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*Małgorzata Kobylińska**

COMPARISON OF SELECTED CRITERIA FOR DETERMINATION OF THE MEASURE OF DEPTH OF AN OBSERVATION IN A TWO-DIMENSIONAL SAMPLE

Abstract. The measure of observation depth in multi-dimensional samples, introduced into statistical practice by Tukey, has become a new tool for data analysis. It is a proposed method for determining multi-dimensional positional statistics, particularly in the analysis of non-typical data with outstanding observations. Applying a rule of depth helps to overcome the difficulties associated with sorting multi-dimensional observations. The notion of data depth has been intensively developed by many scholars, and, consequently, various criteria of the measurement of observation depth in a multi-dimensional samples may be found in literature. This paper contains a comparison of selected criteria of the measurement of observation depth in a two-dimensional case.

Key words: depth measure, measure of depth by Mahalanobis, measure of depth by Tukey, measure of depth by Barnett, measure of simplex depth by Liu.

1. INTRODUCTION

The measure of depth of a point in multi-dimensional sets, introduced to statistical practice by Tukey (1975), has become a new tool for data analysis. Owing to the assignment of a depth of measure to each observation in a sample, it is possible to rank statistical units according to their distance from the central cluster. The notion of the depth of an observation in a sample has been developed by numerous researchers and, consequently, various criteria of determining the depth of an observation in a multi-dimensional sample can be found in literature. In order to compare the criteria, experiments have been performed with data from a two-dimensional sample. The research was also aimed at obtaining answers to the following questions:

* Ph.D., University of Warmia and Mazury, Olsztyn.

- do the employed criteria yield different medians,
- is the determination of a median vector affected by outlying observations,
- which criteria yields the closest and which yield the most varying results of ranging in terms of the distance of an observation from the sample centre,
- which criteria yield the closest and which yield the most varying results of ranging as compared to the results obtained by all the other methods.

2. THE CRITERIA OF DETERMINATION OF MEASURE OF DEPTH OF AN OBSERVATION IN A TWO-DIMENSIONAL SAMPLE

Let $P_n^2 = \{x_1, x_2, ..., x_n\} = \{x_i, i = 1, 2, ..., n\}$ be a system of vectors which express a two-dimensional sample (PD) with size n from a two-dimensional distribution determined by a two-dimensional cumulative distribution function F_2. Let us assume that $\theta \in R^2$ is a vector with values belonging to the set of real numbers. In particular, θ may be any observation from the sample P_n^2.

The following are the criteria of determination of measure of depth of an observation in a two-dimensional sample:

1. Measure of depth by Mahalanobis $Mzan_2$ (1936):

$$Mzan_2(\theta; P_n^2) = [1 + Q(\theta, P_n^2)]^{-1}, \tag{1}$$

where $Q(\theta, P_n^2) = (\theta_1 - \overline{x}_1)^2 s^{11} + 2(\theta_1 - \overline{x}_1)(\theta_2 - \overline{x}_2)s^{12} + (\theta_2 - \overline{x}_2)^2 s^{22}$ while

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \ \overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}, \ \overline{x} = \frac{1}{n}\sum_{j=1}^{n} x_j, \ S = \frac{1}{n-1}\sum_{i=1}^{n}(x_j - \overline{x})', \ S^{-1} = \begin{bmatrix} s^{11} & s^{12} \\ s^{21} & s^{22} \end{bmatrix}.$$

2. Measure of depth by Tukey $Tzan_2$ (1975):

$$Tzan_2(\theta_2, P_n^2) = \inf_H \left\{\frac{n_H}{n}\right\}, \tag{2}$$

based on half planes of depth H such that H is a half plane enclosed inR^2 and $\theta \in H$, whereas n_H is the number of observations from P_n^2 belonging to H.

3. Measure of depth by Barnett $Bzan_2$ (1976):

$$Bzan_2 = (\theta, P_n^2) = W_\theta / W, \tag{3}$$

based on descending convex hulls, where W_θ is a degree of a convex hull to which point θ belongs, whereas W is the number of all the convex hulls which can be created of the elements of sample P_n^2,

4. Measure of simplex depth by Liu $Lzan_2$ (1990):

$$Lzan_2(\theta, P_n^2) = N_3^{-1} \sum_{1 \leqslant i < j < k \leqslant n} 1[\theta \in \Delta(x_i, x_j, x_k)], \tag{4}$$

built on triangles such $\Delta(xx_i, x_j, x_k)$ that $x_i, x_j, x_k \in P_n^2$ and $(i, j, k) \in I_w = \{1, 2, ..., n\}$, where $N_3 = \binom{n}{3}$, whereas $I(A)$ denotes an index function of event A with values

$$I(A) = \begin{cases} 1 & \text{if} \quad \theta \in A, \\ 0 & \text{if} \quad \theta \notin A. \end{cases}$$

Assigning the given observations with respective measures of depth and ordering them according to the growing values of the measure of depth makes it possible to arrange the points of a sample beginning from the most outward to those closest to the "cloud of data". It should be emphasised that the observation with the highest value of the measure of depth determines a two-dimensional median vector (TMV). If there are more of such observations in the sample, it is TMV which is determined as their centre of gravity $(\overline{x}, \overline{y})$.

3. EXPERIMENTS

The results of grouping of n observations in terms of their distance from the centre of the sample with m criteria may be presented as a matrix

$$[z_{ij}] = \begin{bmatrix} z_{11} & z_{12} & \dots & z_{1m} \\ z_{21} & z_{22} & \dots & z_{2m} \\ \dots & \dots & \dots & \dots \\ z_{n1} & z_{n2} & \dots & z_{nm} \end{bmatrix}, \quad i = 1, 2, ..., m, \quad j = 1, 2, ..., m,$$

where z_{ij} denotes the rank of i-th observation, corresponding to the measure of depth of this observation in the sample P_n^2 determined with the use of j-th criterion.

As it is our aim to examine to what extent the result obtained by means of m criteria differ, a matrix was created

$$[R_{ij}] = \begin{bmatrix} 0 & R_{12} & \dots & R_{1m} \\ R_{21} & R_{22} & \dots & R_{2m} \\ \dots & \dots & \dots & \dots \\ R_{m1} & R_{m2} & \dots & 0 \end{bmatrix}, \quad k, l = 1, 2, ..., m,$$

where R_{kl} expresses the sum of absolute differences of the ranks of observations corresponding to the measures of depth determined based on k-th and l-th criterion, which is described as

$$R_{kl} = \sum_{i=1}^{n} |z_{ik} - z_{il}|. \tag{5}$$

The values of $R_{kl} \in \langle 0, R_0 \rangle$, while $R_{kl} = 0$, when the measures of depth for all observations of a sample are identical with k-th and l-th criteria

$$R_0 = \begin{cases} 2q^2, \text{ for even values of } n \text{ and } q = n/2, \\ 2q(q+1), \text{ for odd values of } n \text{ and } q = (n-1)/2. \end{cases} \tag{6}$$

The values of R_{kl} are placed in the matrix of diversities $[R_{kl}]$, which is a symmetrical matrix, dimensions $m \times m$, where $R_{kl} = R_{lk}$ for $k \neq 1$ and $R_{kl} = 0$ for $k = l$. Only $m = 4$ criteria of determining measures of depth of an observation in a sample were taken into account, which means that 6 values of elements of the matrix of differences are under consideration. Based on the matrix $[R_{kl}]$ an answer may be given to the question of which pairs of criteria yield the closest, and which yield the most distant results of ordering observation P_n^2 in relation to the centre of a sample.

There is another question of which criteria yields results which are closest to the results obtained based on all the other criteria. The problem may be decided if the values of elements of the following vector

$$[R_k] = \begin{bmatrix} R_1 \\ R_2 \\ \dots \\ R_m \end{bmatrix}, \quad k = 1, 2, \dots, m,$$

are known; the components of the vector are obtained with the formula

$$R_k = \sum_{l=1}^{m} R_{kl}. \tag{7}$$

The values of $R_k \in \langle 0, (m-1)R_0 \rangle$, where R_0 is determined by (6). This means that the value of the element R_1 is a sum of diversities of ranks of observations P_n^2 obtained with the use of the first and second, first and third, ..., first and m-th criterion. It is obvious that the lower value of $[R_k]$ is proof that the k-th criterion yields a similar result of classification of observations in relation to the distance from the central cluster as compared to the other criteria considered as a whole. The lowest value of a component of vector R_k indicates a criterion which yields the results of ranking which are relatively closest to those obtained with the other methods.

In order to answer the earlier questions, two-dimensional samples were examined empirically, each having the size of 25. Experiments were conducted in the course of the research and different configuration of the tested samples was adopted each time:

- sample 1 (PD1) comes from the standard two-dimensional normal distribution,
- sample 2 (PD2) comes from the population with right-side asymmetric distribution,
- sample 3 (PD3) comes from a two-dimensional normal distribution with a zero-value vector of expected values and the matrix of covariance $\begin{bmatrix} 1 & 0.87 \\ 0.87 & 1 \end{bmatrix}$,
- sample 4 (PD4) comes from a two-dimensional normal distribution with a zero-value vector of expected values and the matrix of covariance $\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, for which one observation was shifted along the axis OY.

Table 1 contains the values of observations X and Y for each sample.

Table 1. Values of observations for two-dimensional samples

No. of obs.	PD1		PD2		PD3		PD4	
	X	Y	X	Y	X	Y	X	Y
1	–0.529	–0.854	0.063	0.361	0.754	1.647	1.503	0.846
2	–0.025	0.370	1.426	0.527	0.008	0.126	–0.795	–1.533
3	1.521	–0.677	0.402	0.009	0.390	0.975	0.188	–0.422
4	–0.247	–0.938	0.131	0.006	1.073	0.903	1.622	1.815
5	0.389	0.579	1.900	2.565	0.494	0.828	0.066	0.721
6	–0.508	0.380	0.000	0.015	–0.973	–0.392	–1.259	–1.819
7	0.252	1.000	0.130	0.741	–0.683	0.470	–0.544	–1.158
8	1.083	0.788	1.586	0.527	0.166	0.920	–1.867	–1.397
9	–0.294	1.462	6.647	4.655	1.209	2.412	–1.885	–1.396
10	0.400	1.597	0.000	0.017	–2.703	–3.899	–1.005	–1.331
11	0.200	1.161	5.079	1.450	1.942	4.735	–0.899	–1.148
12	–0.454	0.491	0.000	0.004	1.652	2.889	–0.559	0.067
13	–1.107	–0.273	0.037	0.011	–1.729	–1.230	0.094	–0.121
14	1.488	–1.896	1.020	1.871	–0.967	–1.633	0.829	0.909
15	0.475	0.855	0.823	1.624	–0.304	0.221	–1.857	–2.246
16	–1.107	–1.329	0.299	0.669	–1.516	–2.991	–0.222	0.066
17	0.296	–1.746	–0.003	0.234	–1.890	–1.853	–0.765	12.000
18	–0.526	0.154	1.426	1.755	–1.277	–1.172	–0.655	–0.188
19	–0.918	0.400	0.000	0.708	–1.953	–3.317	2.707	2.022
20	–0.184	–0.284	3.723	3.756	2.259	2.438	–1.467	–1.210
21	–0.919	1.540	0.555	1.280	–0.508	–0.385	–0.633	–0.743
22	–0.118	1.117	–0.048	–0.508	1.201	2.117	0.007	–0.585
23	–1.205	–1.324	0.079	0.351	1.919	3.189	–1.416	–0.528
24	–0.312	–1.837	6.238	2.549	–0.615	0.161	1.488	1.327
25	1.084	–0.189	0.564	2.697	–0.853	–1.644	–1.184	–1.443

The values of the determined digital characteristics of samples are presented in Table 2.

Table 2. Digital characteristics of two-dimensional samples

Characteristics	PD1		PD2		PD3		PD4	
	X	Y	X	Y	X	Y	X	Y
The lowest observation	1.20–5	–1.896	–0.048	–0.508	–2.703	–3.899	–1.885	–2.246
The highest observation	1.521	1.597	6.647	4.655	2.259	4.735	2.707	12.000
Span	2.726	3.493	6.695	5.163	4.962	8.633	4.592	14.246
Arithmetic mean	–0.051	0.022	1.283	1.115	–0.217	0.081	–0.340	0.100
Standard deviation	0.776	1.096	1.984	1.301	1.388	2.158	1.198	2.731
Lower quartile	–0.526	–0.854	0.037	0.017	–1.277	–1.230	–1.184	–1.331
Median	–0.184	0.370	0.402	0.669	–0.304	0.221	–0.633	–0.528
Upper quartile	0.389	0.855	1.426	1.755	0.754	0.975	0.094	0.721
Quartile deviation	0.4575	0.8545	0.6945	0.869	1.015	1.102	0.639	1.026
Slanted index	0.507	–0.354	1.865	1.215	0.109	0.039	0.914	3.682
Flattening index	2.653	1.972	5.473	4.042	–0.872	–0.079	3.403	18.992

Graphical presentation of the analyzed two-dimensional samples is given in Figures 1–4.

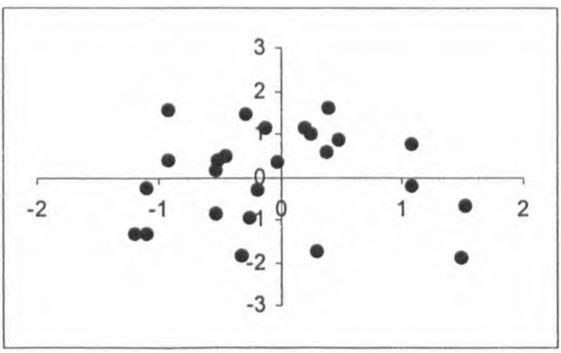

Fig. 1. Correlation chart for PD1

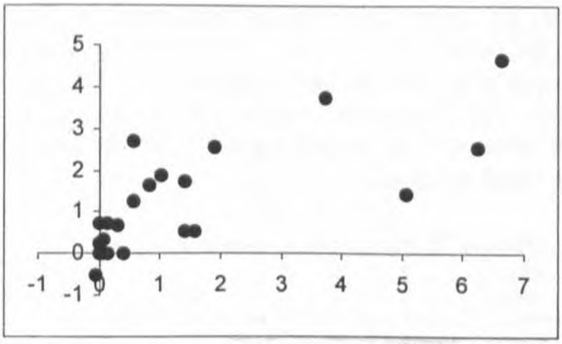

Fig. 2. Correlation chart for PD2

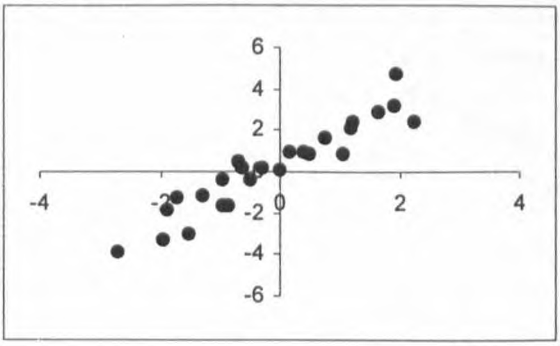

Fig. 3. Correlation chart for PD3

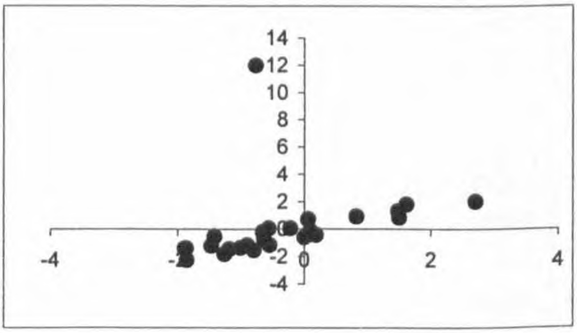

Fig. 4. Correlation chart for PD4

The measurements of depth of points in two-dimensional samples, calculated based on the above criteria, are presented in Tables 3 and 4. In order to clarify the view, the criteria of determining the measure of depth of an observation in a sample PD are designated as follows: Tukey's criterion (T), Liu's criterion (L), Barnett's criterion (B), Mahalanobis' criterion (M). The numbers of observation, according to Table 1, are given next to the value of the measure of depth.

Table 3. Measures of depth of observations for samples PD1 and PD2

PD1					PD2				
No. of obs.	$Lzan_2$	$Mzan_2$	$Bzan_2$	$Tzan_2$	No. of obs.	$Lzan_2$	$Mzan_2$	$Bzan_2$	$Tzan_2$
1	0.1196	0.494	0.75	0.24	1	0.0949	0.7105	0.75	0.2
2	0.2757	0.9074	1	0.44	2	0.1294	0.6175	0.75	0.2
3	0	0.1819	0.25	0.08	3	0.0504	0.5488	0.5	0.12
4	0.1374	0.5454	0.75	0.24	4	0.0756	0.5754	0.75	0.16
5	0.1769	0.6319	0.75	0.28	5	0.0296	0.3405	0.5	0.12
6	0.163	0.6885	0.75	0.24	6	0.0479	0.5832	0.75	0.16
7	0.1383	0.5122	0.75	0.24	7	0.0721	0.7163	0.5	0.16
8	0	0.2751	0.25	0.08	8	0.0751	0.5569	0.5	0.16
9	0.0109	0.3546	0.5	0.12	9	0	0.107	0.25	0.08
10	0	0.2931	0.25	0.08	10	0.0588	0.5841	0.75	0.16
11	0.0741	0.457	0.5	0.16	11	0	0.1232	0.25	0.08
12	0.1665	0.6891	0.75	0.24	12	0.0375	0.5783	0.5	0.12
13	0	0.3412	0.25	0.08	13	0.0781	0.5811	0.75	0.16
14	0	0.1257	0.25	0.08	14	0.0504	0.4694	0.5	0.12
15	0.0771	0.4896	0.5	0.16	15	0.0721	0.5478	0.5	0.16
16	0.0296	0.2277	0.5	0.12	16	0.2342	0.8005	0.75	0.32
17	0.0109	0.2636	0.5	0.12	17	0.0109	0.6666	0.5	0.12
18	0.1917	0.7198	0.75	0.28	18	0.1601	0.6841	1	0.28
19	0.0296	0.4228	0.5	0.12	19	0	0.6683	0.25	0.08
20	0.246	0.9026	1	0.4	20	0	0.1865	0.25	0.08
21	0	0.2407	0.25	0.08	21	0.1013	0.6534	0.75	0.2
22	0.1107	0.4988	0.75	0.2	22	0	0.3645	0.25	0.08
23	0	0.2109	0.25	0.08	23	0.1897	0.7116	1	0.28
24	0	0.2502	0.25	0.08	24	0	0.1139	0.25	0.08
25	0.0603	0.3152	0.5	0.16	25	0	0.1549	0.25	0.08

Table 4. Measures of depth of observations for samples PD3 and PD4

PD3					PD4				
No. of obs.	$Lzan_2$	$Mzan_2$	$Bzan_2$	$Tzan_2$	No. of obs.	$Lzan_2$	$Mzan_2$	$Bzan_2$	$Tzan_2$
1	0.2011	0.6914	0.8	0.28	1	0.0212	0.2925	0.4	0.12
2	0.2297	0.8589	0.8	0.36	2	0	0.7151	0.2	0.08
3	0.2609	0.8805	1	0.36	3	0.0588	0.7601	0.6	0.16
4	0.0296	0.2472	0.4	0.12	4	0.0109	0.2704	0.4	0.12
5	0.1764	0.7325	0.8	0.28	5	0.0504	0.884	0.4	0.12
6	0.1067	0.4476	0.4	0.2	6	0.0109	0.5481	0.4	0.12
7	0.0109	0.2815	0.2	0.08	7	0.0583	0.8244	0.4	0.12
8	0.2075	0.8275	0.6	0.32	8	0.0375	0.3775	0.4	0.12
9	0.0632	0.4869	0.4	0.16	9	0	0.3724	0.2	0.08
10	0	0.2104	0.2	0.08	10	0.1581	0.6928	0.6	0.24
11	0	0.1281	0.2	0.08	11	0.2446	0.7551	0.8	0.32
12	0.0395	0.3754	0.6	0.16	12	0.083	0.9656	0.6	0.16
13	0	0.245	0.2	0.08	13	0.2391	0.8519	0.8	0.36
14	0.1433	0.4692	0.6	0.24	14	0.1245	0.5119	0.8	0.24
15	0.2451	0.8519	0.8	0.36	15	0	0.3523	0.2	0.08
16	0	0.2094	0.2	0.08	16	0.2085	0.9883	1	0.32
17	0.0109	0.3095	0.4	0.12	17	0	0.0429	0.2	0.08
18	0.1255	0.5241	0.6	0.2	18	0.1734	0.9349	0.8	0.24
19	0.0109	0.2361	0.4	0.12	19	0	0.1336	0.2	0.08
20	0	0.1379	0.2	0.08	20	0.1275	0.5202	0.6	0.2
21	0.2549	0.9233	0.8	0.36	21	0.2717	0.8934	1	0.44
22	0.1418	0.521	0.8	0.24	22	0.082	0.8236	0.6	0.2
23	0.0109	0.3135	0.4	0.12	23	0.0375	0.5526	0.4	0.12
24	0.0909	0.4872	0.4	0.16	24	0.0395	0.3002	0.6	0.16
25	0.0884	0.3936	0.6	0.16	25	0.1206	0.6146	0.6	0.24

Two-dimensional median vectors (TMV), determined with the use of the given criteria are presented in Table 5. The last line contains the coordinates of TMV, determined with the use of the boundary distribution criterion. The respective numbers of observations are placed next to the vectors in the corresponding columns.

Table 5. Median vectors for two-dimensional samples

Criterion	PD1		PD2		PD3		PD4	
	DWM	no.	DWM	no.	DWM	no.	DWM	no.
L	(–0.025, 0.370)	2	(0.299, 0.669)	16	(0.390, 0.975)	3	(–0.633, –0.743)	21
M	(–0.025, 0.370)	2	(0.299, 0.669)	16	(–0.508, –0.385)	21	(–0.222, 0.066)	16
B	(–0.105, 0.043)	2, 20	(0.753, 1.053)	18, 23	(0.390, 0.975)	3	(–0.633, –0.743)	21
T	(–0.025, 0.370)	2	(0.299, 0.669)	16	(–0.304, 0.234)	2, 3, 15, 21	(–0.633, –0.743)	21
R boundary	(–0.184, 0.370)	–	(0.402, 0.669)	–	(–0.304, 0.221)	15	(–0.633, –0.528)	–

The results in Table 5 indicate that in the conducted experiments, the smallest differences between TMV, which were determined based on the given criteria, were achieved for PD1 and PD2; the same vectors were yielded by criteria L, M, T. The vectors determined according to the criterion B are situated in the smallest convex hull and for the said samples they have the form of two points. The greatest differences between TMV are recorded for PD3. According to Tukey's criterion, the highest value of the measure of depth is observed for four observations. TMV = (–0.304, 0.234) was determined as the centre of gravity for the observations nos. 2, 3, 15, and 21, according to Table 1. The same median vectors for PD4 are yielded by criteria L, B and T. This sample contains a significantly outlying observation with the coordinates (–0.765, 12), to which criterion M reacts. The greatest differences between TMV established based on the given criteria and the vector of boundary medians can be seen for PD2 for criterion B, for PD3 for L and B, and for PD4 for criterion M.

These considerations lead to a preliminary conclusion: for a two-dimensional sample with low correlation and with no outlying observations, the discussed criteria are equivalent. The smallest differences between TMV determined with the use of the given criteria and the vector of boundary medians are recorded for samples PD1 and PD2. Considering the above, and the numerical aspect, the criterion based on a boundary distribution may be recommended. Research illustrated with calculations leads one to the conclusion that criteria B and L should not be applied in the case of strong correlation, (sample PD3), and in the case of outlying observations – criterion M should not be applied (sample PD4).

An answer to the question – which pairs of criteria yield similar results in the ranking of observations of two-dimensional samples – is made possible by analysing the results presented in Tables 6, 7, 8 and 9. They express the values of R_{kl} determined according to formula 6. An analysis of the results contained in them indicates that the closest results of ranking the observation of sample PD1 in relation to the distance from the sample centre are achieved with the pair of criteria L and T. The coefficient of correlation of Spearman's ranks, determined for measures of depth of the pair is 0.9937. The most varied results of ranking for the sample in question were achieved between the criteria B and M, as well as T and M, for which coefficients of correlation of ranks are 0.8356 and 0.8371, respectively. In 13 cases, the ranks of observations obtained with the use of criterion T are equal to those obtained with criterion L. This supports the earlier comment that the closest results of ranking of observation PD1 were achieved with the use of these criteria.

Table 6. The value of R_{kl} for PD1

k \ l	M	L	B	T
M	0	49	62	56
L		0	29	13
B			0	25
T				0

Table 7. The value of R_{kl} for PD2

k \ l	M	L	B	T
M	0	101	99	82
L		0	51	25
B			0	47
T				0

Table 8. The value of R_{kl} for PD3

k \ l	M	L	B	T
M	0	37	68	41
L		0	53	22
B			0	46
T				0

Table 9. The value of R_{kl} for PD4

k \ l	M	L	B	T
M	0	120	118	133
L		0	41	32
B			0	24
T				0

All the results in Table 7 make it possible to compare the effects of the ranking of observation PD2. As was shown earlier, it may be noticed that the lowest degree of diversification of ranking of observations corresponds to criteria L and T. For this pair, in 10 cases the difference of ranks of the corresponding correlations is 0, whereas the index of correlation of Spearman's ranks is 0.9756. The sample is characterized by a strong right-sided symmetry, to which Mahalanobis criterion reacts. The values obtained for the pairs containing this criterion are more than twice as large as the diversities for the other pairs. The values of indexes of correlation of ranks are for these pairs 0.72, 0.71 and 0.77, respectively. Comparison of the results presented in Table 8 lead to the conclusion that the closest results of ranking of observations of sample PD3 are achieved by the pair of criteria L and T, whereas the most distant results are for pairs M and B, and L and B. Considering the values contained in Table 9, it is easy to notice considerable differences between them. Three pairs: B and T, L and T, and L and B achieve relatively low values as compared to the others. Apart from the pairs of criteria for which the results of ranking of observation PD3 differ little, there are such for which the diversification is over three times as high as for the group specified earlier. The following pairs of criteria belong to this group: M and L, M and B, and M and T. It should be noted that the first group is made up of three criteria, whereas the second group contains criterion M. Criterion M appears the most frequently in these pairs.

In order to obtain the answer to the question which of the criteria yields the closest results in relation to all the others which were included in the analysis, components of vectors $[R_k]$ were determined for each of the two-dimensional samples

$$[R_k] = \begin{bmatrix} 167 \\ 91 \\ 116 \\ 94 \end{bmatrix}, \quad [R_k] = \begin{bmatrix} 2827 \\ 177 \\ 197 \\ 154 \end{bmatrix}, \quad [R_k] = \begin{bmatrix} 146 \\ 112 \\ 167 \\ 109 \end{bmatrix}, \quad [R_k] = \begin{bmatrix} 371 \\ 193 \\ 191 \\ 189 \end{bmatrix}.$$

Components of these vectors correspond to criteria M, L, B and T in consecutive lines. The sequence will be maintained below. They indicate that the best results for all the others are yielded by: criterion L for sample PD1, criterion T for samples PD2, PD3 and PD4. Maximum values of the vectors suggest that those results of ranking of observations which differ from the results obtained with the other criteria for samples PD1, PD2 and PD4 were obtained based on criterion M and for sample PD3 – based on criterion B.

The analysis so far has dealt mainly with a comparison of criteria of determining the measurement of depth of observations for each of two-dimensional samples. The results of the comparison, considered for all the samples together, are presented in Table 10. The numbers in the table were obtained by adding the respective values of R_{kl} for each sample.

Table 10. The value of R_{kl} for two-dimensional samples

k \ l	M	L	B	T
M	0	307	347	312
L		0	174	92
B			0	142
T				0

The vector $[R_l]$, whose components are the sum of the respective values of vectors $[R_k]$ for each sample, has the form:

$$[R_k] = \begin{bmatrix} 966 \\ 573 \\ 663 \\ 546 \end{bmatrix}.$$

The listed results indicate that the least varied results of ranking of observations with the use of the measure of depth of observations in two-dimensional samples were obtained for the pair of criteria L and T, whereas the most varied were obtained for the pairs B and M, and for T and M. The minimum values of the vector $[R_k]$ correspond to the criterion T and L, the maximum value corresponds to criterion M.

4. RESULTS AND CONCLUSIONS

The experiments lead to several conclusions:

1. The results of ranking of observations and determining TMV with the use of the analysed criteria depend on the presence of outlying observations and correlation.

2. In each of the presented experiments, the Mahalanobis criterion comes first in terms of the most varied results of ranking of observations in

relation to the distance of the sample centre, as compared to the results obtained with the other criteria.

3. The Mahalanobis criterion is particularly sensitive to a sample asymmetry and to the presence of outlying observations.

4. With the assumption that there is a strong correlation interrelation between the examined variables, the most varied results were achieved are yielded by Barnett's criterion.

5. Criteria L and T yield the most diversified results of ranking as compared to the other criteria. Because of this, they are recommended for use in experiments.

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Małgorzata Kobylińska

PORÓWNANIE WYBRANYCH KRYTERIÓW WYZNACZANIA MIAR ZANURZANIA OBSERWACJI W PRÓBIE DWUWYMIAROWEJ

(Streszczenie)

W artykule dokonano porównania wybranych kryteriów wyznaczania miary zanurzania obserwacji w statystycznej próbie dwuwymiarowej. Wnioski dotyczące porównania tych kryteriów wyciągnięto na podstawie własnych badań empirycznych na próbach dwuwymiarowych.