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REMARKS ON THE GENERALIZED PROBABILITY OF THE BIFUZZY EVENT

Abstract. The presentation is a continuation of a paper at MSA'04 (T. Gerstenkorn, J. Gerstenkorn (2007)). In 1978 Ph. Smets proposed the so-called *g*-probability of a fuzzy event as a generalization of the L. Zadeh's probability of 1968. In 1980 S. Heilpern also discussed *g*-probability and analysed its properties. In 1992 Ph. Smets discussed once again the same his own problem and demonstrated its axiomatic properties. In this elaboration we desire to discuss the *g*-probability of the bifuzzy (intuitionistic) event and its properties as consistent with Kolmogoroff axiomatics.
Key words: bifuzzy (intuitionistic) event, generalized-probability, fuzzy set.

I. INTRODUCTION

In 1965 L. Zadeh introduced the notion of a fuzzy set as a generalization of the Cantor's set that was dominated to this moment in science. Conception of the fuzzy set allowed the mathematical modelling of not sharp formulated notions, usable very often in the so-called soft-sciences, as e.g. economy, humanistics, law, medicine. The characteristics of the fuzzy set comes after the introducing the so-called membership function of an element x of a considered space X to a fuzzy set A, defined as follows:

$$A = \{(x, \mu_A(x)) / x : x \in X\}, \tag{1}$$

where $X \to [0,1]$, i.e. $\mu_A(x)$ is a characteristic function of the set A, with this difference to the characteristic function $\varphi_A(x)$ of the Cantor's set A that it can take all the values of the interval [0, 1] and not only the values 0 or 1.

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Already in 1968 L. Zadeh introduced also and discussed the idea of probability of the fuzzy event in relation to the fuzzy set. He has brought out the difference between the fuzziness and accidentalness.

Assuming a probability space $(X, \mathcal{A}, P)$, the probability of the fuzzy event $A \in \mathcal{A}$ has been defined by

$$P(A) = \int_X \mu_A(x) P(dx), \tag{2}$$

where the meaning of $\mu_A(x)$ and of the space X is as above and $\mathcal{A}$ is a σ-Algebra.

One can easy see that the membership function $\mu_A(x)$ has replaced in (2) the characteristic function $\varphi_A(x)$ of the normal (crisp) set A.

The considerations relating to the probability of the fuzzy event were continued in following years by many authors. A review of these problems one can find in papers of T. Gerstenkorn and J. Mańko (1994, 1996).

The idea of the fuzzy set has been developed, extended and generalized for the so-called *bifuzzy set*, called by K. Atanassov (1983, 1985, 1986) the *intuitionistic set*.

The generalization consisted in introducing to considerations, besides the membership function, also the so-called non-membership function of the element x to the set A, i.e. one has proposed the following definition of that set:

$$A = \{(\mu_A(x), \nu_A(x)) / x : \ x \in X\}, \tag{3}$$

where $\mu_A(x)$, $\nu_A(x) : X \to [0,1]$ with the condition

$$0 \le \mu_A(x) + \nu_A(x) \le 1 \ \text{ for } x \in X \, . \tag{4}$$

The above statement assumes the existence of the function

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x) \tag{5}$$

called the *intuitionistic index* and the number $\pi_A(x) \in [0,1]$ is treated as a measure of the hesitancy (indecision) connected with a valuation of the degree of membership or non-membership of the element x to the set A. Examples of such interpretation and procedure one can find, e.g. in T. Gerstenkorn and J. Mańko (2006a, 2006b).

Alike as for the fuzzy set, the probability of the intuitionistic fuzzy set has been introduced. The review of some standpoints in this question was presented, e.g. in T. Gerstenkorn and J. Gerstenkorn (2007) and in papers of T. Gerstenkorn and J. Mańko (1995–2006).

II. THE GENERALIZED PROBABILITY OF THE FUZZY EVENT

In 1978 Philippe Smets proposed the so-called *g*-probability of the fuzzy event as a generalization of the probability of that event given by L. Zadeh (1968, (5), p. 423). The definition is the following:

$$P(A) = \int_X g(\mu_A(x))dF(x), \tag{6}$$

where g is a monotonic and non-decreasing function with conditions: $g(0) = 0,\ g(1) = 1$ and $F(x)$ is a distribution function in the probability space $(X, \mathcal{A}, P)$.

But two years later Stanisław Heilpern considered also the *g*-probability and analysed inquiringly its properties.

In 1982 Ph. Smets came back to his considerations with the *g*-probability but in this case he presented the axiomatic grounds of its correctness and concordance with the axiomatics of Kolmogoroff.

III. THE GENERALIZED PROBABILITY OF THE BIFUZZY EVENT

Definition. Let X be any set and $\mathcal{A}$ a σ-Algebra of its bifuzzy sets in X. Then by *g*-probability of the bifuzzy event $A \in \mathcal{A}$ we call a non-negative function $\tilde{P}$ determined on A with the values on [0,1], as follows:

$$\tilde{P}(A) = \int g(\mu_A(x), \nu_A(x))P(dx), \tag{7}$$

where g is a monotonic, non-decreasing function with conditions: $g(0,0) = 0$, $g(1,1) = 1$ and P is a probabilistic measure on X.

Similarly as it was done by Ph. Smets, we show that the function (7) fulfils the Kolmogoroff's axioms.

Axiom 1: $0 \le \tilde{P}(A) \le 1$.

The fulfilment of this condition is evident in view of the postulated conditions for $\mu(x)$ and $\nu(x)$.

Axiom 2: $\tilde{P}(X)=1$.

It suffices to notice that $\mu_x(x)=1$ and $\nu_x(x)=0$ and to take (7).

Axiom 3: Let $A\subseteq X$ and $B\subseteq X$ and $A\cap B=\{\emptyset\}$, then $\tilde{P}(A\cup B)=\tilde{P}(A)+\tilde{P}(B)$

(the events A and B are bifuzzy ones and being excluded.).

We assume that:

$$\mu_{A\cap B}(x)=\mu_A(x)\wedge\mu_B(x) \text{ - the operation minimum,} \tag{8}$$

$$\mu_{A\cup B}(x)=\mu_A(x)\vee\mu_B(x) \text{ - the operation maximum.} \tag{9}$$

Proof. The bifuzzy events A and B are excluded by assumption therefore the sets A and B are disjoint. It means that $\forall x\in X$ the membership function $\mu_{A\cap B}(x)=0$, i.e. $\mu_A(x)\wedge\mu_B(x)=0$, whereas the non-membership function $\nu_{A\cap B}(x)$ has the value 1 for the product of the bifuzzy events A and B. If we assume that the element $x\notin A$ then we form the set $X_A=\{x:\mu_A(x)=0$ and $\nu_A(x)=1\}$. But if we take $x\notin B$ then we set up the set $X_B=\{x:\mu_B(x)=0$ and $\nu_B(x)=1\}$. Let us notice that in account of $\forall x\in X_A\cap X_B$ it will be $\mu_{A\cup B}(x)=0$ and at the same time $\nu_{A\cup B}(x)=1$. Then also $g(\mu_{A\cup B}(x),\nu_{A\cup B}(x))=0$. Next $\forall x\in X_A$ there is $\mu_{A\cup B}(x)=\mu_B(x)$ and $\forall x\in X_B$ there is $\mu_{A\cup B}(x)=\mu_A(x)$ and similarly for $\nu_{A\cup B}(x)$ (it is evident regarding (9)).

Therefore

$$\tilde{P}(A)=\int_X g(\mu_A(x),\nu_A(x))P(dx)=\int_{X_B} g(\mu_A(x),\nu_A(x))P(dx)=$$

$$=\int g(\mu_{A\cup B}(x),\nu_{A\cup B}(x))P(dx),$$

($\mu_A(x)>0$ only if $\mu_B(x)=0$, i.e. if $x\in X_B$).

Similarly

$$\tilde{P}(B)=\int_{X_A} g(\mu_{A\cup B}(x),\nu_{A\cup B}(x))P(dx).$$

Therefore

$$\tilde{P}(A \cup B) = \left\{ \int_{X_A} + \int_{X_B} \right\} g(\mu_{A\cup B}(x), \nu_{A\cup B}(x)) P(dx) = \tilde{P}(A) + \tilde{P}(B).$$

IV. THE GENERALIZED CONDITIONAL PROBABILITY OF THE BIFUZZY EVENT

If there are given two bifuzzy events *A* and *B* defined on *X* then we present the *g*-probability of the event *A* under the given event *B* traditionally , i.e.

$$\tilde{P}(A \mid B) = \frac{\tilde{P}(A \cap B)}{\tilde{P}(B)}, \tag{10}$$

assuming the foregoing conditions on *g* and $\tilde{P}(B) > 0$.

Proof. It is sufficient to notice that the probability $\tilde{P}(B)$ fulfils the axioms of Kolmogoroff, as it was shown above and in the domain of bifuzzy sets, we have

$$A \cap B = \{(\mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x)) / x : \ x \in X\}.$$

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UWAGI O UOGÓLNIONYM PRAWDOPODOBIEŃSTWIE ZDARZENIA DWOISTOROZMYTEGO

Niniejsza prezentacja jest kontynuacją pracy pt. *Probability of fuzzy event. Review of problems* (Prawdopodobieństwo zdarzenia rozmytego. Przegląd zagadnień), przedstawionej na WAS'05 Acta Univ. Lodz., Folia Oeconomica 2007.

W 1978 r. Philippe Smets zaproponował tzw. *g*-prawdopodobieństwo zdarzenia rozmytego jako pewne uogólnienie prawdopodobieństwa tegoż zdarzenia podanego przez Lotfi Zadeha w 1968 r.

W 1980 r. Stanisław Heilpern także rozważał *g*-prawdopodobieństwo i analizował jego własności.

W 1982 r. Ph. Smets ponownie i szeroko rozpatrywał *g*-prawdopodobieństwo i dowodził jego aksjomatycznych własności.

W przedstawianym opracowaniu pragniemy rozpatrzyć *g*-prawdopodobieństwo zdarzenia dwoistorozmytego (intuicjonistycznego) i jego własności jako zgodne z aksjomatyką Kołmogorowa.