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Jacek Białek[*], *Andrzej Czajkowski*[**]

A PROPOSITION OF THE SYSTEM OF WEIGHTS FOR AGGREGATIVE INDEXES ON THE EXAMPLE OF THE INDEX OF WORK EFFICIENCY[1]

ABSTRACT. In this paper we propose a construction of the aggregative index of work efficiency. The proposed system of weights is based on theoretical considerations over the situation in which the number of observations – coming from some of the considered enterprises – is insufficient. In the first part of this paper we consider a group of N – enterprises and two periods of their activity. We propose a construction of index to compare the periods taking into consideration the work efficiency. Next we consider the case when we intend to measure the average, one-period dynamics of the efficiency of work, having data from $T > 2$ periods. We construct a new index which is a more general version of the previous index.

Key words: aggregative index, arithmetic mean, harmonic mean, work efficiency.

I. INTRODUCTION

The contemporary economy makes use of lots of statistical indexes to calculate the dynamics of prices, quantities, and in particular, work efficiency. For example: Laspeyres and Paasche indexes have been known since 19-th century (see Diewert (1976), Shell (1998)). Depending on the type of an economic problem we may also use one of the following indexes: Fisher ideal index (see Fisher (1972)), Törnqvist index (Törnqvist (1936)), Lexis index and other indexes (see Zając (1994), Domański (2001)). Indexes are also used to calculate national income (see Moutlon, Seskin (1999), Seskin, Parker (1998)). Balk (1995) wrote about axiomatic price index theory, Diewert (1978) showed that the Törnqvist index and Fisher ideal index approximate each other. But it is really hard to indicate the best of the statistical indexes (see Dumagan (2002)). The choice of index

[*] Ph. D., Chair of Statistical Methods, University of Łódź.
[**] Ph. D., Chair of Statistical Methods, University of Łódź.
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depends on the information we want to get. Unfortunately, most of indexes take into account no event from the inside of the considered time interval. So if we want to consider also the omitted periods we should use a different formula.

A system of weights for the index of work efficiency (next we consider only this type of indexes) should satisfy all economic postulates (see Gajek, Kałuszka (2000)). But the construction of index, based on economic postulates, have to take into consideration the accidental noise of partial indexes. The partial indexes of work efficiency, based on small number of observations, can lead to wrong conclusions about the global work efficiency. In this paper the proposed system of weights is based on theoretical considerations over the situation in which the number of observations – coming from some of the considered enterprises – is insufficient. We are going to construct the aggregative index, which strongly limit the influence of partial indexes of work efficiency connected with the small number of observations. In the first part of this paper we consider a group of N – enterprises and two periods of their activity. Next, we consider the case when we want to measure the average, one-period dynamics of the work efficiency, having data from $T > 2$ periods.

II. CONSTRUCTION OF INDEX IN THE CASE OF TWO PERIODS

Let us consider a group of $N-$enterprises observed in discrete moments: s (base period) and t (testing period). Let us signify:

W_i^s – work efficiency of $i-$th enterprise at time s, $i \in \{1,2,...,N\}$,

W_i^t – work efficiency of $i-$th enterprise at time t, $i \in \{1,2,...,N\}$,

$I_j(s,t) \equiv I_j = \frac{W_j^t}{W_j^s}$ – partial index of work efficiency of $i-$th enterprise

comparing

periods s and t, where $j \in \{1,2,...,N\}$,

n_i^s – number of employees of $i-$th enterprise at time s,

n_i^t – number of employees of $i-$th enterprise at time t.

We are going to find the right vector of weights $\{g_1, g_2, ..., g_N\}$. Using the above significations we can write the index of work efficiency as follows:

$$\overline{I} = \frac{\sum_{i=1}^{N} g_i I_i}{\sum_{i=1}^{N} g_i}. \tag{1}$$

When we treat each I_i as a random variable on some probabilistic space (Ω, F, P) and each g_i as a real number, we must treat also $\bar{I}$ as a random variable. We are interested in the differences among the calculated, noised by small number of observations from some enterprises index $\bar{I}$ and its theoretical, expected value $\bar{I}_0$[2]. We will calculate the influence of accidental noise of partial indexes on the global index $\bar{I}$ as:

$$d\bar{I} = \bar{I} - \bar{I}_0 = \bar{I} - E\bar{I}. \tag{2}$$

so we are going to minimize the value of dispersion of random variable in (1):

$$\sigma_{\bar{I}}^2 = E(d\bar{I})^2. \tag{3}$$

Let us signify

$$\gamma_i = \frac{g_i}{\sum_{i=1}^{N} g_i}. \tag{4}$$

We get from (1) and (4) that

$$\bar{I} = \sum_{i=1}^{N} \gamma_i I_i, \tag{5}$$

$$I_0 = E\bar{I} = \sum_{i=1}^{N} \gamma_i I_{i0}, \tag{6}$$

where

$$I_{i0} = EI_i,\ i \in \{1,2,...,N\}, \tag{7}$$

$$\sum_{i=1}^{N} \gamma_i = 1. \tag{8}$$

[2] We assume $E\bar{I} < \infty$, $EI_i < \infty$, $Var\bar{I} < \infty$, $VarI_i < \infty$.

Let us assume that dI_i and dI_j are independent random variables for each i and j. Hence, we get the following consequence of this fact:

$$\sigma_{\bar{I}}^2 = E(d\bar{I})^2 = E(\bar{I} - I_0)^2 = E[\sum_{i=1}^{N} \gamma_i (I_i - I_0)^2] =$$
$$= \sum_{i=1}^{N} \gamma_i^2 \sigma_{I_i}^2 . \quad (9)$$

Now we have the optimization task where the aim function is

$$F = \sum_{i=1}^{N} \gamma_i^2 \sigma_{I_i}^2 , \quad (10)$$

with the constraints specified in (8).

The essential and sufficient condition for the optimization task defined by (8) and (10) is formulated as follows:

$$\frac{\partial F}{\partial \gamma_i} = \frac{\partial F}{\partial \gamma_k}, \text{ for each } i \text{ and } k. \quad (11)$$

The formula (11) leads to

$$\gamma_1 \sigma_{I_1}^2 = \gamma_2 \sigma_{I_2}^2 = \ldots = \gamma_N \sigma_{I_N}^2 , \text{ with } \sum_{i=1}^{N} \gamma_i = 1 \quad (12)$$

or equivalently

$$g_1 \sigma_{I_1}^2 = g_2 \sigma_{I_2}^2 = \ldots = g_N \sigma_{I_N}^2 . \quad (13)$$

From (13) we get that

$$g_i = \frac{1}{\sigma_{I_i}^2} . \quad (14)$$

Under the additional assumption that variation coefficients of work efficiency at the moments s and t are similar, after some technical operations we get that

$$\sigma_{I_i}^2 \approx \frac{1}{2}(\frac{1}{n_i^s}+\frac{1}{n_i^t}). \tag{15}$$

Using (14) and (15) we can calculate the weights as follows

$$g_i = \frac{2}{\frac{1}{n_i^s}+\frac{1}{n_i^t}}. \tag{16}$$

From (16) we can get the following conclusion: each weight g_i is a harmonic mean of number of employees of $i-$th enterprise at considered moments s and t. It is easy to verify that if we assumed weights as an arithmetic mean of these numbers:

$$\overline{g}_i = \frac{n_i^s + n_i^t}{2}, \tag{17}$$

we would not solve the optimization task for function F under the constraints specified in (8). In our opinion, this fact recommends the definition g_i over $\overline{g}_i$.

III. CONSTRUCTION OF INDEX IN CASE OF MORE THAN TWO PERIODS

Let us consider a group of $N-$enterprises observed in discrete moments: $\{1,2,..,T\}$. We are going to measure the average, one-period work efficiency. As in the previous case we expect that the new index $\hat{I}$ strongly limits the influence of partial indexes of work efficiency connected with the small number of observations. We propose a list of postulates for this aggregative index:

Postulate 1

$\forall i, i \;\; I_i(t-1,t)=1 \Rightarrow \hat{I}=1$.

This postulate says that in case when partial indexes show no change of work efficiency of given enterprises during the time interval, then the global index must absolutely inform us about no change of work efficiency of the group.

Postulate 2

The influence of enterprises with relatively small number of employees on the average one-period work efficiency is asymptotically negligible.

Postulate 3

If all partial indexes of work efficiency grew by about the same $m\%$ then the value of global index $\hat{I}$ would increase by about the same $m\%$.

Postulate 4

If we increased the number of employees of each enterprise by about the same $m\%$ the index $\hat{I}$ would not change.

Under the above assumptions and significations we propose the following index of the average work efficiency on time interval:

$$\hat{I} = \sum_{i=1}^{N} \beta_i \cdot \sum_{t=2}^{T} \alpha_i(t) \cdot I_i(t-1,t), \tag{18}$$

where

$$\beta_i = \frac{\dfrac{T-1}{\sum_{t=2}^{T} \dfrac{1}{n_i^t}}}{\sum_{j=1}^{N} \dfrac{T-1}{\sum_{t=2}^{T} \dfrac{1}{n_j^t}}} = \frac{\dfrac{1}{\sum_{t=2}^{T} \dfrac{1}{n_i^t}}}{\sum_{j=1}^{N} \dfrac{1}{\sum_{t=2}^{T} \dfrac{1}{n_j^t}}}, \tag{19}$$

and

$$\alpha_i(t) = \frac{\dfrac{2}{\dfrac{1}{n_i^{t-1}} + \dfrac{1}{n_i^t}}}{\sum_{u=2}^{T} \dfrac{2}{\dfrac{1}{n_i^{u-1}} + \dfrac{1}{n_i^u}}} = \frac{\dfrac{1}{\dfrac{1}{n_i^{t-1}} + \dfrac{1}{n_i^t}}}{\sum_{u=2}^{T} \dfrac{1}{\dfrac{1}{n_i^{u-1}} + \dfrac{1}{n_i^u}}}. \tag{20}$$

The formulas (19) and (20) are the alternative proposition for results coming from the paper by Białek (2005). We have the following interpretation of these coefficients: β_i informs the producer how important is a share of $i-$th enterprise taking into consideration the number of employees and α_i^u informs the producer how important is $u-$th moment in the case of $i-$th enterprise.

We can also notice that the numerators and denominators of formulas (19) and (20) are the harmonic mean of right numbers of employees.

It is easy to verify that

$$\sum_{j=1}^{N} \beta_j = 1, \tag{21}$$

$$\sum_{t=2}^{T} \alpha_i(t) = 1. \tag{22}$$

From (21) and (22) we get the index $\hat{I}$ as a weighted mean of all $I_i(t-1,t)$. Besides, the following theorems are true:

Theorem 1

Index $\hat{I}$, defined in (18), satisfies all postulates 1–4 (proof is omitted).

Theorems 2

In the special case, when $T = 2$ (two periods), the formula $\hat{I}$ leads to $\bar{I}$. It is an immediate consequence of the fact that for $T = 2$ we have

$$\alpha_i \equiv 1, \tag{23}$$

$$\beta_i = \frac{\dfrac{2}{n_i^s + n_i^t}}{\sum_{j=1}^{N} \dfrac{2}{n_j^s + n_j^t}} = \gamma_i, \tag{24}$$

and finally

$$\hat{I} = \sum_{i=1}^{N} \beta_i I_i(s,t) = \sum_{i=1}^{N} \gamma_i I_i = \bar{I}\,. \tag{25}$$

Conclusion

Index $\hat{I}$ proposed in (18) is a more general version of index $\hat{I}$. Both indexes have the required properties and strongly limit the influence of partial indexes of work efficiency connected with the small number of observations.

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Jacek Białek, Andrzej Czajkowski

PROPOZYCJA SYSTEMU WAG DLA INDEKSÓW AGREGATOWYCH NA PRZYKŁADZIE INDEKSU WYDAJNOŚCI PRACY

W pracy zaproponowano konstrukcję agregatowego indeksu wydajności pracy. Proponowany system wag wynika z teoretycznych rozważań nad sytuacją, gdy liczba obserwacji pochodzących od któregoś z analizowanych przedsiębiorstw jest niewystarczająca.

W pierwszej części pracy rozważania dotyczą grupy N – przedsiębiorstw i dwóch okresów ich funkcjonowania. Podajemy konstrukcję indeksu dla porównania tych okresów z punktu widzenia wydajności pracy. Następnie rozważamy przypadek, gdy chcemy zmierzyć przeciętną, jedno-okresową dynamikę wydajności pracy posiadając dane pochodzące z $T > 2$ okresów. Konstruujemy nowy indeks stanowiący ogólniejszą wersję poprzedniego indeksu.