ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 216, 2008

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PORTFOLIO CONSTRUCTION WITH MODIFIED SHARPE'S METHOD

ABSTRACT. In the paper we consider a modification of Sharpe's method used in classical portfolio analysis for optimal portfolio building.

The key idea of the paper is the modification of the classical approach by application of the errors-in-variable model. We assume that both independent (market portfolio return) as well as dependent (given asset's return) variables are randomly distributed values related with each other by linear relationship and we build the model used for parameters' estimation.

For model evaluation we made a comparison of portfolios comprising nine stocks from Warsaw Stock Exchange, which are built using classical Sharpe's and proposed method.

Key words: Sharpe's model, errors-in-variable model, estimation, comparison.

I. CLASSIC SHARPE'S MODEL

In classic Sharpe's model return of *k-th* asset's R^k is explained by market portfolio return R^m trough characteristic line equation according to the relationship:

$R^k = \beta_k(R^m - r_f) + r_f + \varepsilon_k$, where r_f is riskless rate of interest and ε_k has normal distribution. Knowing the market sensitivity β_k of *k-th* asset we receive the formula for the expected value and the variance of portfolio return ratio R_p :

$$E(R_p) = \beta(E(R^m) - r_f) + r_f, \quad D^2(R_p) = \beta^2 D^2(R^m) + \sum_{k=1}^{k=K} x_k^2 D^2(\varepsilon_k), \qquad (1)$$

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where $\beta = \sum_{k=1}^{k=K} x_k \beta_k$; x_k - k-th stock's share in investment portfolio. The problem of optimal portfolio choice may be reducted to finding $\max\left[E(R_p) - \lambda D^2(R_p)\right]$ with condition: $x_k \geq 0$, $\sum_{k=1}^{k=K} x_k = 1$, where λ – risk/profit exchange ratio coefficient. In this model the β_k coefficient is estimated by least square method.

II. MODIFIED SHARPE'S MODEL

Let us consider the situation, in which both: dependent variable, being the certain asset's surplus return ratio and denoted R^k as well as independent variable – the market portfolio surplus return ratio R^m are disturbed observations related with each other by linear equation, i.e.:

$$R^m - r_f = \xi + \varepsilon; \qquad R^k - r_f = \beta_k \xi + \alpha_k + \delta_k, \tag{2}$$

If we assume that random variables ε, δ_k have normal distribution with unknown parameters then model (2) is unidentifable. The proof was given by Reiersol(1950).

In that case, replication of measurement of each pair of observation: dependent and independent; m times overcomes the nonidentifiability (see Bunke & Bunke(1989)).

The following approach was therefore proposed: returns of given asset and market portfolio are analysed within periods of t. To be more specific the period of one month has been assumed as t.

Let n denotes number of historical months, m number of monthly returns during given month and K number of assets in portfolio. Let R_{ij}^m and R_{ij}^k are *j-th* monthly return ratios in the *i-th* month of portfolio market and *k–th* asset, respectively and $X_{ij} = R_{ij}^m - r_f$, $Y_{ij}^k = R_{ij}^k - r_f$.

We consider the model:

$$X_{ij} = \xi_{ij} + \varepsilon_{ij}, \quad Y_{ij}^k = \beta_k \xi_{ij} + \alpha_k + \delta_{ij}^k, \quad i = 1, \ldots, n, \; j = 1, \ldots, m, \; k = 1, \ldots, K, \tag{3}$$

It has been assumed that $\delta_{ij} \sim N(0, \sigma_{\delta_k}^2)$, $\xi_{ij} \sim N(s_i, \sigma_s^2)$, $\varepsilon_{ij} \sim N(0, \sigma_\varepsilon^2)$.

All the parameters of aforementioned distributions are unknown.

Now the issue of optimal portfolio choice is finding: $\max\left[E(R_p)-\lambda D^2(R_p)\right]$ under conditions: $x_k \geq 0$, $\sum_{k=1}^{k=K} x_k = 1$ with:

$$E(R_p)=\sum_{k=1}^{k=K} x_k(\beta_k s + r_f), \quad D^2(R_p)=(\sum_{k=1}^{k=K} x_k\beta_k)^2\sigma_s^2+\sum_{k=1}^{k=K} x_k^2\sigma_{\delta_k}^2. \qquad (4)$$

where s is the expected value for distribution of monthly returns of market portfolio during the period of the prognosis.

III. ESTIMATION OF UNKNOWN PARAMETERS

First step – Estimation of parameters for given asset

For given k in model (3) the number of unknown parameters to be estimated increases with n. Unknown parameters are: β_k, α_k, $\sigma_{\delta k}^2$, σ_s^2, σ_ε^2, $s_{1,\dots,}s_n$. It can be shown, that in this model the maximum likelihood estimators have normal distribution asymptotically with respect to m and n (see Dolby(1976)).

Let: $X_{i.}=\frac{1}{m}\sum_{j=1}^{m}X_{ij}$, $Y_{i.}^k=\frac{1}{m}\sum_{j=1}^{m}Y_{ij}^k$, $X_{..}=\frac{1}{n}\sum_{j=1}^{n}X_{i.}$, $Y_{..}^k=\frac{1}{n}\sum_{j=1}^{n}Y_{i.}^k$

and subsequently:

$$w_{xx}=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}(X_{ij}-X_{i.})^2 \qquad w_{yy}^k=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}(Y_{ij}^k-Y_{i.}^k)^2,$$

$$w_{xy}^k=\frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m}(X_{ij}-X_{i.})(Y_{ij}^k-Y_{i.}^k),$$

$$s_{xx}=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(X_{i.}-X_{..}\right)^2 \qquad s_{yy}^k=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}(Y_{i.}^k-Y_{..})^2,$$

$$s_{xy}^k=\frac{1}{n}\sum_{i=1}^{n}\sum_{j=1}^{m}(X_{i.}-X_{..})(Y_{i.}^k-Y_{..}),$$

the maximum likelihood estimator of β_k and α_k coefficients have forms:

$$\hat{\beta}_k = \frac{w_{yy}^k s_{xx} - w_{xx} s_{yy}^k \ - \sqrt{\Delta}}{2(w_{xy}^k \ s_{xx} - w_{xx} s_{xy}^k)} \quad \hat{\alpha}_k = Y_{ij}^k - \hat{\beta}_k X_{ij},$$

where $\Delta = (w_{xx} s_{yy}^k - s_{xx} w_{yy}^k \ )^2 - 4(s_{xy}^k \ w_{yy}^k - s_{yy}^k \ w_{xy}^k)(s_{xx} w_{xy}^k - s_{xy}^k w_{xx})$.

The estimators of all the other values are expressed in terms of $\hat{\beta}_k$ (Cox(1976)).These estimators have forms of:

$$\hat{\sigma}_{s_k}^2 = w_{xy}^k / \hat{\beta}_k - p_k q_k B(\hat{\beta}_k) / \hat{\beta}_k \qquad \hat{\sigma}_{\delta_k}^2 = q_k (W(\hat{\beta}_k) + B(\hat{\beta}_k))$$

where, $B(\hat{\beta}_k) = s_{yy}^k - 2\hat{\beta}_k s_{xy}^k + \hat{\beta}_k^2 s_{xx}$, $\quad W(\hat{\beta}_k) = w_{yy}^k - 2\hat{\beta}_k w_{xy}^k + \hat{\beta}_k^2 \ w_{xx}$ and $p_k = (\hat{\beta}_k w_{xx} - w_{xy}^k) / W(\hat{\beta}_k)$, $\qquad q_k = (w_{yy}^k - \hat{\beta}_k w_{xy}^k) / W(\hat{\beta}_k)$.

Second step - Estimation of unknown parameters for the whole model

For given j we construct vectors: x_j^T, y_j^{kT} as:

$$x_j^{\ T} = (X_{1j}, \ldots X_{nj}), \qquad y_j^{kT} = (Y^k{}_{1j}, \ldots Y^k{}_{nj}), \quad k = 1 \ldots K.$$

The random variables $\mu_j = (x_j^T, y_j{}^{1T}, y_j^{2T} \ldots y_j^{KT})$ are independent with the same normal distribution and the vector of expected values $\mu = (s^T, (\beta_1 s + \alpha_k)^T$, ..., $(\beta_K s + \alpha_k)^T)$ where: $s^T = (s_1, \ldots, s_n)$ and with covariance matrix **V**:

$$V = \begin{bmatrix} \sigma_s^2 + \sigma_\varepsilon^2 & \beta_1 \sigma_s^2 & \cdots & \beta_K \sigma_s^2 \\ \beta_1 \sigma_s^2 & \beta_1^2 \sigma_s^2 + \sigma_{\delta_1}^2 & \cdots & \beta_1 \beta_K \sigma_\varepsilon^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_K \sigma_s^2 & \beta_1 \beta_K \sigma_\varepsilon^2 & \ldots & \beta_K^2 \sigma_s^2 + \sigma_{\delta_K}^2 \end{bmatrix} \otimes I_n = \sum \otimes I_n \,.$$

The logarithm of likehood function for variable μ_j with unknown parameters of: $\beta_1, \ldots, \beta_k$, $\alpha_1, \ldots, \alpha_k$, σ_s^2, σ_ε^2, $\sigma_{\delta_1}^2, \ldots, \sigma_{\delta_K}^2$, $s_1, \ldots, s_n$, has a form of:

$$\log L(\mu_j, \Psi) = C - \frac{m}{2}\log(\det(V)) - \frac{1}{2}\sum_{j=1}^{m} d_j V^{-1} d_j,$$

where $d_j = (\mu_j - \mu)$ and Ψ is the set of unknown parameters.

Let V_ψ denote a matrix composed of matrix V elements after derivation with respect to ψ parameters (ψ being the arbitrary element of Ψ set). The **V** matrix is symmetric thus:

$$\frac{\partial}{\partial \psi}\log L(\mu_j, \Psi) = m\left\{\frac{1}{2}tr(PV_\psi) - d_\psi^T V^{-1} d\right\}, \text{ with } P = V^{-1}(D - V)V^{-1}$$

and $D = \frac{1}{m}\sum_{j=1}^{m} d_j d_j^T$.

It comes that: $d_{\alpha k}^T V^{-1} d = 0$, $\frac{1}{2}Tr(PV_{\beta_k}) - d_{\beta_k}^T V^{-1} d = 0$,

$d_{s_i}^T V^{-1} d = 0$, $Tr(PV_{\sigma_\varepsilon^2}) = 0$, $Tr(PV_{\sigma_{\delta_k}^2}) = 0$, $Tr(PV_{\sigma_s^2}) = 0$.

From equations $d_{\alpha k}^T V^{-1} d = 0$, $d_{s_i}^T V^{-1} d = 0$ we received relationships:

$\alpha_k = Y_{..}^k - \beta_k X_{..}$ and

$$s_i = \frac{X_{i.} + A_1(Y_{.}^1 - Y_{..}^1 + \beta_1 X_{..}) + \ldots + A_K(Y_{.}^K - Y_{..}^K + \beta_K X_{..})}{1 + A_1\beta_1 + \ldots A_K\beta_K} \tag{5}$$

where: $A_k = \beta_k \sigma_\varepsilon^2 / \sigma_{\delta_k}^2$.

From: $Tr(PV_{\sigma_\varepsilon^2}) = 0$, $Tr(PV_{\sigma_{\delta_k}^2}) = 0$, exploiting certain features of matrix algebra we obtain:

$$\sigma_\varepsilon^2 + \sigma_s^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{(X_{ij} - s_i)^2}{mn} \quad \sigma_{\delta_k}^2 + \beta_k^2\sigma_s^2 = \sum_{i=1}^{n}\sum_{j=1}^{m}\frac{(Y_{ij} - \beta_k s_i - \alpha_k)^2}{mn}. \tag{6}$$

From equations (5), (6) and relationship $Tr(PV_{\sigma_s^2}) = 0$ we calculated σ_ε^2:

$$\sigma_\varepsilon^2 = \sum_{k=1}^{K}\left(1 - \frac{T^k(\beta_k)}{\sigma_{\delta_k}^2}\right) \Bigg/ \left(\frac{1}{2}\sum_{i=1}^{K}\sum_{k=1}^{K}\frac{T^{kl}(\beta_k, \beta_l)}{\sigma_{\delta_k}^2\sigma_{\delta_l}^2} - K\sum_{k=1}^{K}\frac{\beta_k^2}{\sigma_{\delta_k}^2}\right), \tag{7}$$

with:

$$T^k(\beta_k) = (s_{yy}^k + w_{yy}^k) - 2\beta_k(s_{xy}^k + w_{xy}^k) + \beta_k^2(s_{xx} + w_{xx}),$$

$$T^{kl}(\beta_k, \beta_l) = \beta_k^2(s_{yy}^l + w_{yy}^l) - 2\beta_k\beta_l(s_{yy}^{kl} + w_{yy}^{kl}) + \beta_l^2(s_{yy}^k + w_{yy}^k),$$

$$w_{yy}^{kl} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m}(Y_{ij}^k - Y_{i.}^k)(Y_{ij}^l - Y_{i.}^l) \qquad s_{yy}^{kl} = \frac{1}{n}\sum_{i=1}^{n}(Y_{i.}^k - Y_{..}^k)(Y_{i.}^l - Y_{..}^l).$$

From relationships (6) eliminating σ_s^2 parameters , we received k of square equations with σ_ε^2 as variable. Applying the Viete's formulas for each of them we received the $\sigma_{\delta_k}^2$ in terms of $\sigma_{\delta_1}^2$ for $k=2,...,K$:

$$\sigma_{\delta_k}^2 = \frac{\beta_k^2}{\beta_1^2}\left(\sigma_{\delta_1}^2 - T^1(\beta_1) - 2\beta_1 w_{xy}^1\right) + T^k(\beta_k) + 2\beta_k w_{xy}^k. \qquad (8)$$

Substituting $\sigma_{\delta_k}^2$ calculated according to (8) into equation (7), we received relationship between σ_ε^2 and $\sigma_{\delta_1}^2$. The parameter σ_ε^2 obtained in such a way we substitute into one of k square equations, which gives a basis for calculating $\sigma_{\delta_1}^2$.

Ultimately the maximum of likehood function depends only on unknown $\beta_1,...,\beta_K$. The last may be determined numerically starting from initial values equal to their estimators calculated in the first step.

IV. AN EXAMPLE PORTFOLIO

An example portfolio consists of nine companies listed on the Warsaw Stock Exchange (debica, KGHM, krosno, orbis, PKNorlen, sokolow, TPSA, wolczanka, zywiec - to be more specific). As the market portfolio the portfolio being the base of WIG index was assumed. The analysis was performed on archival data starting from January 2000 until March 2006.

In practice in portfolio construction it is sufficient to take estimators calculated in the first step. Slight differences in σ_s^2 estimated for various assets have been corrected by averaging. For estimation of unknown value of s, $\sum_{i=1}^{n} s_i$ was assumed. It can be shown that: $\sum_{i=1}^{n} s_i = x_{..}$

To ilustrate the features of portfolio constructed with use of modified method we compared it with that constructed using classical method. It appears that both methods construct quite different contents. Moreover for lambda values, for which there is no significant further portfolio diversification classical method "prefers" (in decreasing order): zywiec, krosno, debica, PKNorlen, sokolow; while the modified one: PKNorlen, zywiec, TPSA, sokolow.
For illustration the expected value and the variance of the portfolio being rebuilt during 14 months have been presented.

Figures 1 and 2 present expected return ratio and its variance for both methods, respectively. It turned out that the expected value is greater than resulting from analysis of the classic model and the variance is lower. To depict the difference in constructing models we present the real monthly return ratio for optimal contents for both portfolios for chosen λ coefficient. The verification was done using past market price (exchange ratio) of the asset in the month, for which the prognosis was made. Picture 3 contains the comparison of real profits for both discussed methods. The WIG return Rm is also included as the background reference. It may be noticed that low-risk portfolio built using the proposed method gives per average profits greater than that of classic Sharpe's portfolio.

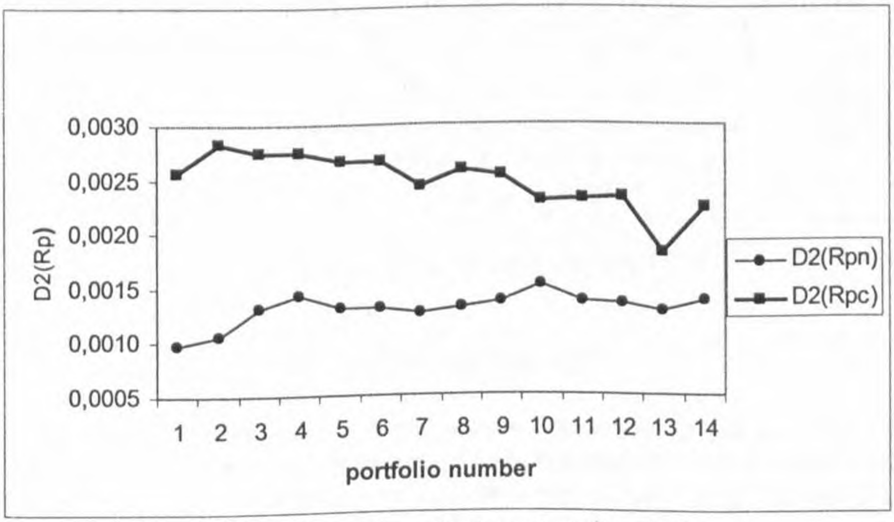

Fig. 1. Comparison of variance for both methods

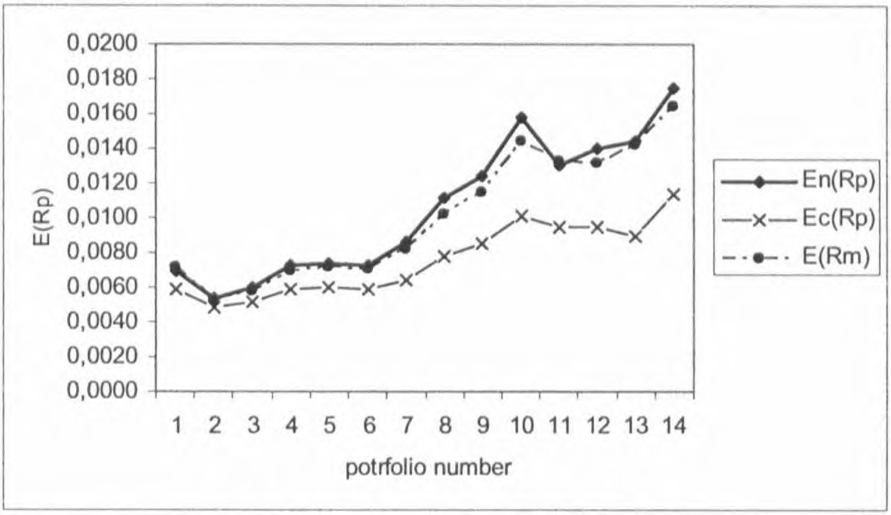

Fig. 2. Expected return ratio for both methods

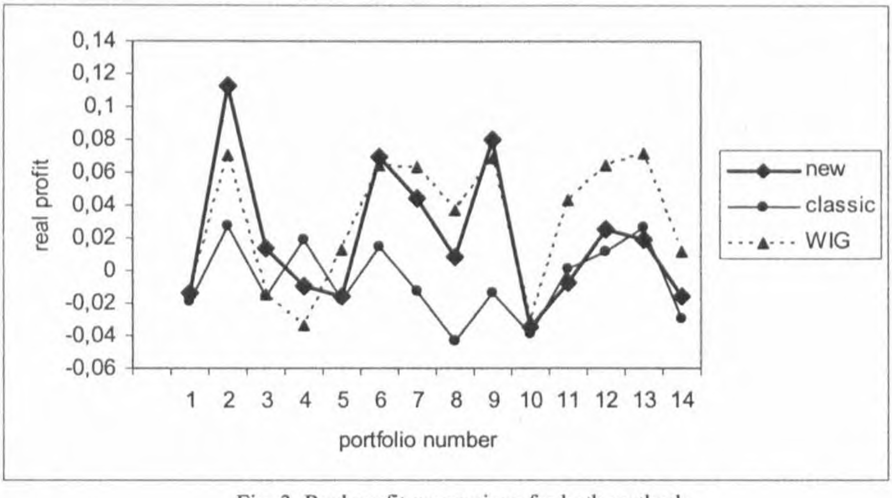

Fig. 3. Real profit comparison for both methods.

V. CONCLUSIONS

A new method of portfolio construction has been proposed. The main assumptions is that, the variable explaining the assets' return ratios is a random variable biased by a random error with normal distribution. The method of estimators' construction in errors-in-variable model extended for many variables has been shown. Comparing parameters of portfolios constructed with classic Sharpe's method and the proposed one, we found that expected returns for our method are bigger, while its variance is lower. In terms of real profits it means that for higher λ (portfolios' diversification is practically "saturated") are greater for the presented model.

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BUDOWA PORTFELA AKCJI ZA POMOCĄ ZMODYFIKOWANEJ METODY SHARPE'A

W klasycznej jednoczynnikowej analizie portfelowej konstruując optymalny portfel papierów wartościowych, wykorzystuje się model jego budowy zaproponowany przez Sharpe'a. Podstawą tej teorii jest założenie, że stopa zwrotu danego waloru jest objaśniana stopą zwrotu portfela rynkowego poprzez zależność liniową. Wiadomo, że na zmienność cen walorów wpływ mają również inne (często trudne do zmierzenia) czynniki rynku. W klasycznym podejściu parametry zależności pomiędzy stopą zwrotu danego waloru a stopą zwrotu portfela rynkowego wyznaczane są z modelu prostej regresji, gdzie zaburzenie losowe jest dopuszczane tylko na wartości zmiennej objaśnianej. W proponowanym w pracy modelu obecność tych czynników uwzględniona jest jako zaburzenie na obu zmiennych losowych wchodzących do klasycznego modelu Sharpe'a.

Przyjęto, że zarówno stopa zwrotu danego waloru jak i stopa zwrotu portfela rynkowego są pewnymi zaburzonymi już wartościami, między którymi istnieje zależność liniowa.

Dla ilustracji zagadnienia porównano portfele składające się z dziewięciu spółek zbudowane w oparciu o klasyczną metodę Sharpe'a i proponowaną jej modyfikację. Jako portfel rynkowy przyjęto portfel leżący u podstaw indeksu giełdowego WIG. Analizę przeprowadzono na podstawie notowań archiwalnych od stycznia 2000 do marca 2006. Budując wyżej wymienione portfele miesięczne dokonano porównań przebudowując je co miesiąc uzyskując w ten sposób wektor składów do analizy porównawczej.