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THE MARKET AS THE MINORITY GAME AND THE STATISTICAL PHYSICS

ABSTRACT. The simplest version of minority game is introduced. It is shown how the minority game can result from the behaviour of individuals. The stability analysis of stationary state is briefly discussed. The modification of the game is described which leads to Nash equilibra as stationary states.

Key words: minority game, stationary states, stability, Nash equilibria.

I. THE MINORITY GAME – DEFINITION

In recent years a growing interest has been observed concerning the application of ideas and methods of statistical physics to the study of economical systems (for recent review see Ref. [1]). It is not surprising since in both cases one attempts to understand how the effect of interactions at the microscopic scale can build up to the macroscopic scale. In spite of basic differences in the microscopic behaviour of interacting atoms and agents entering the economical game one finds striking similarities in global behaviour of physical and financial systems.

The present contribution is aimed at the brief exposition of one of the simplest versions of the minority game [2] studied by Marsili [3]. Due to the lack of space no technical details are given; this concerns also my study of stability of stationary states [4].

The minority game is a model of speculative trading in financial market where agents buy and sell asset shares with the only goal of profiting from price fluctuations. The basic idea is that when most traders are buying it is profitable to sell and vice-versa, so the minority group always win.

We will consider the following situation. We have N agents and each of them formulate at every time step "t" a binary bid (sell/buy) $a_i(t)=\{-1,1\}$. We define

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$$A(t) = \sum_i a_i(t). \tag{1}$$

$A(t)$ is sum of decisions of all agents and will be called the excess demand. The payoff received at time t by each agent depends both on his action and on the aggregate action A(t) and it is given by

$$\pi_i(t) = -a_i(t)A(t); \tag{2}$$

hence the majority of agents, who have $a_i = sign[A(t)]$, receives a negative payoff -|A|, whereas the minority wins a payoff of |A|. Note that the total payoff to agents is allways negative and is given by

$$\sum_i \pi_i(t) = -A^2. \tag{3}$$

It means that this market is not zero – sum game because the part of assets is destined for transaction costs coverage. The maeasures of efficiency are the average excess demand and fluctuations in the steady state:

$$\langle A \rangle = \lim_{T\to\infty} \frac{1}{T - T_{eq}} \sum_{t=T_{eq}}^{T} A(t), \tag{4}$$

$$\sigma^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle A^2 \rangle. \tag{5}$$

where T_{eq} is an equilibration time.

In the stationary state $\langle A \rangle = 0$; if $\langle A \rangle \neq 0$ agents could use this information to increase their profits through the choice of the decision which is opposite to the sign of $\langle A \rangle$. So everybody would do that and in the consequence loose. Notice that if N is fixed the number of agents which choose the actions ±1 are $\frac{N \pm A(t)}{2}$, respectively. Therefore, the number of agents which could have been accommodated in the minority is $\frac{|A|}{2}$; hence σ is measure of the waste of resource. In the stationary state the variance σ^2 of $A(t)$ measures the efficiency with which resources are distributed since the smaller σ^2 the larger is a typical minority group. In other words, σ^2 is a reciprocal measure of global efficiency of the system.

We remark that agents cannot communicate. If communication were possible, agents would have incentives to stipulate contracts. For example two agents agree that they toss a coin and if the outcome is head (tail) they choose $a_1=1$, $a_2=-1$ ($a_1=-1$, $a_2=1$). This means that at every time step one of agents win and one loose. In this situation the balance of both players will be zero $\pi_i(a_1, a_{-1}) + \pi_2(a_2, a_{-2}) = 0$, while the average balance of both players without agreement will be $-\frac{2A^2}{N} < 0$. This contract transforms the negative sum game into zero sum game for these two players. Therefore, the communication is forbidden and agents interact only through a quantity A which is produced by all of them. This reminds the mean field method where spins interact through mean field which is produced collective by all spins.

II. FROM AGENTS' EXPECTATIONS TO THE MINORITY RULE

In this section we show how the behaviour of individual agents can lead to the minority and majority rules. Let us imagine a market in which N agents submit their orders $a_i(t)$ for a certain asset simultaneously at every time step $t = 1, 2,\ldots$ Let $p(t)$ be the price at the time step t and let $a_i(t) > 0$ mean that agent i contributes $a_i(t)$ euro, while $a_i(t) < 0$ mean that agent i sells $\frac{-a_i(t)}{p(t-1)}$. The demand and the supply are given by:

$$D(t) = \frac{N + A(t)}{2}, \tag{6}$$

$$S(t) = \frac{N - A(t)}{2p(t-1)}. \tag{7}$$

The price is fixed by the market clearing condition

$$p(t) = \frac{D(t)}{S(t)} = \frac{(N + A(t))p(t-1)}{N - A}. \tag{8}$$

Take the agent i and assume he must decide whether to buy or sell at time t. To do this, he should compare the expected profit of the two actions, which depends on what the price will be at time $t+1$. If he buys 1 euro of asset at time t the utility he would face at time $t+1$ is given by

$$u_i(t)=\frac{p(t+1)-p(t)}{p(t)}. \tag{9}$$

But the price $p(t+1)$ is unknown to him; therefore, he has to replace $p(t+1)$ by the expectation he has at time t of what the price will be at time $t+1$, denoted by $E_t^{(i)}[p(t+1)]$.

Let us assume that

$$E_t^{(i)}[p(t+1)]=(1-\Psi_i)p(t)+\Psi_i p(t-1); \tag{10}$$

hence the expected price increment is

$$E_t^{(i)}[\Delta p(t+1)]=-\Psi_i\Delta p(t), \tag{11}$$

where $\Delta p(t)=p(t)-p(t-1)$.

The parameter Ψ_i allows to distinguish two types of players:

1) If $\Psi_i > 0$ players believe that market price fluctuate around a fixed value, so that the future price is an average of past prices. According to the equation (11) the future price increment is negatively correlated with the last one; these players are called fundamentalists.

2) If $\Psi_i < 0$ the players believe that the future price increment will occur in the direction of the trend defined by the last two prices, so that the future price increments are positively correlated with the past one. These players are called trend followers.

The equation (9), after using (8) and (10), becomes

$$E_t^{(i)}\left[u_i(t)\,|\,a_i(t)=+1\right]=\frac{-2\Psi_i A(t)}{N+A(t)}. \tag{12}$$

A similar calculation can be carried out for the expectated utility for selling at time t. The net result is that the expected utility for action $a_i(t)$ a time t can be written as

$$E_i^{(i)}\left[u_i(t)\right]=\frac{-2\Psi_i a_i(t)A(t)}{N+A(t)}. \tag{13}$$

Accordingly, agents who took the majority actions, $a_i = sign[A(t)]$, expect to receive a payoff $\frac{-2\Psi_i|A(t)|}{N+A(t)}$, whereas agents in the minority group expect to get $\frac{2\Psi_i|A(t)|}{N+A(t)}$.

This means that the fundamentalists get the positive payoff only if they are in the minority group, while the trend followers get the positive payoff only if they are in the majority group. So in the minority game model we assume that the most players are fundamentalists. In real markets, both groups are present and the resulting price dynamics stems from a competition between the two groups.

III. THE SIMPLEST MINORITY GAME

In the simplest minority game, agents base their choice only on their past experience. Let us suppose that agents employ a probabilistic rule of the form

$$\Pr ob\{b_i(t)=b\}=C(t)\exp\left[b\Delta_i(t)\right],\quad b\in\{-1,1\}, \tag{14}$$

where *C(t)* is a normalization factor, $\Delta_i(t)$ accounts for the agent's expectations about what will be the winning action; if $\Delta_i(t) > 0$ he will choose $b_i(t) = 1$ with higher probability. The score function Δ_i is updated according to

$$\Delta_i(t+1)-\Delta_i(t)=-\Gamma A(t)/N, \tag{15}$$

with $\Gamma > 0$ a constant so that if $A(t) < 0$ agents increase Δ_i and the probability of choosing the action 1 and if $A(t) > 0$ the choice of action –1 is more probable. It means that the probability of choosing some action increase if this action was winning in the last time step. When agents choose the first decision they have no information so we must assume that the initial conditions $\Delta_i(0)$ are drawn from a distribution $p_0(\Delta)$ which is assumed to be the same for all agents. Notice that the equations (14) and (15) have the same form for all agents so we can introduce the new variable $y(t)=\Delta_i(t)-\Delta_i(0)$ which does not depend on *i*, for all times.

As all $b_i(t)$ are independent and have the same probability distribution we can, for $N \to \infty$, adopt the law of large numbers and write $\frac{A(t)}{N} \equiv \frac{1}{N}\sum_i b_i(t) \underset{N\to\infty}{=} \langle b(t)\rangle = \langle th(y(t)+\Delta(0))\rangle_0$, where the average $\langle \ldots \rangle_0$ is on the distribution p_0 of initial conditions.

The equation (19) in new variables has form

$$y(t+1) = y(t) - \Gamma\langle \tanh[y(t)+\Delta(0)] \rangle_0. \tag{16}$$

We consider the stationary state of equation (16), i.e. the solution of the form $y(t)=y^*$. This implies that $\langle th(y(t)+\Delta(0))\rangle_0 \equiv \langle A\rangle = 0$. We could show that the solution y^* is stable under small deviations provided

$$\Gamma < \Gamma_c = \frac{2}{1-\left\langle th\left[y^* + \Delta(0)\right]^2\right\rangle_0}. \tag{17}$$

According to the equation (15) with increasing Γ the probability of choosing decision, which was winning in the last time step, grows. For the small Γ ($\Gamma < \Gamma_c$) the probability is so small that the number of agents, which belong to a minority, does not change considerably and the solutions for $t \to \infty$ tend toward y^*. We can check that in this case the variance σ^2 is proportional to N. For the large Γ ($\Gamma > \Gamma_c$) many people choose a strategy which was the minority startegy in the last time step and this strategy will become the majority strategy. Hence the solution fluctuate around the stationary position and, indeed, we can show that asymptotically $y(t) = y^* + z^*(-1)^t$. Now $\sigma^2 \sim N^2$, so the efficiency of the system is smaller than in the previous case.

IV. THE NASH EQUILIBRIUM

We can ask whether the steady state of the model which we consider is the Nash equlibrium, i.e. an optimal state in which the change of decision of one of agents does not improve his situation. The answer is a no, because in our case $\sigma^2 \sim N$, hence the minority group is so small that many players could have been accomodated to this group and it will be still minority. We want to explain why inductive agents are playing sub-optimally. The non-optimality is the result of the fact that agents in the minority game over-estimate the performance of the strategies they do not play. In connection with this agents often change their strategy and this disturbs the quantity A(t). The efficiency of the system will be better if the virtual gain of the strategy will be equal the real gain. This is possible if agent takes into account only the aggregate action of other agents and will not react to his own action. In real market agents don't make this correction because they think that in a system of N agents every single agent weights 1/N and is thus negligible in the statistical limit $N \to \infty$. Once this assumption is dropped and agents account for their own impact, the resulting steady state improves dramatically and eventually a Nash equilibrium may be reached.

Let us consider the role of market impact in our simplest minority game. We introduce the following modyfication of the learning dynamics (15):

$$\Delta_i(t+1) - \Delta_i(t) = -\frac{\Gamma}{N}\left[A(t) - \eta a_i(t)\right]. \tag{18}$$

One indeed sees that (18) reduces to (15) for $\eta = 0$, whereas for $\eta = 1$ agent *i* considers only the aggregate action of other agents and does not react to his own action $a_i(t)$. In what follows we consider only $\eta = 1$ case. Let us take the average of (18) in the steady state and define $m_i = \langle a_i \rangle$. We note that

$$\langle \Delta_i(t+1) \rangle - \langle \Delta_i(t) \rangle = -\frac{\Gamma}{N}\left[\sum_j m_j - \eta m_i\right] = -\frac{\Gamma}{N}\frac{\partial H_1}{\partial m_i}, \tag{19}$$

where

$$H_1 = \frac{1}{2}\left(\sum_i m_i\right)^2 - \frac{1}{2}\sum_i m_i^2 . \tag{20}$$

We want to find the stationary values of the m_i. One can show that the function H_1 does not grow in time; this implies that the stationary values of the m_i's are given by the minima of H_1. We look, therefore, for the minima of H_1 in the hypercube $[-1,1]^N$. After simple calculation we find that H_1 attains its minima only at the vertices; hence $m_i = \pm 1$ and we can write $H_1 = \left(\sum_i m_i\right)^2 - N$ so we must minimize $\left(\sum_i m_i\right)^2$.

Hence

1. if N is even, $\sum_i m_i = 0$, which means that half of the players take $a_i = 1$ and the other half take a_i=-1, so the waste of sources is equal zero;

2. if N is odd, $\sum_i m_i = \pm 1$, which means that $\frac{N+1}{2}$ players take $a_i = 1$ and $\frac{N-1}{2}$ players take a_i=-1 or inversely.

In both cases the change of decision by anybody doesn't improve his situation. Hence, as soon as agents start to account for their market impact, the collective behavior of the system changes abruptly and the stationary states are indeed Nash equilibria of the associated N persons minority game.

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GIEŁDA JAKO GRA MNIEJSZOŚCIOWA I FIZYKA STATYSTYCZNA

W ostatnich latach obserwuje się wzrastające zainteresowanie zastosowaniami metod fizyki statystycznej w matematycznej teorii rynków finansowych. Okazuje się, że takie idee i metody mechaniki statystycznej, jak równania stochastyczne, rozkłady Gibbsa, przejścia fazowe czy teoria fluktuacji znakomicie nadają się do opisu zjawisk cechujących rynki finansowe.

W referacie podaję zwięzłą dyskusję analizy metodami fizyki matematycznej prostego modelu giełdy – gry mniejszościowej, zdefiniowanej jako gra, w której w każdym kroku zyskuje gracz, który podejmuje decyzję taką jak mniejszość graczy. Okazuje się, że standardowe metody fizyki (funkcja Ljapunowa, równanie Langevina, teoria fluktuacji) pozwalają dokładnie opisać stany stacjonarne gry łącznie z takimi własnościami, jak jednoznaczność, minimalizacja przewidywalności, duże fluktuacje w fazie symetrycznej itp.