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THE AVERAGE PRICE DYNAMICS AND INDEXES OF PRICE DYNAMICS – DISCRETE TIME STOCHASTIC MODEL

Abstract. In this paper we define two indexes of the average price dynamics in a discrete time stochastic model. Several properties of these indexes are proven, the other are presented by examples. In particular, it is shown that one of the indexes is a martingale provides the prices of products form a (vector) martingale. In addition it is also shown that only one definition satisfies all given postulates. We compare this definition with price indexes.

Key words: Törnqvist index, Laspeyres index, Paasche index, Fisher ideal index, Lexis index, martingale.

1. INTRODUCTION

A producer whose enterprise produces a certain group of articles must estimate – in advance of at least one period – all prices and quantities of the products. His success depends on the precise estimation. Certainly, the producer uses lots of methods to calculate the efficiency of an investment (cf. Pielichaty, Poszwa 1999). But he also uses statistical indexes to discover the price and quantity dynamics (Zając 1994). The indexes make it possible to compare two periods of production. The production in next periods depends on the former growth of prices, quantities and the profitability.

The contemporary economy makes use of lots of statistical indexes to calculate the above mentioned dynamics. And for example: Laspeyres and Paasche indexes have been known since 19th century (cf. Diewert 1976, Shell 1998). Depending on the type of an economic problem we may also use one of the following indexes: Fisher (1972) ideal index, Törnqvist (1936) index, Lexis index and other indexes (cf. Zając 1994). Indexes are also used

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to calculate national income (cf. Moutlon 1999, Seskin 1998). Balk (1995) wrote about axiomatic price index theory, Diewert (1978) showed that the Törnqvist index and Fisher ideal index approximate each other. But it is really hard to indicate the best one of the statistical indexes (Dumagan 2002). The choice of index depends on the information we want to get. If we are interested in dynamics of money in time we should use Fisher or Lexis indexes (Zając 1994). Unfortunately all indexes (next we will consider only price indexes) take into account no event from the inside of the considered time interval. Indexes compare two periods: base period T_1 and testing period T_2. No index depends on periods $T_1+1, T_1+2, ..., T_2-1$. So if we want to consider also the omitted periods we should use a different formula. The arithmetic or geometric mean of values $I[T_1, T_1+1]$, $I[T_1+1, T_1+2]$, ..., $I[T_2-1, T_2]$ (where I denotes some price index) seems to be unsuitable here.

In this paper we propose a definition of the average price dynamics taking into account all the time interval $[T_1, T_2]$. We present several properties of this new index. We present also an alternative definition but only one definition – as it will be proven in this paper – will satisfy our postulates. In the last part of this paper we compare the new (proper) definition and statistical indexes in case when the time interval consists of two periods.

2. PRICE DYNAMICS IN A DISCRETE TIME STOCHASIC MODEL

Consider an enterprise which produces and sells a group of N products ($N > 1$). Let $(\Omega, \Im, P)$ be a complete probability space. We observe at discrete time moments $t = 0, 1, 2, ...$ the following random variables on the space $(\Omega, \Im, P)$:

$p_i(t)$ – value (price) of i-th product at time t,

$q_i(t)$ – quantity of i-th product at time t,

$S_i(t)$ – income from selling of i-th product at time t it means:

$$S_i(t) = p_i(t) \times q_i(t), \quad i = 1, 2, 3, ..., N, \quad t = 0, 1, 2, ... \tag{1}$$

The value shares of the commodities at time t are defined by

$$S_i^*(t) = \frac{S_i(t)}{\sum_{i=1}^{N} S_i(t)}. \tag{2}$$

It is obvious that

$$\sum_{i=1}^{N} S_i^*(t) = 1 \quad \text{for each} \quad t = 0, 1, 2, ... \tag{3}$$

In addition we assume that there is no problem in selling of each product.

Let $\mathbf{F} = \{\mathfrak{I}_0, \mathfrak{I}_1, \mathfrak{I}_2, ...\}$ be a filtration, i.e. each $\mathfrak{I}_t$ is an σ-algebra of subsets of Ω with $\mathfrak{I}_0 \subseteq \mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq ... \subseteq \mathfrak{I}$. Without loss of generality, we assume $\mathfrak{I}_0 = \{\emptyset, \Omega\}$. The filtration $\mathbf{F}$ describes how information (about capital market) is revealed to the producer. We assume that each random variable $p_i(t)$, $q_i(t)$ is adapted to $\mathbf{F}$, which means that $p_i(t)$ and $q_i(t)$ are $\mathfrak{I}_t$ – measurable (for each i, t). As results from the above we are allowing the investor to produce and sell components after the market observation.

Here and subsequently, the symbol $X = Y (X < Y)$ means that the random variables X, Y are defined on $(\Omega, \mathfrak{I}, P)$ and $P(X = Y) = 1$ (resp. $P(X < Y) = 1$.

3. DEFINITION OF THE AVERAGE PRICE DYNAMICS

Let $[T_1, T_2]$ be the time interval of monitoring our enterprise $(T_1, T_2 \in N \cup \{0\}, T_1 < T_2)$.

Our definition of the average price dynamics in discrete-time model (in the above time period) is as follows

$$I^P[T_1, T_2] = \sum_{i=1}^{N} \left[\frac{\sum_{u=T_1}^{T_2} p_i(u) q_i(u)}{\sum_{k=1}^{N} \sum_{u=T_1}^{T_2} p_i(u) q_i(u)} \cdot \sum_{u=T_1+1}^{T_2} \frac{1}{2} \cdot \left(\frac{p_i(u-1) q_i(u-1)}{\sum_{y=T_1+1}^{T_2} p_i(y-1) q_i(y-1)} + \frac{p_i(u) q_i(u)}{\sum_{y=T_1+1}^{T_2} p_i(y) q_i(y)} \right) \cdot \frac{p_i(u)}{p_i(u-1)} \right]. \tag{4}$$

Let us signify additionally

$$\alpha_i^u[T_1, T_2] \equiv \alpha_i^u = \frac{1}{2} \left(\frac{p_i(u-1) q_i(u-1)}{\sum_{y=T_1+1}^{T_2} p_i(y-1) q_i(y-1)} + \frac{p_i(u) q_i(u)}{\sum_{y=T_1+1}^{T_2} p_i(y) q_i(y)} \right) \tag{5}$$

for $i = 1, 2, ..., N, \quad u = T_1 + 1, T_1 + 1, ..., T_2$,

$$\beta_i[T_1, T_2] \equiv \beta_i = \frac{\sum\limits_{u=T_1}^{T_2} p_i(u)q_i(u)}{\sum\limits_{k=1}^{N}\sum\limits_{u=T_1}^{T_2} p_k(u)q_k(u)}, \quad i = 1, 2, ..., N, \tag{6}$$

and denote the relative growth of price of i-th product within the time period $[u-1, u]$ by

$$p_i^w(u) = \frac{p_i(u) - p_i(u-1)}{p_i(u-1)}, \quad u = T_1 + 1, T_1 + 1, ..., T_2. \tag{7}$$

Now the definition (4) is as follows

$$I^p[T_1, T_2] = \sum_{i=1}^{N} \beta_i \cdot \sum_{u=T_1+1}^{T_2} \alpha_i^u \cdot \frac{p_i(u)}{p_i(u-1)}. \tag{8}$$

Using (7) we obtain

$$I^p[\mathrm{T}_1, \mathrm{T}_2] = \sum_{i=1}^{N} \beta_i \cdot \sum_{u=T_1+1}^{T_2} \alpha_i^u \cdot (1 + p_i^w(u)), \tag{9}$$

where β_i informs the producer how profitable is i-th product on a global scale, and α_i^u informs the producer how important is u-th moment in case of i-th product.

4. THE INTERPRETATION OF THE AVERAGE PRICE DYNAMICS

Notice that from (5) we have for any t-th product

$$\sum_{u=T_1+1}^{T_2} \alpha_i^u = 1, \quad \alpha_i^u \geqslant 0. \tag{10}$$

Let us signify

$$\chi_i = \sum_{u=T_1+1}^{T_2} \alpha_i^u(1 + p_i^w(u)). \tag{11}$$

By (10) we have the following interpretation of χ_i: it equals the average growth of price of i-th product observed during the time period $[T_1, T_2]$. It means that treating the moment of observation as a random variable U, we get

$$\chi_{I=i} = \mathrm{E}_{U/I}[(1 + p_I^*(U))/I = i], \tag{12}$$

where:

I – is a random variable describing the choice of the moment of observation,

U – as above, with conditional distribution as follows

$$P(U = u/I = i)\alpha_i^u, \quad i = 1, 2, ..., N, \quad u = T_1 + 1, T_1 + 1, ..., T_2. \tag{13}$$

In addition notice that

$$\sum_{i=1}^{N} \beta_i = 1, \quad \forall i\ \beta_i \geqslant 0, \tag{14}$$

so by (9) we have

$$I^p[T_1, T_2] = E_I[E_{U/I}[(1 + p_I^w(U))/I]], \tag{15}$$

where random variable I has a distribution as follows

$$P(I = i) = \beta_i, \quad \mathrm{i} = 1, 2, ..., N. \tag{16}$$

Formula (15) can be written as

$$I^p[T_1, T_2] = E_I\left[E_{U/I}\left[\frac{p_I(U)}{p_I(U-1)}/I\right]\right]. \tag{17}$$

This means that if we repeat the procedure of "choosing" the product number n_0 and if the price of this product (p_0) comes – sequentially – from different moments t, then the average price of the product at the time T_2 will be some random variable P. By (17) and well known formula: $E[E[\mathrm{X}/\mathrm{Y}] = EX$ we get

$$E[P] = p_0 \cdot I^p[T_1, T_2], \tag{18}$$

which means

$$I^p[T_1, T_2] = E\left[\frac{P}{p_0}\right]. \tag{19}$$

5. BASIC PROPERTIES OF THE AVERAGE PRICE DYNAMICS

Next we formulate a list of properties of the average price dynamics defined by (4). Since the proofs are simple they will be omitted here. Before the presentation of the properties notice that if the enterprise produces only one product n_0 then our definition is as follows

$$I^p_{n_0}[T_1, T_2] = \sum_{u=T_1+1}^{T_2} \alpha^u_{n_0}(1 + p^w_{n_0}(u)), \tag{20}$$

where

$$\alpha^u_{n_0} = \frac{1}{2}\left(\frac{p_{n_0}(u-1)q_{n_0}(u-1)}{\sum_{y=T_1+1}^{T_2} p_{n_0}(y-1)q_{n_0}(y-1)} + \frac{p_{n_0}(u)q_{n_0}(u)}{\sum_{y=T_1+1}^{T_2} p_{n_0}(y)q_{n_0}(y)}\right). \tag{21}$$

Property 1. Certainly we have

$$\forall T_1 \leqslant t \leqslant T_2 \quad p_i(t) = c_i \Rightarrow I^p[T_1, T_2] = 1. \tag{22}$$

This property has almost axiomatic character. It says that in case the price of each product is constant during the time interval $[T_1, T_2]$ then the index defined by (4) must absolutely inform us about that situation.

Property 2. Assume that all products are infinitely divisible. If for some $k \in \{1, 2, 3, ..., N\}$ holds

$$\max_{i \in \{1,2,...,N\}\setminus\{k\}} S^*_i(u) \leqslant \theta \cdot S^*_k(u), \quad \text{for each} \quad u = T_1, ..., T_2 \tag{23}$$

then we get

$$\lim_{\theta \to 0} I^p[T_1, T_2] = I^p_k[T_1, T_2]. \tag{24}$$

This property says that the influence of unprofitable products on the average price dynamics is asymptotically negligible.

Property 3. If all prices grew at about the same m% then the value of our average price dynamics would not change. Similarly, if all quantities grew at about the same s% then the index defined by (4) would have the same value before and after the growth.

Property 4. With probability one we have

$$\min_k I^p_k[T_1, T_2] \leqslant I^p[T_1, T_2] \leqslant \max_k I^p_k[\mathrm{T}_1, \mathrm{T}_2]. \tag{25}$$

Property 4 means that the average price dynamics is not greater than the highest price dynamics of a single product, and not smaller than the smallest price dynamics of a single product.

Property 5. The following implication is a more general version of Property 1

$$(\forall i \in \{1, 2, ..., N\} \quad I_i^p[T_1, T_2] \approx 1) \Rightarrow I^p[T_1, T_2] \approx 1. \tag{26}$$

When the average price dynamics of each single product is approximately constant then the average price dynamics (on a global scale) is also approximately constant.

6. EXAMPLES

Example 1. Let us consider an enterprise producing $N = 5$ products. During the last $T_2 = 4$ periods production was persented in Table 1.

Table 1. Production in the first enterprise

Product No.	Price				Quantity			
	period 1	period 2	period 3	period 4	period 1	period 2	period 3	period 4
1	15	20	25	18	100	110	90	100
2	15	20	25	19	100	120	110	100
3	15	20	24	16	100	80	110	100
4	15	20	24	15	100	90	110	100
5	15	14	12	5	100	100	90	80

Source: own data.

We can see that in case of the first four products the situation was similar. The quantity was near 100 units and the growth of prices of the products had the same trend. At the beginning prices of the first four products grew by about $\approx 33\%$ (periods 1, 2 and 2, 3) and after that prices decreased by about $\approx 47\%$ (periods 3, 4). The price of the fifth product had the smaller growth during the first three periods but a period no. 4 was completely different. Then the price had the highest decrease ($\approx 58\%$). Using statistical price indexes to compare periods 1 and 4 we get: Paasche index (0.99), Laspeyres index (0.973), Fisher index (0.986), Lexis index (0.986), Törnqvist index (0.961). Indexes inform us about the decrease of prices (1–4%). In case we want to consider every event from the period no. 2 or 3 we should use I^p formula. After calculation we obtain $I^p = 1.038$, so this index informs about almost 4% growth of prices. Comparing periods 1 and 4 (first four products had identical quantity) we can see that prices of the first four products grew up a little. So the decrease in the price of the fifth product had a huge influence on all price indexes. Considering periods 2 and 3 we can see that the growth of prices was high enough to obtain the average price dynamics $I^p = 1.038$. So the average growth of prices after four periods was almost 4%.

Example 2. Let us consider an enterprise producing $N = 5$ products. During the last $T_2 = 4$ periods production was presented in Table 2.

Table 2. Production in the second enterprise

Product No.	Price				Quantity			
	period 1	period 2	period 3	period 4	period 1	period 2	period 3	period 4
1	15	20	22	17	100	110	90	100
2	100	105	111	105	17	20	20	25
3	515	520	525	530	1000	900	1000	1040
4	1150	1170	1170	1100	100	90	110	100
5	315	320	330	315	200	220	230	225

Source: own data.

Comparing all adjacent periods we obtain the following results (Table 3).

Table 3. Considered indexes for the second enterprise

Index \ Compared periods	[1, 2]	[2, 3]	[3, 4]
Paasche	1.0125	1.0104	0.9921
Laspeyres	1.0123	1.0108	0.9908
Fisher	1.0124	1.0106	0.9914
Lexis	1.0124	1.0106	0.9915
Törnqvist	1.0191	1.0107	0.9578
I^p	1.0125	1.0137	0.9977

Source: calculations based on Table 2.

Conclusion. If for all adjacent periods $[t, t+1]$ the following formula is true

$$\forall i \in \{1, 2, ..., N\} \quad S_i(t) \approx S_i(t-1) \tag{27}$$

and prices of all products do not change the value rapidly, then statistical indexes and I^p index accept very similar values. Our mathematical proof of this conclusion is presented in (Section 9).

Example 3. It seems interesting to observe all indexes when prices and quantities (coming from different periods) differ strongly. When the difference is small – as in the previous example – indexes accept very similar values. Now we are going to observe the situation when some prices and quantities have the value described by parameters a and b we will manipulate both parameters.

But firstly let us consider the enterprise producing $N = 5$ products. During the last two periods (t and $t+1$) production was presented in Table 4.

Table 4. Production in the third enterprise

Product No.	Price		Quantity	
	period: t	period: $t+1$	period: t	period: $t+1$
1	225	223	90	100
2	25	23	110	100
3	44	50	210	200
4	104	110	50	60
5	70	80	90	100

Source: own data.

We can see that prices and quantities of the majority of products differed. For example, in case of products no. 1 and no. 3 we had almost five-time the difference of prices. In case of products no. 3 and no. 4 we noticed almost four-time the difference of quantities. After calculation we get the following values of statistical price indexes (resp. Lexis index, Laspeyres index, Paasche index, Fischer index, Törnqvist index):

$$I^P_{Lex} = 1.0488, \quad I^P_L = 1.0496, \quad I^P_{Pa} = 1.04804, \quad I^P_F = 1.0488, \quad I^P_T = 1.0406.$$

And the average price dynamics $I^P = 1.0511$.

As in the first case all values do not differ much again. Next we are going to manipulate the values of some prices and quantities using the above mentioned parameters a and b. The maximum relative change of any value will equal even 60–100%.

Let us use the following signification:

$$\Delta I = I^P - I^P_{Lex},\ \Delta I_2 = I^P - I^P_L,\ \Delta I_3 = I^P - I^P_{Pa},\ \Delta I_4 = I^P - I^P_F,\ \Delta I_5 = I^P_{Lex} - I^P_F.$$

For $-30 \leqslant a,\ b \leqslant 30$ we make computer simulations described by Tables 5–8 and Figures $1-4$.

Example 3.1.

Table 5. Production depends on parameters *a*, *b*

Product No.	Price		Quantity	
	period: t	period: $t+1$	period: t	period: $t+1$
1	225	223	90	100
2	25	23	110	100
3	44	50	210	200
4	104	110	50	60
5	70	$80+a$	90	$100+b$

Source: own data.

We get the following dependences:

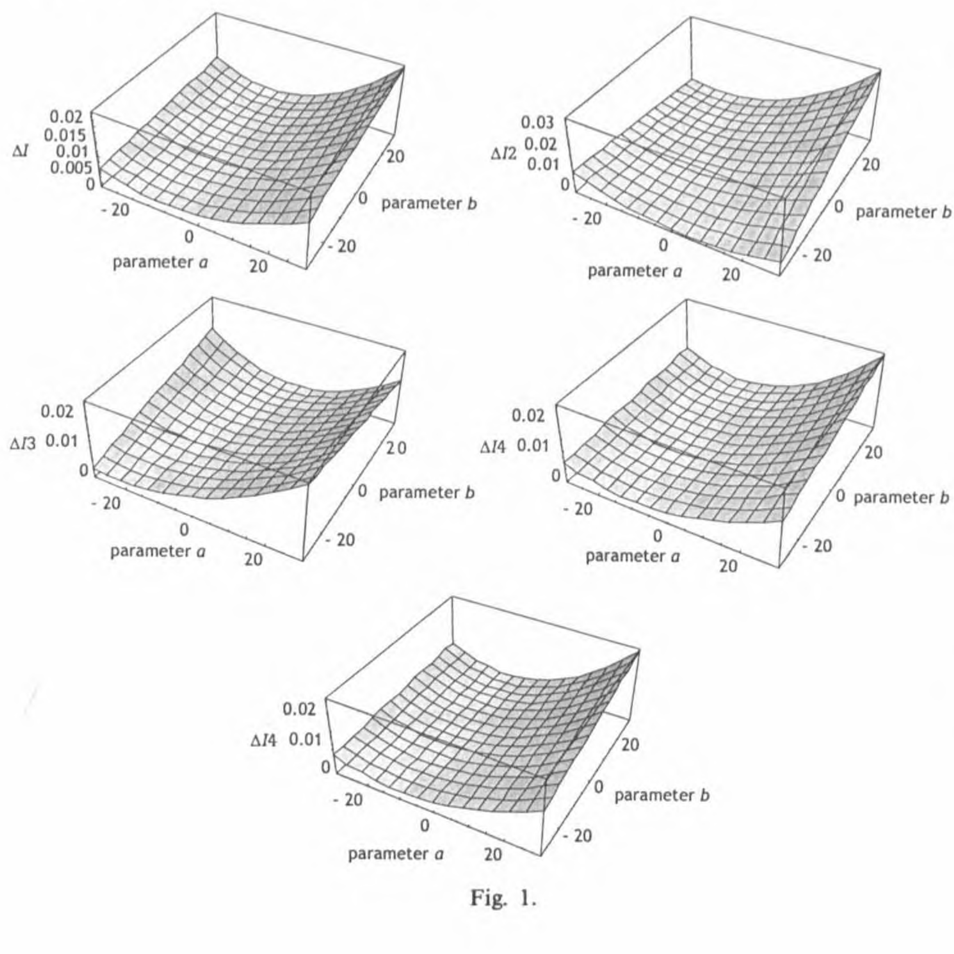

Fig. 1.

Example 3.2.

Table 6. Production depends on parameters a, b

Product No.	Price		Quantity	
	period: t	period: $t+1$	period: t	period: $t+1$
1	225	223	90	100
2	25	23	110	100
3	44	50	210	200
4	104	110	50	60
5	70	$80+a$	$90+b$	100

Source: own data.

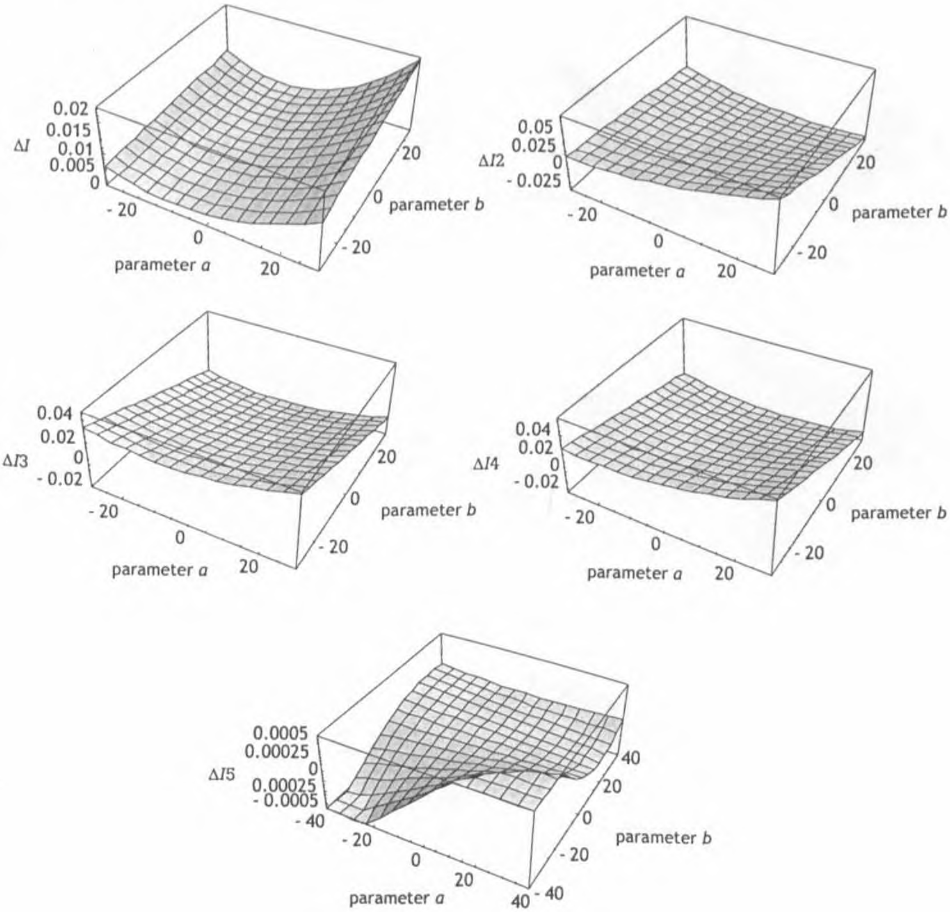

Fig. 2.

Example 3.2.

Table 7. Production depends on parameters a, b

Product No.	Price		Quantity	
	period: t	period: $t+1$	period: t	period: $t+1$
1	225	223	90	100
2	25	23	110	100
3	44	50	210	200
4	104	$110+a$	50	60
5	70	80	90	$100+b$

Source: own data.

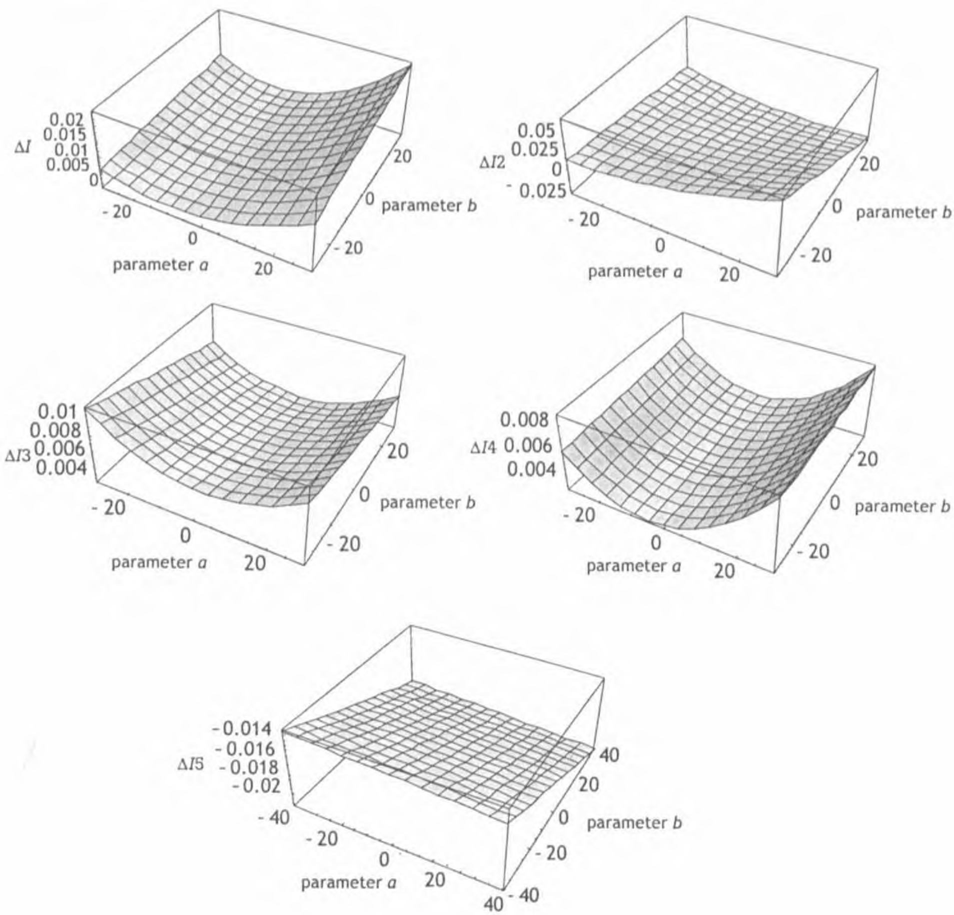

Fig. 3.

Example 3.3.

Table 8. Production depends on parameters a, b

Product No.	Price		Quantity	
	period: t	period: $t+1$	period: t	period: $t+1$
1	225	223	90	100
2	25	23	110	100
3	44	50	210	200
4	$104+a$	110	50	60
5	70	80	90	$100+b$

Source: own data.

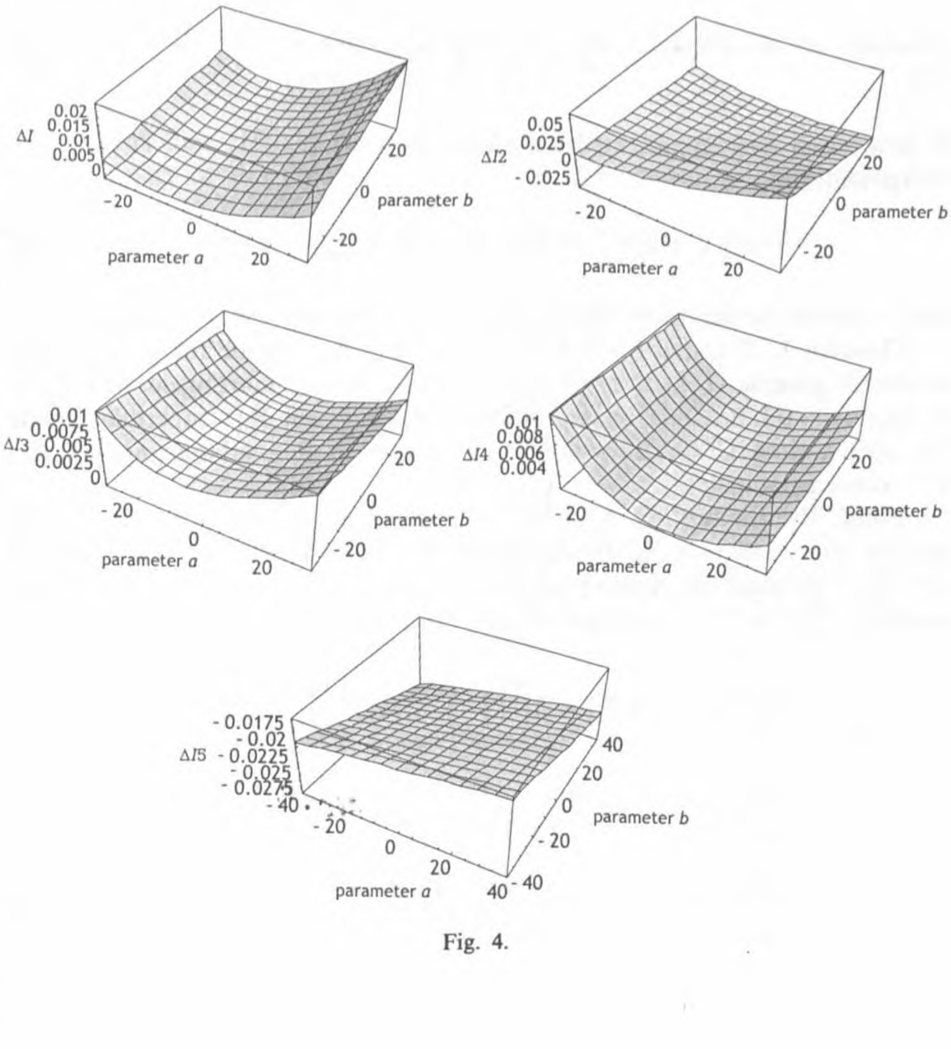

Fig. 4.

Conclusion. The bigger differences can be seen among the prices and among the quantities (comparing the adjacent periods) the larger differences can be observed among all statistical price indexes and I^P index. But let us notice that the difference between I^P index and other statistical indexes is often smaller than the differences among statistical price indexes (cf. $\Delta I_5(a, b)$ Examples 3.2 and 3.3). If we manipulated other products we would get similar conclusions.

7. AN ALTERNATIVE DEFINITION

At the first sight the following definition of the average price dynamics seems to be better than the definition (4):

$$\hat{I}^P[T_1, T_2] = \prod_{u=T_1}^{T_2-1} \sum_{i=1}^{N} S_i^*(u) \cdot \frac{p_i(u-1)}{p_i(u)} \tag{28}$$

It is a much less complicated definition than (4).

In addition

$$\forall T_1 < T < T_2 \quad \hat{I}^P[T_1, T_2] = \hat{I}^P[T_1, t] \cdot \hat{I}^P[t, T_2], \tag{29}$$

and – wwhat is the most interesting – the following theorem is true:

Theorem 1. If $\{p_i(t) : t = 0, 1, 2, ...]$ is an **F** – martingale for each i, them stochastic process $\{\hat{I}^P[0, t] : t = 0, 1, 2, ...]$ is also an **F** – martingale. Moreover, if $\{p_i(t) : t = 0\, 1, 2, ...\}$ is an **F** – submartingale (resp. **F** – supermartingale) for each i, then $\{\hat{I}^P[0, t] : t = 0, 1, 2, ...\}$ is also an **F** – submartingale (resp. **F** – supermartingale).

Proof. By assumption both random variables $p_i(t)$, $q_i(t)$ are $\Im_t$ – measurable (for each i, t). So by definition (28) the random variable $\hat{I}^P[0, t]$ is also $\Im_t$ – measurable. Analogically we can prove the fact that each random variable $S_i^*(t)$ is $\Im_t$ – measurable too. Notice that

$$E(\hat{I}^P[0, t+1]/\Im_t) = E\left(\prod_{u=0}^{t} \sum_{i=0}^{N} S_i^*(u) \cdot \frac{p_i(u+1)}{p_i(u)} / \Im_t\right) =$$

$$= E\left[\left(\prod_{u=0}^{t-1} \sum_{i=1}^{N} S_i^*(u) \cdot \frac{p_i(u+1)}{p_i(u)}\right) \cdot \left(\sum_{i=1}^{N} S_i^*(t) \cdot \frac{p_i(t+1)}{p_i(t)}\right) / \Im_t\right] =$$

$$= \left(\prod_{u=0}^{t-1} \sum_{i=1}^{N} S_i^*(u) \cdot \frac{p_i(u+1)}{p_i(u)}\right) \cdot E\left[\left(\sum_{i=1}^{N} S_i^*(t) \cdot \frac{p_i(t+1)}{p_i(t)}\right) / \Im_t\right] \tag{30}$$

By assumption process $[p_i(t): t = 0, 1, 2, ...]$ is an $\mathbf{F}$ – martingale for each i. Because of the fact $S_i^*(t)$ is $\Im_t$ – measurable we can write.

$$E\left[\sum_{i=1}^{N} S_i^*(u)\cdot\frac{p_i(t+1)}{p_i(t)}/\Im_t\right] = \sum_{i=1}^{N} E\left[\left(S_i^*(t)\frac{p_i(t+1)}{p_i(t)}\right)/\Im_t\right] =$$

$$= \sum_{i=1}^{N}\frac{S_i^*(t)}{p_i(t)}\cdot E[p_i(t-1)/\Im_t] = \sum_{i=1}^{N}\frac{S_i^*(t)}{p_i(t)}\cdot p_i(t) = \sum_{i=1}^{N} S_i^*(t) = 1. \tag{31}$$

Summing up by (30) and (31) we get

$$E[\hat{I}^P[0, t-1]/\Im_t] = \hat{I}^P[0, t] \tag{32}$$

It is obvious that.

$E[\hat{I}^P[0,\mathrm{t}]] < \infty$ (we cannot imagine a different situation on the market). (33)

This means that $\{\hat{I}^P[0, t]: t = 0, 1, 2, ...\}$ is an $\mathbf{F}$ – martingale. The proof of the first part of the theorem is completed. The proof of the second part is analogous so it will be omitted.

Remark 1. Unfortunately, if for some $k \in \{1, 2, 3, ..., N\}$ holds

$$\max_{i\in\{1,2,...,N\}\setminus\{k\}} S_i^*(u) \leqslant \theta\cdot S_i^*(u), \quad \text{for all} \quad u = T_1, ..., T_2$$

then

$$\lim_{\theta\to 0} \hat{I}^P[T_1, T_2] = \prod_{u=T_1}^{T_2-1}\frac{p_k(u+1)}{p_k(u)} = \frac{p_k(T_2)}{p_k(T_1)}.$$

So if one of the products dominated (strongly) with regard to incomes, then $\hat{I}^P$ index does not depend on random variables $p_k(T_1+1), ..., p_k(T_2-1)$. So in this case $\hat{I}^P$ index does not inform us about the price dynamics during the time interval $[T_1, T_2]$. This index does not take into account periods $T_1+1, ..., T_2-1$ so in our opinion I^P definition is better.

8. COMPARISON OF THE I^P DEFINITION AND STATISTICAL PRICE INDEXES

Let us denote by

I_T^P – Törnqvist price index, I_F^P – Fisher ideal price index.

Consider two adjacent periods t, $t+1$. By (4) we get in this case

$$I^P[t, t+1] \equiv I^P = \sum_{t=1}^{N} \left(\frac{p_i(t)q_i(t) + p_i(t-1)q_i(t-1)}{\sum_{i=1}^{N} p_i(t)q_i(t) + \sum_{i=1}^{N} p_i(t-1)q_i(t-1)} \right) \cdot \frac{p_i(t+1)}{p_i(t)} \tag{34}$$

Let us assume that

$$\forall_i \, S_i(t) \approx S_i(t-1), \quad q_i(t) \approx q_i(t-1). \tag{35}$$

So we assume that incomes and quantities of every product have similar values in the considered periods. Diewert (1978) proved that under this assumption the following approximation is true:

$$\ln I_T^P \approx I_F^P - 1 \tag{36}$$

Using the following implication

$$a_i \approx b_i \Rightarrow \frac{a_i - b_i}{\sum_{i \leqslant M} (a_i + b_i)} \approx \frac{1}{2} \left(\frac{a_i}{\sum_{i \leqslant M} a_i} + \frac{b_i}{\sum_{i \leqslant M} b_i} \right), \tag{37}$$

from (34) we get

$$I^P[t, t+1] \approx \sum_{i=1}^{N} \frac{1}{2} \left(\frac{p_i(t)q_i(t)}{\sum_{i=1}^{N} p_i(t)q_i(t)} + \frac{p_i(t+1)q_i(t+1)}{\sum_{i=1}^{N} p_i(t+1)q_i(t+1)} \right) \cdot \frac{p_i(t+1)}{p_i(t)} =$$

$$= \sum_{i=1}^{N} \frac{1}{2} (S_i^*(t) + S_i^*(t+1) \cdot \frac{p_i(t+1)}{p_i(t)}. \tag{38}$$

We know that (a dependence on time is omitted in the below formulas)

$$I_T^P = \prod_{i=1}^{N} \left(\frac{p_i(t+1)}{p_i(t)} \right)^{\frac{S_i^*(t) + S_i^*(t+1)}{2}}. \tag{39}$$

From (39) we get

$$\ln I_T^P = \sum_{i=1}^{N} \left(\frac{S_i^*(t) + S_i^*(t+1)}{2} \right) \cdot \ln(1 + p_i^w(t)). \tag{40}$$

Because of the fact that for small values of x we get (from Taylor's theory)

$$\ln(1+x) \approx x, \tag{41}$$

we obtain by (35)

$$\ln(1+p_i^w(t)) \approx p_i^w(t) = \frac{p_i(t+1)q_i(t)}{p_i(t)}. \tag{42}$$

Using (41) and (40) we get

$$\ln I_T^P \approx \sum_{i=1}^{N} \left(\frac{S_i^*(t) - S_i^*(t+1)}{2} \cdot \frac{p_i(t+1)}{p_i(t)} - 1 \right). \tag{43}$$

So by (43) and (38) we have

$$\ln I_T^P \approx I^P - 1. \tag{44}$$

Now, using (36), we obtain

$$I^P \approx I_F^P. \tag{45}$$

Finally, the well known Bortkiewicz formula (e.g. Zając (1994) allows us to state additionally (by assumption (35)).

$$I^P \approx I_L^P \approx I_{Pa}^P. \tag{46}$$

where I_L^P, I_{Pa}^P denote Laspeyres and Paasche price indexes respectively.

9. CONCLUSIONS

All presented properties of the average price dynamics (I^P) prove the proper construction of the definition (4). This index differs from any other statistical price index not only because of the structure but also the application. I^P index has to take into account any event in production within the considered time interval $[T_1\, T_3]$. But we have also proved that in case the time interval consists of only two periods, all indexes and our index approximate each other. An additional argument is that the I^P index has a specific construction so it is easy to check which period has the biggest influence on the average price dynamics.

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PRZECIĘTNA DYNAMIKA CEN PRODUKTÓW A INDEKSY AGREGATOWE CEN – MATEMATYCZNY MODEL STOCHASTYCZNY Z CZASEM DYSKRETNYM

(Streszczenie)

W artykule zaprezentowano dwie definicje indeksu przeciętnej dynamiki cen produktów w modelu stochastycznym z czasem dyskretnym. Przedstawiono ich podstawowe własności; niektóre z nich zostały udowodnione, inne – poparte przykładami. Ponadto udowodniono, iż jedna z definicji stanowi martyngał, jeśli tylko procesy cen poszczególnych produktów również są martyngałami. Jednocześnie okazało się, iż tylko jedna definicja posiada wszystkie wymagane własności. Na końcu niniejszego artykułu dokonano porównania rekomendowanej definicji z klasycznymi agregatowymi indeksami cen. Okazało się, iż w przypadku gdy rozważany interwał czasowy zawiera dwa okresy produkcyjne, to przy pewnych dodatkowych założeniach proponowana definicja daje się aproksymować klasycznymi indeksami Fishera, Tõrnqvista i innymi.