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MULTIVALUED STOCHASTIC PROCESSES

Abstract. Multivalued random variables and stochastic processes can be use in integral geometry, mathematical economics or stochastic optimization. In the study of multivalued stochastic processes the some clue problem is the question of existing the vector-valued selection processes. Using the methods of selection operators it is possible to show the existence of convergence in distribution selections and stationary selections for multivalued stochastic processes.

Key words: mutivalued random variable, mutivalued stochastic processes.

1. INTRODUCTION

We present a concept of selection operators for multivalued random variables. For multivalued stochastic processes the some important problem is the question of existing the vector-valued selection processes. In this paper we continue our work on properties of multivalued random variables (Trzpiot 1995a, b, c, 1999, 2002). First two sections contain basic definitions, next characterizations of identically distributed multivalued random variables and the selection problem of multivalued random variables converging in distribution. We show the existence of convergence in distribution selections and stationary selections for multivalued stochastic processes.

2. MULTIVALUED RANDOM VARIABLE

Given a probability measure space (Ω, A, μ) random variable in classical definition is a mapping from Ω to $\mathbf{R}$. Multivalued random variable is a mapping from Ω to all closed subset of X.

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We have a real Banach space X with metric d. For any nonempty and closed sets $A, B \subset X$ we define the Hausdorff distance $h(A, B)$ of A and B.

Definition 1. *The excess* for two nonempty and closed sets be defined by

$$e(A, B) = \sup_{x \in A} d(x, B), \quad \text{where} \quad d(x, B) = \inf_{y \in B} \| x - y \| \tag{1}$$

the Hausdorff distance of A and B is given by

$$h(A, B) = \max \{e(A, B), e(B, A)\}, \tag{2}$$

the norm $\| A \|$ of set A we get as

$$\| A \| = h(A, \{0\}) = \sup_{x \in A} \| x \| \tag{3}$$

The set of all nonempty and closed subsets of X is a metric space with the Hausdorff distance. The set of all nonempty and compact subsets of X is a complete, separable metric space with the metric h.

Definition 2. A *multivalued function* $\varphi : \Omega \to 2^X$ with nonempty and closed values, is said to be (weakly) *measurable* if φ satisfies the following equivalent conditions:

- $\varphi^{-1}(C) = \{\omega \in \Omega : \varphi(\omega) \cap G \neq \emptyset\} \in A$ for every G open subset of X,
- $d(x, \varphi(\omega))$ is measurable in ω for every $x \in X$,
- there exists a sequence $\{f_n\}$ of measurable functions $\{f_n\} : \Omega \to X$ such that $\omega(\omega) = c1 \{f_n(\omega)\}$ for all $\omega \in \Omega$.

Definition 3. A measurable multivalued function $\varphi : \Omega \to 2^X$ with nonempty and closed values is called a *multivalued random variable*.

A multivalued function φ is called strongly measurable, if there exist a sequence $\{\varphi_n\}$ of simple functions (measurable functions having a finite number of values in 2^X), such that $h(\varphi_n(\omega), \varphi(\omega)) \to 0$ a.e.

Since set of all nonempty and compact (or convex and compact) subsets of X is a complete separable metric space with the metric h, so multifunction $\varphi : \Omega \to 2^X$ is measurable if and only if is strongly measurable. This is equivalent to the Borel measurability of φ.

Let $K(X)$ denote all nonempty and closed subsets of X. As the σ-field on $K(X)$, we get the σ-field generated by $\varphi^{-1}(G) = \{\omega \in \Omega : \varphi(\omega) \cap G \neq \emptyset\}$, for every open subset G of X. The smallest σ-algebra containing these $\varphi^{-1}(G)$ we de noted by $A\varphi$

1. Two multifunctions φ and ψ are independent if $A\varphi$ and $A\psi$ are independent.

2. Two multifunctions φ and ψ are identically distributed if $\mu(\varphi^{-1}(C)) = \mu(\psi^{-1}(C))$ for all closed $C \subset X$.

Definition 4. We say that a sequence of multivalued random variables $\varphi_n : \Omega \rightarrow 2^{K(X)}$ is independent if so is $\{\varphi_n\}$ considered as measurable functions from (Ω, A, μ) to $(K(X), G)$.

Definition 5. Two multivalued random variables $\varphi \psi : \Omega \rightarrow 2^{K(X)}$ are identically distributed if $\varphi(\omega) = \psi(\omega)$ a.e.

Particularly for φ_n with compact values independence (identical distributedness) of $\{\varphi_n\}$ coincides with that considered as Borel measurable functions to all nonempty, compact subsets of X.

Definition 6. *A selection* of the measurable multifunction $\varphi : \Omega \rightarrow 2^X$ is a measurable function $f : \Omega \rightarrow X$, such that $f(\omega) \in \varphi(\omega)$ for all $\omega \in \Omega$.

Let $\varphi, \psi : \Omega \rightarrow 2^{K(X)}$ be two multivalued random variables, we define the following operation (Castaing, Valadier 1997):

- $(\varphi \cup \psi)(\omega) = cl(\varphi(\omega) + \psi(\omega)), \quad \omega \in \Omega.$
- for a measurable real-valued function g:

$$(g\varphi)(\omega) = g(\omega)\varphi(\omega), \quad \omega \in \Omega.$$

- $(\overline{co}\,\varphi)(\omega) = \overline{co}\,\varphi(\omega), \quad \omega \in \Omega,$

($\overline{co}$-denote the closed convex hull).

3. MEAN OF MULTIVALUED RANDOM VARIABLE

Let $L^p(\Omega, A)$, for $1 \leqslant p \leqslant \infty$, denote the X - valued L^p - space. We introduce the multivalued L^p space.

Definition 7. The multivalued space $L^p[\Omega, K(X)]$, for $1 \leqslant p \leqslant \infty$ denote the space of all measurable multivalued functions $\varphi : \Omega \rightarrow 2^{K(X)}$, such that $\|\varphi\| = \|\varphi(\cdot)\|$ is in L^p.

Then $L^p[\Omega, K(X)]$ becomes a complete metric space with the metric H_p given by

$$H_p(\varphi, \psi) = \{\textstyle\int_\Omega h(\varphi(\omega), \psi(\omega)^p d\mu\}^{1/p}, \quad \text{for} \quad 1 \leqslant p \leqslant \infty$$
$$H_\infty(\varphi, \psi) = \operatorname*{ess\,sup}_{\omega \in \Omega} h(\varphi(\omega), \psi(\omega), \tag{4}$$

where φ and ψ are considered to be identical if $\varphi(\omega) = \psi(\omega)$ a.e.

We can define similarly other L^p space for set of different subsets of X (convex and closed, weakly compact or compact). We denote by $L^p[\Omega, K(X)]$ the space of all strongly measurable functions in $L^p[\Omega, K(X)]$. Then all this space is complete metric space with the metric H_p.

Definition 8. The mean $E(\varphi)$, for a multivalued random variables $\varphi:\Omega \rightarrow 2^{K(X)}$ is given as the integral $\int_\Omega \varphi d\mu$ of φ defined by

$$E(\varphi) = \int_\Omega \varphi d\mu \{\int_\Omega f d\mu : f \in S(\varphi)\}, \tag{5}$$

where

$$S(\varphi) = \{f \in \mathrm{L}^1[\Omega, X] : \mathrm{f}(\omega) \in \varphi(\omega) \in \varphi(\omega) \text{ a.e.}\}$$

The mean $E(\varphi)$ exists, if $S(\varphi)$ is nonempty. Multifunction φ is an integrable, if $\|\varphi(\omega)\|$ is an integrable. If φ have an integral, then $E(\varphi)$ is compact. If μ is atomless, then $E(\varphi)$ is convex. If φ have an integral and $E(\varphi)$ is nonempty, then co $E(\varphi) = E(\text{co}\,\varphi)$, (co – denote convex hull of the set).

This multivalued integral was introduced by Aumann (1965). For detailed arguments concerning the measurability and integration of multifunction we refer to Castaing and Valadier (1977), Debreu (1967), Rockefellar (1976). Now we present some properties of mean of multivalued random variables.

Let $\varphi, \psi:\Omega \rightarrow 2^{K(X)}$ be two multivalued random variables with nonempty $S(\varphi)$ and $S(\psi)$ then:

- $cl\,E(\varphi \cup \psi) = cl(E(\varphi) + E(\psi))$, where $(\varphi \cup \psi)(\omega) = cl(\varphi(\omega) + \psi(\omega))$.
- $cl\,E(\overline{co}\,\varphi) = \overline{co}\,E(\varphi)$, where $(\overline{co}\,\varphi)(\omega) = \overline{co}\,\varphi(\omega)$, the closed convex hull.
- $h(cl\,E(\varphi), cl\,E(\psi)) = H_1(\varphi, \psi)$.

Lemat 1. [2] Let $\varphi:\Omega \rightarrow 2^{K(X)}$ and $1 \leqslant p \leqslant \infty$. If

$$S^p(\varphi) = \{f \in L^p[\Omega, X]:\ f(\omega) < \varphi(\omega) \text{ a.e.}\} \tag{6}$$

then exists a sequence $\{f_n\}$ contained in $S^p(\varphi)$ such that $\varphi(\omega) = cl\{f_n(\omega)\}$ for all $\omega \in \Omega$.

Lemat 2. [2] Let $\varphi, \psi:\Omega \rightarrow 2^{K(X)}$ and $1 \leqslant p \leqslant \infty$. If $S^p(\varphi) = S^p(\psi) \neq \emptyset$ then $\varphi(\omega) = \psi(\omega)$ a.e.

This properties of mean of multivalued random variables are in fact the properties of the multivalued Aumann's integral.

4. MULTIVALUED STOCHASTIC PROCESS

Let T denote the set of positive integers or nonnegative real numbers.

Definition 9. *Multivalued stochastic process* is a family of multivalued random variables indexed by T $\{\varphi_n, t \in T\}$.

Supposing that **P** are the certain properties of stochastic processes.

Definition 10. *A vector valued stochastic process* $\{f_n, t \in T\}$ will be called a **P** selection of $\{\varphi_n, n \geqslant 1\}$, if $\{f_n, t \in T\}$ has the properties **P** and $f_n \in \varphi_n$, a.e. for each $t \in T$.

Let $\{A_t, t \in T\}$ be an increasing family of sub-σ-algebras of A.

A multivalued stochastic process $\{\varphi_n, t \in T\}$ is said to be integrable if for each $t \in T$ is integrable bounded (respectively, A_t measurable)

Definition 11. Let X be a separable Banach space. The map $\Gamma : K(X) \rightarrow X$ is called a *selection operator* if $\Gamma(A) \in A$, for all $A \in K(X)$.

1) Γ is called a continuous selection operator (or measurable operator) if $\Gamma$is continuous with respect to topology on $K(X)$ generated by the subbase $\{A \in K(X),\ a < d(x, A) < b\}$ $(a, b \in R, x \in X\}$. Denote Borel σ-algebra of this topology by **B**. This is separable and completely mertizable topology space $(K(X), W)$.

2) Γ is called a linear selection operator if for any A, $B \in K(X)$

$$\Gamma(\alpha_1 A + \alpha_2 B) = \alpha_1 \Gamma(A) + \alpha_2 \Gamma(B). \tag{7}$$

3) Γ is called a Lipschitz selection operator if there exists a constant $k > 0$ such that for any $A, B \in K(X)$

$$\| \Gamma(A) - \Gamma(B) \| \leqslant k d(A, B) \tag{8}$$

Theorem 1. Let X be a separable Banach space. Then there exists a sequence of measurable selection operators $\{\Gamma_n\}$ such that for each $A \in K(X)$

$$A = \mathrm{cl}\{\Gamma_n(A)\}. \tag{9}$$

Salinetti and Wets (1979) studied the distribution theory of multivalued random variables in finite dimensional Banach spaces. They proved that multivalued random variables φ_1 and φ_2 are identically distributed if and only if the real-valued stochastic process $\{d(x, \varphi_1), x \in X\}$ and $\{d(x, \varphi_2), x \in X$ have the same finite dimensional distribution.

If a sequence of multivalued random variables $\{\varphi_n\}$ converges in distribution to φ, then there exist selections $\{f_n\}$ of $\{\varphi_n\}$ such that $\{\|f_n\|\}$ converges in distribution to $\|f\|$, where f is a vector valued random variables with $f \in \varphi$ a.e.

Theorem 2. Let X be a finite-dimensional Banach space, and let φ_1 and φ_2 be two multivalued random variables. Then the following are equivalent:

1. φ_1 and φ_2 are identically distributed.

2. There exist selection sequences $\{f_n^1\}$ and $\{f_n^2\}$ of φ_1 and φ_2 such that $\varphi_1(\overline{\omega}) = cl\{f_n^i(\overline{\omega})\}$, $i = 1, 2$.

3. The real-valued stochastic process $\{d(x,\varphi_1), x\in X\}$ and $\{d(x,\varphi_2), x\in X\}$ have the same finite dimensional distribution.

Proof $(1\Rightarrow 2)$ Suppose that φ_1 and φ_2 are identically distributed, and let $\{\Gamma_n\}$ be the sequences of measurable selection operators as in Theorem 1.

We define $f_n^i=\Gamma_n(\varphi^i)$ for $i=1,2$, so we have $\Gamma_i(\overline{\omega})=cl\{f_n^i(\overline{\omega})\}$.

To prove that $\{f_n^1\}$ and $\{f_n^2\}$ have the same finite dimensional distribution, it is sufficient to show that for any positive integer $k\geqslant 1$, $n_1,\dots,n_k\geqslant 1$ and open sets $G_1,\dots,,G_k$ one has

$$P\{\omega\in\Omega: f_{nj}^1\in G_j,\ 1\leqslant j\leqslant k\}=P\{\omega\in\Omega: f_{nk}^2\in G_j,\ 1\leqslant j\leqslant k\}. \tag{10}$$

According to definition we get

$$P\{\omega\in\Omega: f_{nj}^i\in G_j,\ 1\leqslant j\leqslant k\}=P\{\omega\in\Omega:\Gamma_{nk}(\varphi_i)\in G_j,\ 1\leqslant j\leqslant k\}=$$

$$=P(\omega\in\Omega:\varphi_i\in\Gamma_{nj}^{-1}(G_j),\ 1\leqslant j\leqslant k\}=P\{\omega\in\Omega:\varphi_i\in\bigcap_{j=1}^{k}\Gamma_{nj}^{-1}(G_j)\}.$$

Since $\Gamma_{nj}^{-1}(G_j)\in\mathbf{B}(1\leqslant j\leqslant k)$ according to definition of Γ_{nj}^{-1}, so equation (10) follows from the assumption.

Proof $(2\Rightarrow 3)$. Suppose that (2) is true, so we need to show that for any $k\geqslant 1$, $x_1,\dots,x_k\in X$ and nonnegative number $a_1,\dots,a_k$ one has

$$P\{\omega\in\Omega: d(x_k,\varphi_1)<a_k\}=P\{\omega\in\Omega: d(x_k,\varphi_2)<a_k\}. \tag{11}$$

We check this firstly for $k=2$, generalization is easy.

Because $\Gamma_i(\omega)=cl\{f_n^i(\overline{\omega})\}$, for $i=1,2$, so we know that $d(x_k,\varphi_i)=\inf_n\|x_k-f_n^i\|$.

Let

$$A_1^{ip}=\{\omega\in\Omega:\|x_k-f_1^i\|<a_k\} \tag{12}$$

and let

$$A_n^{ip}=\{\omega\in\Omega:\|x_k-f_1^i\|<a_k\}\cap\bigcap_{i=1}^{n-1}A_1^{ij},\ n\geqslant 1. \tag{13}$$

For each fixed i and j $\{A_n^{ij}\}$ is a sequence of disjoint measurable sets and we have

$$\{\omega\in\Omega: d(x_k,\varphi_i)<a_k, k=1,2)=(\bigcup_{n-1}^{\infty}A_n^{i1})\cap\bigcup_{m=1}^{\infty}A_m^{i2})= \tag{14}$$

$$\bigcup_{n=1}^{\infty}\bigcup_{m=1}^{\infty}(A_n^{i1}-\cap A_m^{i2})\quad\text{for}\quad i=1,2.$$

According to fact that $\{f_n^1\}$ and $\{f_n^2\}$ have the same finite dimensional distribution we get for each pair (n, m) $P(A_n^{11} \cap A_m^{12}) = P(A_n^{21} \cap A_m^{22})$.

The point (11) follows from the tact that $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (A_n^{i1} - \cap A_m^{i2})$ is the union of a sequence of disjoint measurable sets.

Proof $(3 \Rightarrow 1)$. This was proved by Salinetti and Wets (1979).

Theorem 3. Let X be a separable Banach spaces and let $\{\varphi_n\}$ be a sequence of closed and convex multivalued random variables converging to distribution φ. Then, there exist a sequence of selection $\{f_n\}$ of $\{\varphi_n\}$ such that $\{f_n\}$ converges in distribution to $f \in \varphi$ a.e.

Proof. We denote by $Kc(X)$ all closed and convex subsets of X. We claim that $\varphi(\omega) \in Kc(X)$ a.e. Since $Kc(X)$ is a closed subspace of $(K(X), W)$ so $\varphi_n \in K(X)$ a.e. for each $n \geqslant 1$. Let Γ be the continuous selection operators on $Kc(X)$.

Define

$$f = \Gamma(\varphi) \quad \text{and} \quad f_n = \Gamma(\varphi_n). \tag{15}$$

We can see that f_n and f are measurable and $f_n \in \varphi_n$, $f \in \varphi$ a.e. Let μ_{f_n} and μ_f denote the probability measure on X induced by f_n and f respectively. To prove that $\{f_n\}$ converge in distribution to f, it is sufficient to show that for every bounded continuous function $g: X \rightarrow R$ one has

$$\lim_{n \to \infty} \int_X g(x) d\mu_{f_n} = \int_X g(x) d\mu_f \tag{16}$$

Let $F(A) = g(\Gamma(A))$ for each $A \in Kc(X)$. Than F is bounded continuous function on $Kc(X)$.

We can write

$$\int_X g(x) d\mu_{f_n} = \int_\Omega g(f(\omega)) d\mu_f \quad \text{and} \quad \int_X g(x) d\mu_f = \int_\Omega g(f(\omega)) dP. \tag{17}$$

Next

$$\int_{Kc(X)} F(A) d\mu_{f_n} = \int_\Omega F(\varphi_n(\omega)) dP \quad \text{and} \quad \int_{Kc(X)} F(A) d\mu_f = \int_\Omega F(\varphi(\omega)) dP. \tag{18}$$

As $\{\varphi_n\}$ converges in distribution to φ it follows that

$$\lim_{n \to \infty} \int_X g(x) d\mu_{f_n} = \lim_{n \to \infty} \int_\Omega g(f_n(\omega)) dP = \lim_{n \to \infty} \int_\Omega g(\Gamma(\varphi_n(\omega)) dP =$$

$$\lim_{n \to \infty} \int_\Omega F(\varphi_n(\omega) dP = \lim_{n \to \infty} \int_{Kc(X)} F(A) dP = \int_{Kc(X)} F(A) d\mu_f = \int_\Omega F(\varphi(\omega)) dP =$$

$$\int_\Omega g(\Gamma(\varphi(\omega)) dP = \int_\Omega g(f(\omega) dP = \int_X g(x) d\mu_f. \tag{19}$$

This proves (16) and the theorem follows.

Theorem 4. Let X be a separable Banach space and let $\{\varphi_t, t \in R^+\} \subset \mu_f[\Omega, X]$ be a regular and right-continuous with respect to topology space $(K(X), W)$. Then $\{\varphi_r, t \in R^+\}$ has a regular and right-continuous selection.

Proof. Let Γ be the continuous selection operator on $Kc(X)$. Define for each $t \in R^+$ and $\omega \in \Omega$, $f_t(\omega) = \Gamma(\varphi_t(\omega))$. It is easy to check that $\{f_t, t \in R^+\}$ is regular and a right-continuous selection of $\{\varphi_t, t \in R^+\}$.

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WIELOWARTOŚCIOWE PROCESY STOCHASTYCZNE

(Streszczenie)

Wielowartościowe zmienne losowe i wielowartościowe procesy stochastyczne znajdują zastosowanie w geometrii różniczkowej, w matematycznej ekonomii oraz w zadaniach stochastycznej optymalizacji. W teorii wielowartościowych procesów stochastycznych ważnym problemem jest pytanie o istnienie wektora selektorów procesu stochastycznego. W artykule wykorzystując operatory selekcyjne, pokazujemy zbieżność względem dystrybuant oraz stacjonarność selektora wielowartościowego procesu stochastycznego.