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LIMIT PROPERTY OF THE COMPOUND DISTRIBUTION BINOMIAL–GENERALIZED TWO-PARAMETER GAMMA

Abstract. In 1920 M. Greenwood and G. U. Yule presented a compound distribution Poisson–two-parameter gamma and gave some interesting applications of this compounding. In 1973 H. Jakuszenkow published a compound of the Poisson distribution with the generalized two-parameter gamma one. In 1982 T. Gerstenkorn demonstrated a compound binomial-generalized beta distribution and in the limit procedure received a compound binomial-generalized three-parameter gamma one. Now there is presented a limit property of that distribution when one of its parameters takes a special constant value, i.e. if we have a particular two-parameter generalized gamma, giving the theorem of Jakuszenkow.

Key words: limit distributions, compound distributions, generalized gamma distribution.

1. INTRODUCTION

In 1920 M. Greenwood and G. U. Yule presented a compound of the Poisson distribution

$$P(X = x; \lambda) = \frac{\lambda^x}{x!}\exp(-\lambda), \quad \lambda > 0, \quad x = 0, 1, 2, \ldots \tag{1}$$

with the two-parameter gamma one:

$$h(\lambda) = \frac{a^v}{\Gamma(v)}\lambda^{-1}\exp(-a\lambda), \quad 0 < \lambda < \infty, \quad v, a > 0,$$

obtaining the negative binomial distribution

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$$P(X=x)=\binom{v+x-1}{x}p^x q^v, \quad x=0,1,2,..., \quad v>0,$$

where $p=\frac{1}{1+a}$, $q=1-p$.

It was the first paper in the field of compound probability distributions demonstrating also some interesting applications of the given method. Since that time there have been published many papers having respect to the compounding problem. For the most part they are mentioned in the comprehensive elaborations, as e.g. Patil et al. (1968), Johnson et al. (1969), (1992), (1997), Wimmer and Altmann (1999).

In this paper we refer to the paper by Jakuszenkow (1973) who gave a compounding of the Poisson (1) and generalized two-parameter gamma distributions, the last given in the form

$$f(x;a,p)=\frac{2x^{p-1}}{a^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}\exp\left(-\frac{x^2}{a}\right), \quad 0<p<+\infty, \quad a,p>0, \tag{2}$$

obtaining the distribution

$$P(X=x)=\frac{2}{a^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)x!}\left(\frac{2}{a}\right)^{-\frac{p+x}{2}}\cdot\Gamma(p+x)\exp\left(\frac{a}{8}\right)D_{-p-x}\left(\sqrt{\frac{a}{2}}\right), \tag{3}$$

where $x=0,1,2,...,$ a, $p>0$, and $D_p(z)$ is the so-called parabolic cylinder function, i.e. it is a particular solution of the differential equation

$$\frac{d^2u}{dz^2}+(p+\frac{1}{2}-\frac{z^2}{4})u=0.$$

2. LIMIT PROPERTY

In 1982 T. Gerstenkorn published a compound *binomial-generalized beta distribution.* It is known that if one parameter of the generalized beta distribution tends to infinity then one obtains the so-called *generalized three-parameter gamma* distribution:

$$f(x;a,b,p)=\frac{ax^{p-1}}{b^{\frac{p}{2}}\Gamma\left(\frac{p}{2}\right)}\exp\left(-\frac{x^2}{b}\right),\quad x,a,b,p>0, \tag{4}$$

(cf. Stacy 1962, p. 1187; Śródka 1973, p. 77).

The compound *binomial* – (4) distribution

$$P(x;a,b,c,n,p)=\binom{n}{x}\frac{1}{\Gamma\left(\frac{p}{a}\right)}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k(bc^a)^{\frac{x+k}{a}}\Gamma\left[\frac{p+x+k}{a}\right], \tag{5}$$

where a, b, p, $c>0$, n – a fixed natural number, is obtained in the same limit procedure.

For the aim of further discussion, that is for a comparison with the result of H. Jakuszenkow, we simplify that distribution by one parameter, assuming $a=2$, i.e. taking

$$P(x;a,b,c,n,p)=\binom{n}{x}\frac{1}{\Gamma\left(\frac{p}{2}\right)}\sum_{k=0}^{n-x}\binom{n-x}{k}(-1)^k(bc^2)^{\frac{x+k}{2}}\Gamma\left[\frac{p+x+k}{2}\right]. \tag{6}$$

The gamma function occurring in (6) is next presented in the integral form for convenience of further transformations and we obtain

$$\Gamma\left[\frac{p+x+k}{2}\right]=\int_0^\infty\left(y^{\frac{1}{2}}\right)^{p+x+k-2}\exp\left(-y^{\frac{1}{2}}\right)^2dy=\Gamma.$$

Next by putting $y^{\frac{1}{2}}=t$, we have

$$\Gamma=2\int_0^\infty t^{p+x+k-1}e^{-t^2}dt.$$

Then we assume $n=\frac{\lambda}{c}$, where λ is a finite constant greater than zero and we calculate the limit of (6) when $n\to\infty$. We have in the first place

$$(6)=\frac{2b^{\frac{x}{2}}}{\Gamma\left(\frac{p}{2}\right)}\binom{n}{x}c^x\sum_{k=0}^{n-x}\binom{n-x}{k}c^k\int_0^\infty t^{p+x-1}e^{-t^2}(-t\sqrt{b})^kdt.$$

We notice that

$$\lim_{n\to\infty}\binom{n}{x}c^x = \frac{\lambda^x}{x!} \quad \text{and} \quad \lim_{n\to\infty}\binom{n-x}{k}c^k = \frac{\lambda^k}{k!}.$$

In consequence

$$\lim(6) = \frac{2b^{\frac{x}{2}}\lambda^x}{\Gamma\left(\frac{p}{2}\right)x!}\sum_{k=0}^{\infty}\int_0^{\infty} t^{p+x-1}e^{-t^2}\frac{(-\lambda t\sqrt{b})^k}{k!}dt = \frac{2b^{\frac{x}{2}}\lambda^x}{\Gamma\left(\frac{p}{2}\right)x!}\int_0^{\infty} t^{p+x-1}e^{-t^2}\sum_{k=0}^{\infty}\frac{(-\lambda t\sqrt{b})^k}{k!}dt =$$

$$\frac{2b^{\frac{x}{2}}\lambda^x}{\Gamma\left(\frac{p}{2}\right)x!}\int_0^{\infty} t^{p+x-1}\exp(-t^2 - \lambda t\sqrt{b})dt = \frac{2b^{\frac{x}{2}}\lambda^x}{\Gamma\left(\frac{p}{2}\right)x!} 2^{-\frac{p+x}{2}}\Gamma(p+x)\exp\left(\frac{b\lambda^2}{8}\right)D_{-p-x}\left(\lambda\sqrt{\frac{b}{2}}\right).$$

The last relation is obtained by the known formula (Ryżyk, Gradsztejn 1964, 3. 462; Рыжик, Градштейн 1971):

$$\int_0^{\infty} x^{\nu-1}\exp(-\beta x^2 - \gamma x)dx = (2\beta)^{-\frac{\nu}{2}}\Gamma(\nu)\exp\left(\frac{\gamma^2}{8\beta}\right)D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right).$$

$\beta, \nu > 0$, by putting: $\nu = p + x$, $\beta = 1$, $\gamma = \lambda\sqrt{b}$.

In the end, the limit of (6) is expressed by

$$\frac{2\lambda^x}{b^{\frac{p}{a}}\Gamma\left(\frac{p}{2}\right)x!}\left(\frac{2}{b}\right)^{-\frac{p+x}{2}}\Gamma(p+x)\exp\left(\frac{b\lambda^2}{8}\right)D_{-p-x}\left(\lambda\sqrt{\frac{b}{2}}\right),$$

and for $\lambda = 1$ we obtain

$$\frac{2}{b^{\frac{p}{a}}\Gamma\left(\frac{p}{2}\right)x!}\left(\frac{2}{b}\right)^{-\frac{p+x}{2}}\Gamma(p+x)\exp\left(\frac{b}{8}\right)D_{-p-x}\left(\sqrt{\frac{b}{2}}\right),$$

the same formula as given by Jakuszenkow (1973, formula 13, p. 71).

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WŁASNOŚĆ GRANICZNA ZŁOŻONEGO ROZKŁADU DWUMIANOWEGO Z UOGÓLNIONYM DWUPARAMETROWYM GAMMA

(Streszczenie)

W roku 1920 M. Greenwood i G. U. Yule przedstawili złożony rozkład Poisson – dwuparametrowy gamma i podali interesujące zastosowanie tego złożenia. W 1973 r. H. Jakuszenkow podała złożenie rozkładu Poissona z uogólnionym dwuparametrowym rozkładem gamma. W 1982 r. T. Gerstenkorn opublikował złożony rozkład dwumianowy z uogólnionym beta i w przejściu granicznym otrzymał złożony rozkład dwumianowy z uogólnionym trzyparametrowym gamma. Obecnie przedstawiona jest własność graniczna tegoż rozkładu, dająca twierdzenie H. Jakuszenkow, jeśli jeden z parametrów tego rozkładu $a = 2$, tzn. przy ograniczeniu się do szczególnego dwuparametrowego uogólnionego rozkładu gamma.