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STATIC HEDGING OF BARRIER OPTIONS OF TYPE DOWN-AND-OUT CALLS

Abstract. The paper presents path-dependent options with a single barrier in terms of path-independent standard options. The key to providing this result is put-call symmetry, which is assumed to hold when the underlying first reaches the barrier price.

Key words: static hedging, put-call symmetry, single barrier options.

1. INTRODUCTION

In contrast to instruments, used in stock exchange turnover instruments accessible on outside market can be and they usually are fitted to customers' individual needs. For the first time these non-standard instruments appeared in the United States 30 years ago. They were the barrier options of type down-and-out. This type of instruments was called earlier „boutique" or „designer" however at present it defined as exotic. Static hedging for exotic options based on standard options will be introduced. The method consists in examining the relations between European options put and call options which here different exercise prices.

2. PUT-CALL SYMMETRY

Put-call symmetry can be viewed as a result of both an extension and a restriction of put-call parity. The restrictions sufficient to achieve this result are essentially that the underlying price process has a symmetric

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volatility structure as well as zero drift. We assume that the market is frictionless and there are no arbitrage abilities. Let $P(K)$ and $C(K)$ denote the time 0 price of an European put and call, respectively, with both options stuck at K and maturing at T. The maturity is the same for all instruments, and we can suppress dependence on the time to maturity to simplify. Let B denote the time 0 price of a pure discount bond paying one dollar at T. Then Put-Call Parity expressed in terms of the forward price F for time T is expressed:

$$C(K) = [F-K]B + P(K). \tag{1}$$

Put-Call Parity means that if the mutual strike of the put and call is the actual forward price, then the options have the same value. To receive put-call symmetry we assume that the underlying price process is a diffusion, with zero drift under any risk-neutral measure, where the volatility coefficient performs a certain symmetry condition. Thus, we rule out jumps in the price process and assume that the process starts afresh at any stopping time, such as at a first passage time to barrier. We also make an assumption that the volatility of the forward price is a function $\sigma(F_t, t)$ of the forward price F_t and time t. Moreover, we also assume the symmetry condition:

$$\sigma(F_t, t) = \sigma(F^2/F_t, t), \quad \text{for all } F_t \ 0 \text{ and } \quad t \in [0, T], \tag{2}$$

where F is the current forward price. The symmetry condition is satisfied in the Black (1976) model where volatility is deterministic i.e. $\sigma(F_t, t) = \sigma(t)$. The symmetry arises when the volatility is graphed as a function of $X_t \ln(F_t/F)$. Letting $\nu(X_t, t) \ \sigma(F_t, t)$, the equivalent condition is:

$$\nu(x, t) = \nu(-x, t), \quad \text{for all } x \in \Re \quad \text{and} \quad t \in [0, T] \tag{3}$$

European put-call symmetry: given frictionless markets, no arbitrage, zero drift, and the symmetry condition, the following relationship holds:

$$C(K)K^{-1/2} = \mathrm{P(H)H}^{-1/2}, \tag{4}$$

where the geometric mean of the call strike K and the put strike H is the forward price F:

$$(KH)^{1/2} = F. \tag{5}$$

Put-call symmetry is illustrated of in Figure 1. When the current forward is $ 12, a call struck at 16 has the same value as 4/3 puts struck at $ 9. The reason the call has a much greater value, even though it is further out-of-the-money arithmetically, is that our diffusion process has greater absolute volatility when prices are high than when prices are low. Because call and put payoffs are determined by the arithmetic distance between terminal price and strike, the higher absolute volatility at higher prices leads to higher call values. The fulcrum occurs at the expected value under the risk-neutral distribution, which is the current forward price. Summing the product of density and distance from the wedge on the right of the fulcrum gives the forward price of European call struck at the forward. Summing the product of density and absolute distance from the wedge on the left of the fulcrum gives the forward price of an at-the-money forward European put.

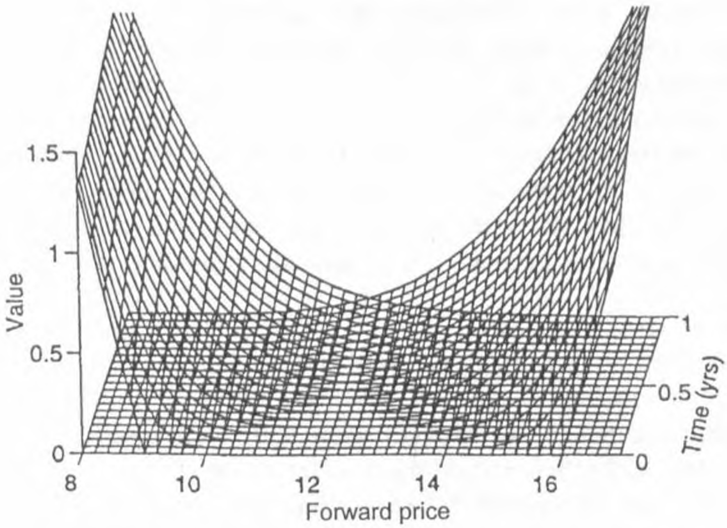

Fig. 1. Put-call symmetry. A call with strike 16 is equal to 4/3 puts with strike 9 when the forward price is 12
Source: Carr et al (1998)

3. SINGLE BARRIER OPTIONS OF TYPE DOWN-AND-OUT CALLS

In case of path-dependent options with a single barrier to providing the put-call symmetry, which is assumed to hold when the underlying reaches the barrier price first for the first time. The axis of the symmetry for volatility is the barrier price. We will present hedging knock-out calls. Such calls behave like regular calls except that they are knocked out the first time the underlying hits a prespecified barrier. In contrast, knock in calls become standard calls when the barrier is hit or otherwise they expire worthless. The result of valuation and hedging strategy for knock-out calls, corresponding to the results for knock-in calls can be presented using the following parity relation (Chriss 1996):

$$OC(K, H) = C(K) - IC(K, H), \tag{6}$$

where $IC(K, H)(OC(K, H))$ is an in-call (out-call) with strike K and barrier H.

Down-and-out call (DOC) with strike K and barrier $H<K$ becomes worthless if H is not hit at any time during its life. If the barrier has not been hit by the expiration date, the terminal payoff is that of a standard call struck at K. The hedge of a down-and-out call is needed to match the terminal payoff and the payoff along the barrier. The first step in constructing a hedge is to match the terminal payoff, which is done by purchasing a standard call $C(K)$. Let is consider option value along the barrier. When $F = H$, the DOC is worthless, while a current hedge $C(K)$ has a positive value. In this way we need to sell off an instrument that has the same value as the European call when the forward price is at the barrier. Using put-call symmetry when $F = H$, we receive:

$$C(K) = KH^{-1}P(H^2K^{-1}). \tag{7}$$

To complete the hedge, we need to write KH^{-1} European puts struck at H^2K^{-1}. The complete replicating portfolio for DOC is buy-and-hold strategy in standard options which is purchased at the initiation of the option

$$DOC(K, H) = C(K) - KH^{-1}P(H^2K^{-1}), \quad H < K \tag{8}$$

If the barrier is hit before expiration, the replicating portfolio should be liquidated with put-call symmetry which guarantees that the proceeds from selling the call are exactly offset by the cost of buying back the puts. If the barrier is not hit before expiration, then the long call gives the desired terminal payoff and the written puts expire worthless, as $H^2K^{-1} < H$ when $H < K$.

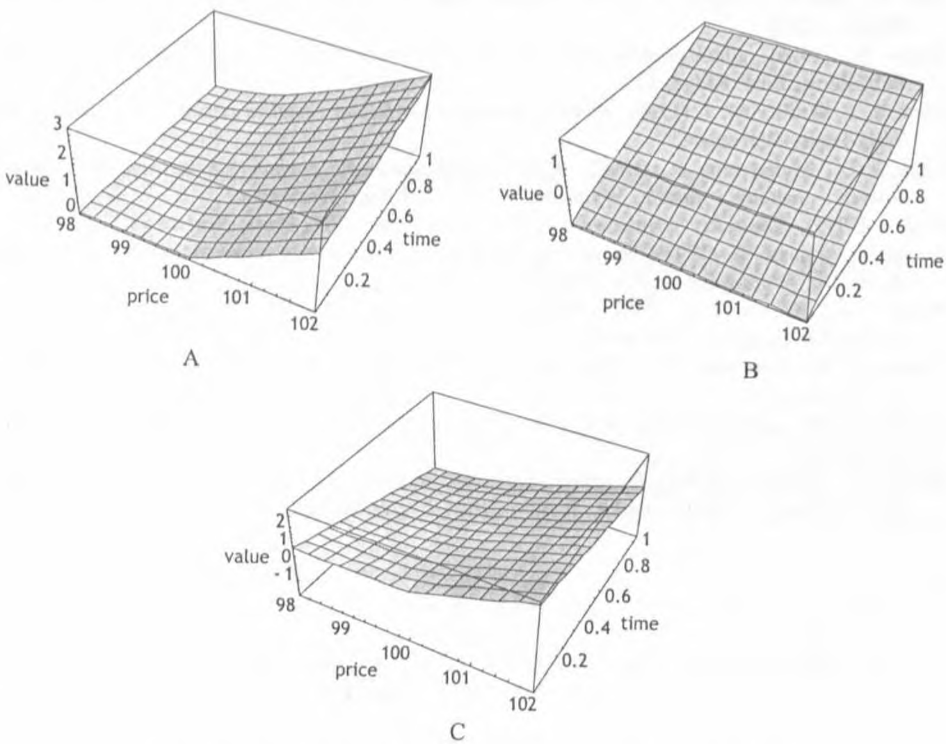

Fig. 2. Static hedge for a down-and-out-call ($K = 100$, $H = 98$)

Figure 2 illustrates the replication of down-and-out-call with strike $K = \$\ 100$, barrier $H = \$\ 98$, and an initial maturity of one year. Panel A is of a standard call with the same strike and maturity as the down-and-out. Along the barrier $F = \$\ 98$, the call has a positive value. Panel B is of $KH^{-1} = 1.0204$ puts struck at $H^2K^{-1} = \$\ 96.04$. Notice that the value of these puts along the barrier $F = \$98$ matches that of the standard call. When Panel B is subtracted from Panel A, the result is Panel C, which shows that the replicating portfolio has zero value along the barrier $F = \$98$ and the payoff of a standard call struck at $\$100$ at expiration.

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STATYCZNE ZABEZPIECZENIE OPCJI BARIEROWYCH KUPNA TYPU DOWN-AND-OUT

(Streszczenie)

W przeciwieństwie do instrumentów znajdujących się w obrocie giełdowym, instrumenty dostępne na rynkach pozagiełdowych mogą być i zazwyczaj są dopasowywane do indywidualnych potrzeb klientów. Takie niestandardowe instrumenty po raz pierwszy pojawiły się 30 lat temu w Stanach Zjednoczonych. Były to barierowe opcje kupna typu *down-and-out*. Tego typu instrumenty nazywano wcześniej butikowymi (*boutique*) lub konstruktorskimi (*designer*), jednak obecnie określa się je mianem egzotycznych. W artykule przedstawiono statyczne zabezpieczenie dla egzotycznych opcji w oparciu o opcje standardowe. Metoda ta polega na relacji między europejskimi opcjami sprzedaży i kupna, które posiadają różne ceny wykonania. Do wyceny i statycznego zabezpieczenia egzotycznych opcji barierowych zależnych od trajektorii (*path-dependent*) zastosowano uogólniony efekt symetryczności sprzedaży-kupna (*put-call symmetry*).