2, λ_1, λ_2 is an arrangement of v elements into b blocks $B_i = \{B_i^{(1)} : B_i^{(2)}\}$ each with $k = k_1 + k_2$ distinct elements, the number of elements in $B_i^{(j)}$ being k_j, $j = 1,2$, $i = 1,2,...,b$ such that each element occurs in r blocks, each pair of distinct elements is linked in exactly λ_1 blocks and $1-$linked in exactly λ_2 blocks. If B is a block with subsets $B^{(1)}$ and $B^{(2)}$ such that $B = \{B^{(1)} : B^{(2)}\}$ where $B^{(1)} = \{a_1^{(1)}, a_2^{(1)}, ..., a_{k_1}^{(1)}\}$, $B^{(2)} = \{a_1^{(2)}, a_2^{(2)}, ..., a_{k_2}^{(2)}\}$ then two elements in B are said to be linked or $1-$linked if and only if they belong to different subsets or the same subsets of

B, respectively. All $v,\ b,\ r,\ k_1,\ k_2,\ \lambda_1,\ \lambda_2$ are the parameters and they are not independent and they are related by the following identities

$$vr = bk,$$
$$b = \frac{\lambda_1 v(v-1)}{2k_1k_2},$$
$$\lambda_2 = \frac{\lambda_1[k_1(k_1-1)+k_2(k_2-1)]}{2k_1k_2},$$
$$r = \frac{\lambda_1 k(v-1)}{2k_1k_2}.$$

In the next part of the paper balanced bipartite weighing design with the parameters $v,\ k_1,\ k_2,\ \lambda_1,\ \lambda_2$ will be written as $v,\ b,\ r,\ k_1,\ k_2,\ \lambda_1,\ \lambda_2$. The existence of the balanced bipartite weighing designs with the parameters $v,\ b,\ r,\ k_{11},\ k_{21},\ \lambda_1,\ \lambda_2$ implies the existence of the balanced bipartite weighing design with the parameters $v,\ b,\ r,\ k_{12}=k_{21},\ k_{22}=k_{11},\ \lambda_1,\ \lambda_2$. In the other words, if in the balanced bipartite weighing design the size of subblocks is changed, the other parameters are the same. Then without lose of generality we can assume that $k_1 \le k_2$.

If in the balanced bipartite weighing design the number of objects in the first subblock is not equal to the number of objects in the second subblock $(k_1 \ne k_2)$ then each object exist in r_1 blocks in the first subblock and in r_2 blocks in the second subblock, $r = r_1 + r_2$. Then:

$$r_1 = \frac{\lambda_1(v-1)}{2k_2},$$
$$r_2 = \frac{\lambda_1(v-1)}{2k_1}.$$

Let $\mathbf{N}^*$ be the incidence matrix of such a design with elements equal to 0 or 1, then:

$$\mathbf{N}^*\mathbf{N}^{*'} = (r-\lambda_1-\lambda_2)\mathbf{I}_v + (\lambda_1+\lambda_2)\mathbf{1}_v\mathbf{1}_v'.$$

V. CONSTRUCTION OF THE DESIGN MATRIX

Let $\mathbf{N}_h^*$ be the incidence matrix of a balanced bipartite weighing design with the parameters v, b_h, r_h, k_{1h}, k_{2h}, λ_{1h}, λ_{2h}, $h = 1,2$. From the matrix $\mathbf{N}_h^*$ we construct the matrix $\mathbf{N}_h$ by replacing k_{1h} elements equal to 1, which corresponds to the elements belonging to the first subblock by elements equal to -1. Thus each column of the matrix $\mathbf{N}_h$ will contain k_{1h} elements equal to -1, k_{2h} elements equal to 1 and $v - k_{1h} - k_{2h}$ elements equal to 0.

Now we define the matrices $\mathbf{X}_1$ and $\mathbf{X}_2$ of the chemical balance weighing designs in the form:

$$\mathbf{X}_1 = \mathbf{N}_1', \tag{6}$$

$$\mathbf{X}_2 = \mathbf{N}_2'. \tag{7}$$

Now we define the matrix $\mathbf{T}$ of the chemical balance weighing design as

$$\mathbf{T} = \begin{bmatrix} \mathbf{N}_1' & \mathbf{1}_{b_1} \\ \mathbf{N}_2' & \mathbf{0}_{b_2} \end{bmatrix}. \tag{8}$$

In this design we have $p = v + 1$ and $n_1 = b_1$, $n_2 = b_2$. Clearly, such design implies that ith object is weighed $r_1 + r_2$ times, $i = 1,2,...,v$ and $(v+1)$th object is weighed b_1 times.

For finding the optimality condition we have to determine the relations between the parameters of the balanced bipartite weighing design, for which the matrix $\mathbf{N}_1$ satisfy the condition (5). From (5) we have $\mathbf{N}_1 \mathbf{1}_{b_1} = \mathbf{0}_p$. This condition is fulfilled if and only if $r_{21} - r_{11} = 0$. The last equation implies that $k_{21} = k_{11}$. This contradicts with the assumption. In other words, we cannot construct the design matrix $\mathbf{T}$ in the form (8).

Now we consider the matrix $\mathbf{X}_1$ of the chemical balance weighing designs as

$$\mathbf{X}_1 = \begin{bmatrix} \mathbf{N}_1' \\ -\mathbf{N}_1' \end{bmatrix}. \tag{9}$$

Then

$$\mathbf{T} = \begin{bmatrix} \mathbf{N}_1' & \mathbf{1}_{b_1} \\ -\mathbf{N}_1' & \mathbf{1}_{b_1} \\ \mathbf{N}_2' & \mathbf{0}_{b_2} \end{bmatrix}. \tag{10}$$

In this design $n_1 = 2b_1$, $n_2 = b_2$, each of the v first columns of $\mathbf{T}$ contains $2r_{11} + r_{12}$ elements equal to -1, $2r_{21} + r_{22}$ elements equal to 1 and $2b_1 + b_2 - 2r_1 - r_2$ elements equal to 0, $(v+1)$th column contains $2b_1$ elements equal to 1 and b_2 elements equal to 0.

Let the dispersion matrix of errors $\sigma^2\mathbf{G}$ be in the form

$$\mathbf{G} = \begin{bmatrix} \frac{1}{a}\mathbf{I}_{b_1} & \mathbf{0}_{b_1}\mathbf{0}_{b_1}' & \mathbf{0}_{b_1}\mathbf{0}_{b_2}' \\ \mathbf{0}_{b_1}\mathbf{0}_{b_1}' & \frac{1}{a}\mathbf{I}_{b_1} & \mathbf{0}_{b_1}\mathbf{0}_{b_2}' \\ \mathbf{0}_{b_2}\mathbf{0}_{b_1}' & \mathbf{0}_{b_2}\mathbf{0}_{b_1}' & \mathbf{I}_{b_2} \end{bmatrix}. \tag{11}$$

Theorem 3 Any nonsingular chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11), is optimal if and only if

$$2a(\lambda_{21} - \lambda_{11}) + (\lambda_{22} - \lambda_{12}) = 0 \tag{12}$$

and

$$2a(b_1 - r_1) - r_2 = 0. \tag{13}$$

Proof. For the design matrix $\mathbf{T}$ given by (10) with the matrix $\mathbf{G}$ given by (11) the condition (5) is always fulfilled. From the condition (4) we have

$$2\mathbf{N}_1\mathbf{N}_1' = [2a(r_1 - \lambda_{21} + \lambda_{11}) + (r_2 - \lambda_{22} + \lambda_{12})]\mathbf{I}_v + \\ + [2a(\lambda_{21} - \lambda_{11}) + (\lambda_{22} - \lambda_{12})]\mathbf{1}_v\mathbf{1}_v'$$

The last equality will be true if and only if (12) and (13) are satisfied. Hence the theorem.

If the chemical balance weighing design given by the matrix $\mathbf{T}$ in the form (10) is optimal then

$$\mathrm{V}(\hat{w}_j) = \frac{\sigma^2}{2ab_1}.$$

We can notice that if the parameters of two balanced bipartite weighing designs satisfy conditions $\lambda_{21} = \lambda_{11}$ and $\lambda_{22} = \lambda_{12}$ then we have

Corollary 1 If $\lambda_{2h} - \lambda_{1h} = 0,\ \ h = 1,2$, then the chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11), is optimal if and only if the condition (13) is fulfilled.

VI. THE BALANCED BIPARTITE WEIGHING DESIGNS LEADING TO THE OPTIMAL DESIGNS

We have seen in Corollary 1 that if the parameters of two balanced bipartite weighing designs satisfy the condition $\lambda_{2h} - \lambda_{1h} = 0,\ \ h = 1,2$, then a chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11), is optimal if and only if the condition (13) is true. Under these condition we have formulated a theorem following from the paper of Ceranka and Graczyk (2005).

Theorem 4 For a given $a = \dfrac{uc^2}{2s(v-c^2)}$ the existence of the balanced bipartite weighing design with the parameters v, $b_1 = \dfrac{2sv(v-1)}{c^2(c^2-1)}$, $r_1 = \dfrac{2s(v-1)}{c^2-1}$, $k_{11} = \dfrac{c(c-1)}{2}$, $k_{21} = \dfrac{c(c+1)}{2}$, $\lambda_{11} = s, \lambda_{21} = s$ and v, $b_2 = \dfrac{2uv(v-1)}{c^2(c^2-1)}$, $r_2 = \dfrac{2u(v-1)}{c^2-1}$, $k_{12} = \dfrac{c(c-1)}{2}$, $k_{22} = \dfrac{c(c+1)}{2}$, $\lambda_{12} = u$ $\lambda_{22} = u$, $c = 2,3,\ldots,$ $s,u = 1,2,\ldots,$ $v > c^2$, implies the existence of the optimum chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11).

Proof. It is easy to prove that the parameters of the balanced bipartite weighing designs satisfy the condition (13) for a given a.

Theorem 5 For a given $a = 0,25$ the balanced bipartite weighing designs with the parameters $v = 13,\ b_1 = 78,\ r_1 = 48,\ k_{11} = 3,\ k_{21} = 5,\ \lambda_{11} = 15,\ \lambda_{21} = 13$ and $v = 13,\ b_2 = 39,\ r_2 = 15,\ k_{12} = 1,\ k_{22} = 4,\ \lambda_{11} = 2,\ \lambda_{22} = 3$ give the optimum chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11).

Theorem 6 For a given $a = 0,5$ the balanced bipartite weighing designs with the parameters

(i) $v = 13,\ b_1 = 78,\ r_1 = 36,\ k_{11} = 2,\ k_{21} = 4,\ \lambda_{11} = 8,\ \lambda_{21} = 7$ and $v = 13,\ b_2 = 78,\ r_2 = 42,\ k_{12} = 2,\ k_{22} = 5,\ \lambda_{12} = 10,\ \lambda_{22} = 11$,

(ii) $v = 13,\ b_1 = 78,\ r_1 = 42,\ k_{11} = 2,\ k_{21} = 5,\ \lambda_{11} = 10,\ \lambda_{21} = 11$ and $v = 13,\ b_2 = 78,\ r_2 = 36,\ k_{12} = 2,\ k_{22} = 4,\ \lambda_{12} = 8,\ \lambda_{22} = 7$,

(iii) $v = 17,\ b_1 = 68,\ r_1 = 20,\ k_{11} = 1,\ k_{21} = 4,\ \lambda_{11} = 2,\ \lambda_{21} = 3$ and $v = 17,\ b_2 = 136,\ r_2 = 48,\ k_{12} = 2,\ k_{22} = 4,\ \lambda_{12} = 8,\ \lambda_{22} = 7$,

(iv) $v = 21,\ b_1 = 42,\ r_1 = 12,\ k_{11} = 1,\ k_{21} = 5,\ \lambda_{11} = 1,\ \lambda_{21} = 2$ and $v = 21,\ b_2 = 210,\ r_2 = 30,\ k_{12} = 1,\ k_{22} = 2,\ \lambda_{12} = 2,\ \lambda_{22} = 1$,

give the optimum chemical balance weighing design with the design matrix $\mathbf{T}$ in the form (10) and with the dispersion matrix of errors $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of the form (11).

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UKŁADY OPTYMALNE DLA $p + 1$ OBIEKTÓW W OPARCIU O UKŁADY OPTYMALNE DLA p OBIEKTÓW

W pracy omawiane jest zagadnienie optymalnej estymacji nieznanych miar obiektów przy wykorzystaniu operacji pomiarowych w modelu chemicznego układu wagowego. Podane zostały relacje wymagane, aby istnienie optymalnego chemicznego układu wagowego dla p obiektów implikowało istnienie optymalnego chemicznego układu wagowego dla $p + 1$ obiektów. W modelu liniowym zakłada się, że błędy pomiarów są nieskorelowane i mają różne wariancje. Do konstrukcji macierzy układu optymalnego wykorzystuje się macierze incydencji dwudzielnych układów bloków.