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A MERGER OF PENSION FUNDS – A STOCHASTIC MODEL[1]

ABSTRACT. In Polish law there exists a definition of the average rate of return of a group of pension funds which, as it was proved by Gajek and Kałuszka (2000), does not satisfy some economic postulates. These authors proposed another definition of the average rate of return. In this paper we consider the problem of a merger of pension funds taking into consideration both measures. We will show that relations between the presented definitions can be different in the case of a merger of any funds.

Key words: average rate of return of a group of pension funds, martingale.

I. INTRODUCTION

Open Pension Funds are institutions which should invest their clients' money in the most effective way. There are lots of measures for the efficiency of these investments. The measures should be well constructed – it means that all changes of fund's assets, connected with any investment, should influence the given measure. It is very important to calculate the average rate of return of a group of pension funds. Firstly, having this result we can compare any fund with the group. The *good* fund should be more effective than, on average, the group. But, first of all, in the Polish law regulations (The Law on Organization and Operation of Pension Funds, Art. 173, Dziennik Ustaw Nr 139 poz. 934, Art 173; for the English translation see *Polish Pension…*, 1997) the definition of the average return of a group of funds determines a minimal rate for any fund. In the case of deficit it is possible that this *weak* fund will have to cover it. It is always a very dangerous situation for funds. In the Polish law the following definition of the average return of a group of n pension funds can be found:

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$$\bar{r}_0(T_1,T_2)=\sum_{i=1}^{n}\frac{1}{2}r_i(T_1,T_2)\cdot\left(\frac{A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_i(T_2)}{\sum_{i=1}^{n}A_i(T_2)}\right), \tag{1}$$

where by $r_i(T_1,T_2)$ we denote the rate of the $i-$th fund during a given time period $[T_1,T_2]$ and by $A_i(t)$ we denote the value of $i-$th fund's assets at time t. After the year 2004 the results of funds for the last 36 months are verified once on half year, it means $[T_1,T_2]=[1,36]$.

II. ALTERNATIVE MEASURE FOR THE AVERAGE RATE OF RETURN

In the paper of Gajek and Kałuszka (2000) the authors showed that the definition (1) does not satisfy a group of economic postulates. For example, it is easy to show that in the case, when the number of units is constant at every fund during the time interval $[T_1,T_2]$, then

$$\bar{r}_0(T_1,T_2)\neq\frac{\sum_{i=1}^{n}A_i(T_2)-\sum_{i=1}^{n}A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}. \tag{2}$$

When none of the clients change the found or come into or out of the business, then any change of the assets A_i should reflect only the investment results of the $i-$th fund. But the conclusion from (2) is opposite. Moreover, considering an even number of funds, where half of them have the return rates equal to 50% and the rest of funds have the return rates equal to (–50%), we should get the real average return rate on the level 0%. But using formula (1) we get 12.5 %. The larger the differences between r_i, the more stranger the values produced by $\bar{r}_0$ (see Białek (2005)). That is the reason for construction an alternative definition of the average rate of return of a group of pension funds. Let us consider a group of n pension funds which start their activity selling accounting units at the same price. We observe them in discrete time moments $t=0,1,2,\ldots$.

Let us define a probability space (Ω, F, ℙ). Let 𝔽 = $\{F_0, F_1, ...\}$ be a filtration, i.e. each F_t is an б-algebra of Ω with $F_0 \subseteq F_1 \subseteq F_2 \subseteq ... \subseteq F$. Without loss of generality, we assume $F_0 = \{\emptyset, \Omega\}$. The filtration 𝔽 describes how information is revealed to the investor.

We consider the following random variables (for given t):

$w_i(t)$ – value of participation unit of the i – th fund at time t,

$k_i(t)$ – number of units of the i – th fund at time t,

$A_i(t) = k_i(t)w_i(t)$ – value of i – th fund's assets at time t,

$A(t) = \sum_{i=1}^{n} A_i(t)$,

$A_i^*(t) = A_i(t) / A(t)$.

Here and subsequently, the symbol $X = Y$ means that the random variables X, Y are defined on (Ω, F, ℙ) and ℙ $(X = Y) = 1$. We assume that each $w_i(t)$ and $k_i(t)$ is adapted to 𝔽, which means that each $w_i(t)$ and $k_i(t)$ is measurable with respect to F_t. Under the above assumptions and significations Gajek and Kałuszka (2002) proposed the following definition of the average rate of return of a group of funds:

$$\bar{r}(T_1, T_2) = \prod_{u=T_1}^{T_2-1}(1 + \sum_{i=1}^{n} A_i^*(u) r_i(u, u+1)) - 1, \tag{3}$$

where

$$r_i(u, u+1) = \frac{r_i(u+1) - r_i(u)}{r_i(u)}. \tag{4}$$

The definition (3) satisfies all economic postulates (see Gajek, Kałuszka (2002)). In the mentioned paper the authors proved the following theorems:

Theorem 1

With the probability one we have

$$\bar{r}(t, t+1) \le \bar{r}_0(t, t+1) \tag{5}$$

and in the natural case of

$$\exists i,j \ \frac{w_i(t+1)}{w_i(t)} \neq \frac{w_j(t+1)}{w_j(t)}, \tag{6}$$

then we obtain

$$\bar{r}(t,t+1) < \bar{r}_0(t,t+1). \tag{7}$$

The inequality (7) suggests that the average return defined in Polish law overestimates the real average rate of return of a group of funds.

Theorem 2

If $\{w_i(t): t = 0,1,2,...\}$ is an $\mathbb{F}$–martingale for each i, then $\{\bar{r}(0,t): t = 0,1,2,...\}$ is also an $\mathbb{F}$–martingale (see Wentzell (1980), Domański, Pruska (2000)). Moreover, in the case of $\{w_i(t): t = 0,1,2,...\}$ is an $\mathbb{F}$–submartingale (resp. $\mathbb{F}$–supermartingale) for each i, then $\{\bar{r}(0,t): t = 0,1,2,...\}$ is an $\mathbb{F}$–submartingale (resp. $\mathbb{F}$–supermartingale).

Remark. The average rate of return defined in Polish law in general is not a martingale provided the values of units are martingale (see Gajek and Kałuszka (2002)).

III. THE PROBLEM OF A MERGER OF FUNDS

We have observed lots of mergers of pension funds since 1999 (the beginning of Open Pension Funds in Poland). We can mention for example the following mergers: 2000 (Norwich Union, Sampo), 2001 (OFE Pocztylion, Arka Invesco OFE), 2002 (Zurich, Generali), etc. The fund, which takes over the other fund, gains its assets and recalculates the gained units according to the value of own units at the moment of merger (see *Dziennik Ustaw nr 139, Rozdz. 5, Art. 71*). The natural question is how to modify the definition of the average return for the case of merger? It is not so obvious from the point of view of the definition (1) because (in the case of merger of any funds during the considered time interval $[T_1, T_2]$), we have the different number of funds at the moments T_1 and T_2. In this paper we propose modifications of both measures in the case of merger. We are going to verify the relations between the modified measures.

MODIFICATION OF POLISH DEFINITION

Suppose there exists n pension funds at time $t = 0,1,2....,\tau$. At time τ the $(n-1)$ – th fund takes over the n – th fund, so both funds form a new fund, say $(n-1)$ – th. The rate of return of
this new fund for time interval $[T_1, \tau)$ can be calculated as

$$r_{n-1}(T_1,\tau) = \prod_{t=T_1}^{\tau-1} \left(\frac{A_{n-1}(t)w_{n-1}(t+1)}{A_{n-1,n}(t)w_{n-1}(t)} + \frac{A_n(t)w_n(t+1)}{A_{n-1,n}(t)w_n(t)}\right) - 1, \tag{8}$$

where, according to *Dziennik Ustaw nr 139, Rozdz. 5, Art. 71,* we have

$$A_{n-1,n}(t) = k_{n-1}(t)w_{n-1}(t) + k_n(t)w_n(t). \tag{9}$$

The rate of return of the new fund for the time interval $[\tau, T]$ can be calculated as

$$r_{n-1}(\tau, T_2) = \frac{w_{n-1}(T_2)}{w_{n-1}(\tau)} - 1. \tag{10}$$

Using the known property of the rate of return r_i:

$$r_i(s,t) + 1 = (r_i(s,u) + 1)\cdot(r_i(u,t) + 1), \text{ for } s < u < t \tag{11}$$

from (9) and (10), under natural assumption (11) for the new fund, we get

$$r_{n-1}(T_1, T_2) = (r_{n-1}(T_1,\tau) + 1)\cdot(r_{n-1}(\tau, T_2) + 1) - 1. \tag{12}$$

After the merger, at time $\tau +$, the assets of the new fund equal to

$$A_{n-1}(\tau +) = k_{n-1}(\tau)w_{n-1}(\tau) + k_n(\tau)w_n(\tau). \tag{13}$$

After the moment τ we observe $n-1$ funds and according to *Dziennik Ustaw nr 139, Rozdz 5, Art. 71* the new fund recalculates the number of units after the merger as follows:

$$k_{n-1}(\tau+)=\frac{A_{n-1}(\tau+)}{w_{n-1}(\tau)}. \tag{14}$$

Finally, the modified average rate of return of the group of funds can be written as follows:

$$\begin{aligned}\bar{r}_0(T_1,T_2)&=\sum_{i=1}^{n-2}\frac{1}{2}r_i(T_1,T_2)\cdot\left(\frac{A_i(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_i(T_2)}{\sum_{i=1}^{n-1}A_i(T_2)}\right)+\\&+\frac{1}{2}r_{n-1}(T_1,T_2)\cdot\left(\frac{A_{n-1}(T_1)+A_n(T_1)}{\sum_{i=1}^{n}A_i(T_1)}+\frac{A_{n-1}(T_2)}{\sum_{i=1}^{n-1}A_i(T_2)}\right),\end{aligned} \tag{15}$$

where $r_{n-1}(T_1,T_2)$ is specified in (12).

MODIFICATION OF GAJEK-KAŁUSZKA DEFINITION

The definition, coming from Gajek and Kałuszka (2002), takes into consideration all moments

$T_1,\ T_1+1,\ \dots,\ T_2$. The specific construction of (3) makes this definition easy to modify. Using the above significations and assumptions we propose the following modification:

$$\bar{r}(T_1,T_2)=\prod_{t=T_1}^{\tau-1}\left(\sum_{i=1}^{n}A_i^*(t)\frac{w_i(t+1)}{w_i(t)}\right)\cdot\left(\sum_{i=1}^{n-2}A_i^*(\tau)\frac{w_i(\tau+1)}{w_i(\tau)}+A_{n-1}^*(\tau+)\frac{w_{n-1}(\tau+1)}{w_{n-1}(\tau+)}\right)\cdot$$

$$\cdot\prod_{t=\tau+1}^{T_2-1}\left(\sum_{i=1}^{n-1}A_i^*(t)\frac{w_i(t+1)}{w_i(t)}\right)-1, \tag{16}$$

where

$$A_{n-1}^*(\tau+)=\frac{A_{n-1}(\tau+)}{\sum_{i=}^{n-2}A_i(\tau)+A_{n-1}(\tau+)}. \tag{17}$$

We can separate the part connected with the *normal* time interval and after the merger.

IV. EMPIRICAL RESULTS

We consider the period November 30, 2005 – January 30, 2006 (three months) for Polish pension funds. In case of Poland we have $n = 15$, $t = 1,2,3$. For this period of time the list of Open Pension Funds with regard to value of assets was as follows:

Table 1

Open Pension Funds in Poland with regard to value of assets, Nov 30, 2005 – Jan 30, 2006

Pension fund	Net assets Nov 2005 (mln PLN)	Net assets Dec 2005 (mln PLN)	Net assets Jan 2006 (mln PLN)
AIG	6 959.12	7 300.77	7 577.76
Allianz	2 118.50	2 200.32	2 281.25
Bankowy	2 619.13	2 749.34	2 851.42
CU	22 535.90	23 457.70	24 259.30
DOM	1 284.23	1 341.95	1 412.99
Ergo Hestia	1 860.22	2 010.79	2 090.69
Generali	2 779.16	3 230.40	3 049.88
ING NN	18 846.00	19 655.10	20 463.80
PeKaO	1 287.20	1 351.02	1 427.68
Pocztylion	1 676.52	1 705.21	1 740.68
Polsat	716.47	751.60	791.36
PZU Złota Jesień	11 261.80	11 757.10	12214
Sampo	2 976.50	3 141.07	3 256.46
Skarbiec Emerytura	2 399.34	2 456.72	2 560.06
Winterthur	2 966.46	3 230.20	3 356.99

Source: www.money.pl.

The values of units of funds were as follows:

Table 2

Open Pension Funds in Poland with regard to value of units, Nov 30, 2005 – Jan 30, 2006

Pension fund	Value of unit 30 XI 2005 (PLN)	Value of unit 30 XII 2005 (PLN)	Value of unit 30 I 2006 (PLN)
1	2	3	4
AIG	21.20	21.71	22.35
Allianz	20.74	21.11	21.70
Bankowy	22.09	22.58	23.21
CU	22.53	23.04	23.63
DOM	22.96	23.54	24.60
Ergo Hestia	22.15	22.59	23.27
Generali	22.68	23.27	24.00

Table 2 (cont.)

1	2	3	4
ING NN	24.10	24.62	25.45
PeKaO	20.41	21.00	22.01
Pocztylion	20.70	21.21	21.91
Polsat	23.77	24.44	25.52
PZU Złota Jesień	22.39	22.85	23.55
Sampo	23.05	23.50	24.15
Skarbiec Emerytura	20.93	21.44	22.17
Winterthur	21.98	22.44	23.10

Source: www.money.pl.

The formulas (15) and (16), for the considered situation, lead to:

$$\bar{r}_0(1,3)=\sum_{i=1}^{13}\frac{1}{2}r_i(1,3)\Big(\frac{A_i(1)}{\sum_{i=1}^{15}A_i(1)}+\frac{A_i(3)}{\sum_{i=1}^{14}A_i(3)}\Big)+\frac{1}{2}r_{14}(1,3)\Big(\frac{A_{14}(1)+A_{15}(1)}{\sum_{i=1}^{15}A_i(1)}+\frac{A_{14}(3)}{\sum_{i=1}^{14}A_i(3)}\Big), \tag{18}$$

$$\bar{r}(1,3)=\Big(\sum_{i=1}^{15}A_i^*(1)\frac{w_i(2)}{w_i(1)}\Big)\cdot\Big(\sum_{i=1}^{13}A_i^*(2)\frac{w_i(3)}{w_i(2)}+A_{14}^*(2+)\frac{w_{14}(3)}{w_{14}(2+)}\Big)-1, \tag{19}$$

where

$$A_{14}(2+)=k_{14}(2)w_{14}(2)+k_{15}(2)w_{15}(2), \tag{20}$$

$$A_{14}^*(2+)=\frac{A_{14}(2+)}{A_{14}(2+)+\sum_{i=1}^{13}A_i(2)}, \tag{21}$$

$$k_{14}(2+)=\frac{A_{14}(2+)}{w_{14}(2)}, \tag{22}$$

$$w_{14}(2+)=w_{14}(2). \tag{23}$$

If we did not observe any merger of funds we would get the following values:

$$\bar{r}_0(1,3) = 5.33\%, \quad \bar{r}(1,3) = 5.31\%$$

so the thesis (7) from theorem 1 is verified, it means $\bar{r}_0(1,3) > \bar{r}(1,3)$.

Now let us consider the situation when the ING NN takes over AIG (the merger of the most powerful funds – just the hypothetical situation). Let us assume that values of units of these funds decreased by about $s\%$ at time $t = 2$. Manipulating the parameter s we get:

Table 3

The average rates of return of the group of Polish pension funds for period Nov 30, 2005 – Jan 30, 2006

measure	$s = 10\%$	$s = 15\%$	$s = 205$
$\bar{r}_0(1,3)$	5.37	5.37	5.37
$\bar{r}(1,3)$	5.43	5.53	5.67

Source: own calculation based on table 1 and table 2.

CONCLUSIONS

The larger the value of parameter s, the larger the difference between definitions.

What is more interesting – in the case of a merger of funds the relation (7) from thesis of the theorem 1 can be the opposite. Having the results from table 3 we can see that it is possible to get the relation: $\bar{r}_0 < \bar{r}$. But we should remember that we treat the average returns $\bar{r}_0, \bar{r}$ according to (15) and (16).

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PROBLEM FUZJI FUNDUSZY EMERYTALNYCH – MODEL STOCHASTYCZNY

W polskim prawie funkcjonuje definicja przeciętnej rentowności grupy funduszy emerytalnych, która – jak pokazali Gajek i Kałuszka (2000) – nie spełnia pewnych ekonomicznie zasadnych postulatów. Jednocześnie zaproponowali oni nową miarę dla przeciętnego zwrotu grupy funduszy. W niniejszym artykule omówiony zostaje problem fuzji funduszy emerytalnych z punktu widzenia tych różnych miar. Okaże się, że relacje zachodzące pomiędzy miarami są inne w przypadku, gdy dochodzi do przejęcia któregoś z funduszy.