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BALANCED BLOCK DESIGNS LEADING TO THE OPTIMUM CHEMICAL BALANCE WEIGHING DESIGN WITH EQUAL CORRELATIONS OF ERRORS

Abstract. The paper is studying the estimation problem of individual weights of objects using the chemical balance weighing design under the restriction on the number times in which each object is weighed. We assume that errors have the same variances and they are equal correlated. The necessary and sufficient conditions under which the lower bound of variance of each of the estimated weights is attained are given. The incidence matrices of the balanced incomplete block designs and balanced bipartite weighing designs are used to construct the matrix of the optimum chemical balance weighing designs.

Key words: balanced bipartite weighing design, balanced incomplete block design, chemical balance weighing design.

1. INTRODUCTION

Let us consider the class $\mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ of the $n \times p$ matrices $\mathbf{X}$ with elements equal to -1, 0 or 1, where m is the maximum number of elements equal to -1 and 1 in each column of the matrix $\mathbf{X}$. The matrices belonging to this class are the design matrices of the chemical balance weighing designs. Suitable model we can write in the form:

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}, \tag{1}$$

where $\mathbf{y}$ is an $n \times 1$ random observed vector of the recorded results of weights, $\mathbf{w}$ is an $p \times 1$ column vector representing the unknown weights of

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objects and $\mathbf{e}$ is an $n \times 1$ random vector of errors. We assume that the errors are equal correlated and they have the same variances, i.e. $\mathbf{E(e)} = \mathbf{0}_n$ and $\mathrm{Var(e)} = \sigma^2\mathbf{G}$, where $\mathbf{0}_n$ is the $n \times 1$ column vector of zeros, $\mathbf{G} = g[(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n']$, where $g > 0$, $\frac{-1}{n-1} < \rho < 1$ are given constants.

For estimating individual unknown weights of the objects we can use the normal equations

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\hat{\mathbf{w}} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}, \tag{2}$$

where $\hat{\mathbf{w}}$ is the vector of the weights estimated by the least squares method.

The chemical balance weighing design is singular or nonsingular depending on whether the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. If $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, the least squares estimator of $\mathbf{w}$ is given in the form

$$\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y} \tag{3}$$

and the variance-covariance matrix of $\hat{\mathbf{w}}$ is given by formula

$$\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}. \tag{4}$$

In the case $\mathbf{G} = \mathbf{I}_n$, Hotelling (1944) has studied some problems connected with chemical balance weighing designs. He has shown that for a chemical balance weighing design the minimum attainable variance for each of the estimated weights is σ^2/n. He proved the theorem that each of the variance of the estimated weights attains the lover bound if and only if $\mathbf{X}'\mathbf{X} = n\mathbf{I}_p$. This design is called the optimum chemical balance weighing design. It implies that for the optimum chemical balance weighing design the elements of matrix $\mathbf{X}$ there are -1 and 1, only. In this case several methods of construction the optimum chemical balance weighing designs are available in Raghavarao (1971) and Banerjee (1975). In the above model of the optimum chemical balance weighing design with equal correlated errors Ceranka and Katulska (1998) gave the necessary and sufficient conditions under which the lower bound of variance of the estimators was attained.

In this paper we study the similar problem of existing of the optimum chemical balance weighing design with equal correlated errors under assumption the elements of the design matrix $\mathbf{X}$ are equal to -1, 1 or 0.

2. VARIANCE LIMIT OF ESTIMATED WEIGHTS

Let assume that the positive definite matrix **G** is given as

$$\mathbf{G} = g[(1-\rho)\mathbf{I}_n + \rho\mathbf{1}_n\mathbf{1}_n'], \quad \frac{-1}{n-1} < \rho < 1, \quad g > 0. \tag{5}$$

For the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ and **G** in (5) Ceranka and Graczyk (2003) have showed the following theorem.

Theorem 1. In the nonsingular chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where **G** is given by (5), the variance of each of the estimated measurements of objects **w** can not be less then

$$\operatorname{Var}(\hat{w}_j) \geqslant \begin{cases} \dfrac{\sigma^2 g(1-\rho)}{m} & \text{if } 0 \leqslant \rho < 1, \\ \dfrac{\sigma^2 g(1-\rho)}{m - \dfrac{\rho}{1+\rho(n-1)}(m-2u)^2} & \text{if } \dfrac{-1}{n-1} < \rho < 0, \end{cases} \quad j = 1, 2, ..., p,$$

where $m = \max\{m_1, m_2, ..., m_p\}$, m_j is the number of objects equal to –1 and 1 in j-th column of **X**, $u = \min\{u_1, u_2, ..., u_p\}$, u_j is the number of elements equal to –1 in j-th column of **X**, $j = 1, 2, ..., p$.

Definition 1. Any nonsingular chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix **G** is of the form (5), is optimal if the variance of each of the estimators attain the lower bound, i.e. if

$$\operatorname{Var}(\hat{w}_j) = \begin{cases} \dfrac{\sigma^2 g(1-\rho)}{m} & \text{if } 0 \leqslant \rho < 1, \\ \dfrac{\sigma^2 g(1-\rho)}{m - \dfrac{\rho}{1+\rho(n-1)}(m-2u)^2} & \text{if } \dfrac{-1}{n-1} < \rho < 0, \end{cases} \quad j = 1, 2, ..., p.$$

Theorem 2. Let $0 \leqslant \rho < 1$. Any nonsingular chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix **G** is of the form (5), is optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p$

and

(ii) $\mathbf{X}'\mathbf{1}_n = \mathbf{0}_p$.

Theorem 3. Let $\frac{-1}{n-1} < \rho < 0$. Any nonsingular chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix $\mathbf{G}$ is of the form (5), is optimal if and only if

(i) $\mathbf{X}'\mathbf{X} = m\mathbf{I}_p \frac{\rho(m-2u)^2}{1+\rho(n-1)}(\mathbf{I}_p - \mathbf{1}_p\mathbf{1}_p')$

(*ii*) $u_1 = u_2 = ... = u_p = u$

and

(iii) $\mathbf{X}'\mathbf{1}_n = \mathbf{z}_p$,

where $\mathbf{z}_p$ *is* $p \times 1$ vector, for which the j-th element is equal $(m-2u)$ or $-(m-2u)$, $j = 1, 2, ..., p$.

In the next section we will construct the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ of the optimum chemical balance weighing design with $\mathbf{G}$ of the form (5) using the incidence matrices of the balanced incomplete block designs and the balanced bipartite weighing designs.

3. BALANCED BLOCK DESIGNS

In this section we remind the definitions of the balanced incomplete block design given in Raghavarao (1971) and of the balanced bipartite weighing design given in Huang (1976).

A balanced incomplete block design there is an arrangement of v treatments into b blocks, each of size k, in such a way, that each treatment occurs at most ones in each block, occurs in exactly r blocks and every pair of treatments occurs together in exactly λ blocks. The integers v, b, r, k, λ are called the parameters of the balanced incomplete block design. Let $\mathbf{N}$ be the incidence matrix of balanced incomplete block design. It is straightforward to verify that

$$vr = bk,$$
$$\lambda(v-1) = r(k-1), \tag{6}$$
$$\mathbf{NN}' = (r-\lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v',$$

where $\mathbf{1}_v$ is the $v \times 1$ vector of units.

A balanced bipartite weighing design there is an arrangement of v treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. Every pair of treatments from different subblocks appears together in λ_1 blocks and every pair of treatments from the same subblock appears together in λ_2 blocks. The integers v, b, r, k_1, k_2, λ_1, λ_2 are called the parameters of the balanced bipartite weighing design. Let $\mathbf{N}^*$ be the incidence matrix of such a design. The parameters are not independent and they are related by the following identities

$$vr = bk,$$
$$b = \frac{\lambda_1 v(v-1)}{2k_1k_2},$$
$$\lambda_2 = \frac{\lambda_1[k_1(k_1-1) + k_2(k_2-1)]}{2k_1k_2}, \tag{7}$$
$$r = \frac{\lambda_1 k(v-1)}{2k_1k_2},$$
$$\mathbf{N}^*\mathbf{N}^{*\prime} = (r-\lambda_1-\lambda_2)\mathbf{I}_v + (\lambda_1+\lambda_2)\mathbf{1}_v\mathbf{1}_v'.$$

4. THE DESIGN MATRIX

In this section we will present new method of construction of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p,m}(-1, 0, 1)$ of the optimum chemical balance weighing design. It is based on the incidence matrices of the balanced incomplete block designs and of the balanced bipartite weighing designs under assumption that the errors are correlated with equal variances.

Let $\mathbf{N}_1$ be the incidence matrix of the balanced incomplete block design with parameters v, b_1, r_1, k_1, λ_1. And let $\mathbf{N}_2^*$ be the incidence matrix of balanced bipartite block design with parameters v, b_2, r_2, k_{12}, k_{22}, λ_{12}, λ_{22}. Using this matrix we built the matrix $\mathbf{N}_2$ by replacing k_{11} elements equal to $+1$ of each column which correspond to the elements belonging to the first subblock by –1. Then each column of the matrix $\mathbf{N}_2$ contains k_{12} elements equal to –1, k_{22} elements equal to 1 and $v - k_{12} - k_{22}$ elements equal to 0. Hence $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ is of the form

$$\mathbf{X} = \begin{bmatrix} 2\mathbf{N}_1' - \mathbf{1}_{b_1}\mathbf{1}_v' \\ \mathbf{N}_2' \end{bmatrix}. \tag{8}$$

In such a design we determine unknown measurements of $p = v$ objects. Each object is weighed $m = b_1 + r_2$ times in $n = b_1 + b_2$ measurement operations. Because $\mathbf{G}$ is the positive definite matrix then $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if $\mathbf{X}'\mathbf{X}$ is nonsingular. Hence we have the following lemma.

Lemma 1. Chemical balance weighing design with the matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ given by the form in (8) is nonsingular if and only if

$$v \neq 2k_1$$

or

$$k_{12} \neq k_{22}.$$

Proof. The thesis is the consequence of the equalities

$$\mathbf{X}'\mathbf{X} = [4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12}]\mathbf{I}_v + [b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12}]\mathbf{1}_v\mathbf{1}_v' \tag{9}$$

and

$$\det(\mathbf{X}'\mathbf{X}) = [4(r_1 - \lambda_1) + r_2 - \lambda_{22} + \lambda_{12}]^{v-1}\left[\frac{r_1}{k_1}(v - 2k_1)^2 + \frac{(v-1)\lambda_{12}}{2k_{12}k_{22}}(k_{12} - k_{22})^2\right]. \tag{10}$$

For $0 \leqslant \rho < 1$ and $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ in the form (8) we consider the optimality conditions given in the Theorem 2. From (i) of Theorem 2. it derivers that $b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12} = 0$ and from (ii) when $k_{12} \neq k_{22}$ we have $b_1 - 2r_1 - \dfrac{\lambda_{12}(v-1)(k_{22} - k_{12})}{2k_{12}k_{22}} = 0$. Thus we get the following theorm.

Theorem 4. Any chemical balance weighing design with the matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p,m}(-1, 0, 1)$ given in (8), for which $k_{12} \neq k_{22}$, and with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix $\mathbf{G}$ is of the form (5), is optimal for the estimation of individual unknown measurements of objects if and only if

(i) $b_1 - 4(r_1 - \lambda_1) + (\lambda_{22} - \lambda_{12}) = 0$

and

(ii) $b_1 - 2r_1 - \dfrac{\lambda_{12}(v-1)(k_{22} - k_{12})}{2k_{12}k_{22}} = 0.$

Corollary 1. Let $0 \leqslant \rho < 1$. If the parameters of the balanced incomplete block designs and the balanced bipartite weighing designs are equal to

(i) $v = 12$, $b_1 = 33$, $r_1 = 11$, $k_1 = 4$, $\lambda_1 = 3$ and $v = 12$, $b_2 = 66$, $r_2 = 33$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$;

(ii) $v = 15$, $b_1 = 42$, $r_1 = 14$, $k_1 = 5$, $\lambda_1 = 4$ and $v = 15$, $b_2 = 105$, $r_2 = 56$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$;

(iii) $v = 16$, $b_1 = 40$, $r_1 = 15$, $k_1 = 6$, $\lambda_1 = 5$ and $v = 16$, $b_2 = 80$, $r_2 = 20$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 2$, $\lambda_{22} = 2$;

(iv) $v = 25$, $b_1 = 40$, $r_1 = 16$, $k_1 = 10$, $\lambda_1 = 6$ and $v = 25$, $b_2 = 100$, $r_2 = 16$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 1$, $\lambda_{22} = 1$,

then $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ in the form (8) is the design matrix of the optimum chemical balance weighing design with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix $\mathbf{G}$ is of the form (5).

Now, we consider the case $\frac{-1}{n-1} < \rho < 0$.

Theorem 5. If $\frac{-1}{n-1} < \rho < 0$ and $k_{12} \neq k_{22}$ then any nonsingular chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ given by the form in (8) with the variance-covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix $\mathbf{G}$ is of the form (5), is optimal for the estimation of individual unknown measurements of objects if and only if

$$\rho = \frac{b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12}}{(2r_1 - b_1 + r_{22} - r_{12})^2 - (b_1 + b_2 - 1)(b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12})}$$

and

$$b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12} < 0,$$

where $r_{12} = \frac{\lambda_{12}(v-1)}{2k_{22}}$, $r_{22} = \frac{\lambda_{12}(v-1)}{2k_{12}}$, $n = b_1 + b_2$.

Proof. From Theorem 3 it derivers that if $\frac{-1}{n-1} < \rho < 0$, then chemical balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ is optimal for the matrix $\sigma^2\mathbf{G}$, where $\mathbf{G}$ is of (5), if and only if the conditions (i)–(iii) are fulfilled. Hence from the condition (iii) we get $\mathbf{c}_j'\mathbf{X}'\mathbf{1}_n = m - 2u$ or $-(m - 2u)$, $j = 1, 2, \ldots, p$, where $m - 2u = 2r_1 - b_1 + r_{22} - r_{12}$, vector $\mathbf{c}_j$ is equal to the j-th column of the identity matrix. From the condition (i) in Theorem 3. and from (10) it derivers $\mathbf{c}_j'\mathbf{X}'\mathbf{X}\mathbf{c}_i = b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12}$, $i \neq j$, and

$b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12} = \frac{\rho(m-2u)^2}{1+\rho(n-1)}$. The last relation implies $\rho < 0$ because $b_1 - 4(r_1 - \lambda_1) + \lambda_{22} - \lambda_{12} < 0$. Hence the thesis.

Theorem 6. Let $\frac{-1}{n-1} < \rho < 0$. If for a given v and ρ the parameters of the balanced incomplete block design and of the balanced bipartite weighing design are equal to

(i) $\rho = \frac{-3}{2(20s^2+5s-4)}$, $v = 4s$, $b_1 = 2(4s-1)$, $r_1 = 4s-1$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s$, $b_2 = 2s(4s-1)$, $r_2 = 3(4s-1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 2, 3, \ldots$,

(ii) $\rho = \frac{-1}{2(6s^2+7s+1)}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = 2s(4s+1)$, $r_2 = 16s$, $k_{12} = 3$, $k_{22} = 5$, $\lambda_{12} = 15$, $\lambda_{22} = 13$, $s = 2, 3, \ldots$, $4s+1$ is prime or prime power,

(iii) $\rho = \frac{-3}{28s^2+22s+7}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = 2s(4s+1)$, $r_2 = 6s$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 2, 3, \ldots$, $4s+1$ is prime or prime power,

(iv) $\rho = \frac{-3}{13s^2+23s+7}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = s(4s+1)$, $r_2 = 5s$, $k_{12} = 2$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 2$, $s = 2, 3, \ldots$, $4s+1$ is prime or prime power,

(v) $\rho = \frac{-3}{40s^2+14s+7}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = 2s(4s+1)$, $r_2 = 12s$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 3, 4, \ldots$, $4s+1$ is prime or prime power,

(vi) $\rho = \frac{-1}{16s^2+2s+3}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = 2s(4s+1)$, $r_2 = 8s$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 3$, $s = 3, 4, \ldots$, $4s+1$ is prime or prime power,

(vii) $\rho = \frac{-1}{13s^2-3s+5}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = s(4s+1)$, $r_2 = 5s$, $k_{12} = 1$, $k_{22} = 4$, $\lambda_{12} = 2$, $\lambda_{22} = 3$, $s = 1, 2, \ldots$, $4s+1$ is prime or prime power,

(viii) $\rho = \frac{-1}{44s^2-14s+5}$, $v = 4s+1$, $b_1 = 2(4s+1)$, $r_1 = 4s$, $k_1 = 2s$, $\lambda_1 = 2s-1$ and $v = 4s+1$, $b_2 = 2s(4s+1)$, $r_2 = 14s$, $k_{12} = 2$, $k_{22} = 5$, $\lambda_{12} = 10$, $\lambda_{22} = 11$, $s = 2, 3, \ldots$, $4s+1$ is prime or prime power,

(ix) $\rho = \dfrac{-1}{10s^2 + 14s + 5}$, $v = 4s + 3$, $b_1 = 4s + 3$, $r_1 = 2s + 1$, $k_1 = 2s + 1$, $\lambda_1 = s$ and $v = 4s + 3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 3(2s + 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 1, 2, \dots$, $4s + 3$ is prime or prime power,

(x) $\rho = \dfrac{-2}{32s^2 + 36s + 11}$, $v = 4s + 3$, $b_1 = 4s + 3$, $r_1 = 2s + 1$, $k_1 = 2s + 1$, $\lambda_1 = s$ and $v = 4s + 3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 6(2s + 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1, 2, \dots$, $4s + 3$ is prime or prime power,

(xi) $\rho = \dfrac{-1}{2(12s^2 + 11s + 3)}$, $v = 4s + 3$, $b_1 = 4s + 3$, $r_1 = 2s + 1$, $k_1 = 2s + 1$, $\lambda_1 = s$ and $v = 4s + 3$, $b_2 = (2s + 1)(4s + 3)$, $r_2 = 4(2s + 1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 3$, $s = 1, 2, \dots$, $4s + 3$ is prime or prime power,

(xii) $\rho = \dfrac{-1}{40s^2 + 68s + 29}$, $v = 8s + 7$, $b_1 = 8s + 7$, $r_1 = 4s + 3$, $k_1 = 4s + 3$, $\lambda_1 = 2s + 1$ and $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 3(4s + 3)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 1, 2, \dots$,

(xiii) $\rho = \dfrac{-2}{128s^2 + 200s + 79}$, $v = 8s + 7$, $b_1 = 8s + 7$, $r_1 = 4s + 3$, $k_1 = 4s + 3$, $\lambda_1 = 2s + 1$ and $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 6(4s + 3)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1, 2, \dots$,

(xiv) $\rho = \dfrac{-1}{2(48s^2 + 70s + 26)}$, $v = 8s + 7$, $b_1 = 8s + 7$, $r_1 = 4s + 3$, $k_1 = 4s + 3$, $\lambda_1 = 2s + 1$ and $v = 8s + 7$, $b_2 = (4s + 3)(8s + 7)$, $r_2 = 4(4s + 3)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 3$, $s = 1, 2, \dots$,

(xv) $\rho = \dfrac{-1}{10s^4 - 6s^2 + 1}$, $v = 4s^2 - 1$, $b_1 = 4s^2 - 1$, $r_1 = 2s^2 - 1$, $k_1 = 2s^2 - 1$, $\lambda_1 = s^2 - 1$ and $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 3(2s^2 - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$, $s = 2, 3, \dots$,

(xvi) $\rho = \dfrac{-2}{32s^2 - 28s^2 + 7}$, $v = 4s^2 - 1$, $b_1 = 4s^2 - 1$, $r_1 = 2s^2 - 1$, $k_1 = 2s^2 - 1$, $\lambda_1 = s^2 - 1$ and $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 6(2s^2 - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1, 2, \dots$,

(xvii) $\rho = \dfrac{-1}{2(12s^4 - 13s^2 + 4)}$, $v = 4s^2 - 1$, $b_1 = 4s^2 - 1$, $r_1 = 2s^2 - 1$, $k_1 = 2s^2 - 1$, $\lambda_1 = s^2 - 1$ and $v = 4s^2 - 1$, $b_2 = (2s^2 - 1)(4s^2 - 1)$, $r_2 = 4(2s^2 - 1)$, $k_{12} = 1$, $k_{22} = 3$, $\lambda_{12} = 3$, $\lambda_{22} = 3$, $s = 1, 2, \dots$,

(xviii) $\rho = \dfrac{-1}{2(12s^4 - 5s^2 + 2t^2 + 2t - 8s^2t + 2st)}$, $v = 4s^2$, $b_1 = 4st$, $r_1 = t(2s - 1$, $k_1 = s(2s - 1)$, $\lambda_1 = (s - 1)t$ and $v = 4s^2$, $b_2 = 2s^2(4s^2 - 1)$, $r_2 = 3(4s^2 - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, t, $s = 2, 3, \dots$, $t \geqslant s$,

(xix) $\rho = \frac{-2}{48s^4 + 16s^3 - 4s^2 - 4s - 1}$, $v = 4s^2$, $b_1 = 4s^2$, $r_1 = 2s^2 + s$, $k_1 = 2s^2 + s$, $\lambda_1 = s^2 + s$ and $v = 4s^2$, $b_2 = 4s^2(4s^2 - 1)$, $r_2 = 3(4s^2 - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 4$, $\lambda_{22} = 1$, $s = 1, 2, \ldots$,

(xx) $\rho = \frac{-2}{48s^4 - 16s^3 - 4s^2 + 4s - 1}$, $v = 4s^2$, $b_1 = 4s^2$, $r_1 = 2s^2 - s$, $k_1 = 2s^2 - s$, $\lambda_1 = s^2 - s$ and $v = 4s^2$, $b_2 = 4s^2(4s^2 - 1)$, $r_2 = 3(4s^2 - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 4$, $\lambda_{22} = 1$, $s = 1, 2, \ldots$,

(xxi) $\rho = \frac{-1}{2s(12s^3 - 8s^2 - s + 2)}$, $v = 4s^2$, $b_1 = 4s^2$, $r_1 = 2s^2 - s$, $k_1 = 2s^2 - s$, $\lambda_1 = s^2 - s$ and $v = 4s^2$, $b_2 = 2s^2(4s^2 - 1)$, $r_2 = 3(4s^2 - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1, 2, \ldots$,

(xxii) $\rho = \frac{-1}{2s(12s^3 + 8s^2 - s - 2)}$, $v = 4s^2$, $b_1 = 4s^2$, $r_1 = 2s^2 + s$, $k_1 = 2s^2 + s$, $\lambda_1 = s^2 + s$ and $v = 4s^2$, $b_2 = 2s^2(4s^2 - 1)$, $r_2 = 3(4s^2 - 1)$, $k_{12} = 2$, $k_{22} = 4$, $\lambda_{12} = 8$, $\lambda_{22} = 7$, $s = 1, 2, \ldots$,

(xxiii) $\rho = \frac{-4}{720s^2 + 24s - 11}$, $v = 12s$, $b_1 = 2(12s - 1)$, $r_1 = 12s - 1$, $k_1 = 6s$, $\lambda_1 = 6s - 1$ and $v = 12s$, $b_2 = 12s(12s - 1)$, $r_2 = 3(12s - 1)$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 4$, $\lambda_{22} = 2$, $s = 1, 2, \ldots$,

(xxiiv) $\rho = \frac{-1}{416s^4 + 20s + 5}$, $v = 40s + 1$, $b_1 = 2(40s + 1)$, $r_1 = 40s$, $k_1 = 20s$, $\lambda_1 = 20s - 1$ and $v = 40s + 1$, $b_2 = 4s(40s + 1)$, $r_2 = 24s$, $k_{12} = 1$, $k_{22} = 5$, $\lambda_{12} = 1$, $\lambda_{22} = 2$, $s = 1, 2, \ldots$, $40s + 1$ is prime or prime power,

then the chemical balance weighing design with the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n \times p, m}(-1, 0, 1)$ given by the form in (8) with the variance – covariance matrix of errors $\sigma^2\mathbf{G}$, where the matrix $\mathbf{G}$ is of the form (5), is optimal for the estimation of individual unknown measurements of objects.

5. THE EXAMPLE

Let us consider the experiment in which we want to determine the unknown measurements of $p = 7$ objects using $n = 28$ operations. We additionally assume that each object is weighed at least $m = 16$ times and the parameter $\rho = \frac{-1}{29}$. Let $\mathbf{N}_1$ be the incidence matrix of the balanced incomplete block design with parameters $v = 7$, $b_1 = 7$, $r_1 = 3$, $k_1 = 3$,

$\lambda_1 = 1$ and $\mathbf{N}_2^*$ be the incidence matrix of the balanced bipartite weighing design with the parameters $v = 7$, $b_2 = 21$, $r_2 = 9$, $k_{12} = 1$, $k_{22} = 2$, $\lambda_{12} = 2$, $\lambda_{22} = 1$

$$\mathrm{N}_1 = \begin{bmatrix} 1&0&0&0&1&0&1 \\ 1&1&0&0&0&1&0 \\ 0&1&1&0&0&0&1 \\ 1&0&1&1&0&0&1 \\ 0&1&0&1&1&0&0 \\ 0&0&1&0&1&1&0 \\ 0&0&0&1&0&1&1 \end{bmatrix},$$

$$\mathrm{N}_2^* = \begin{bmatrix} 1_1&0&0&1_2&0&0&1_2&1_1&0&1_2&0&1_2&0&0&1_1&0&0&0&0&1_2&1_2 \\ 1_2&1_1&0&0&1_2&0&0&0&1_1&0&1_2&0&1_2&0&1_2&1_1&0&0&0&0&1_2 \\ 0&1_2&1_1&0&0&1_2&0&0&0&1_1&0&1_2&0&1_2&1_2&1_2&1_1&0&0&0&0 \\ 0&0&1_2&1_1&0&0&1_2&1_2&0&0&1_1&0&1_2&0&0&1_2&1_2&1_1&0&0&0 \\ 1_2&0&0&1_2&1_1&0&0&0&1_2&0&0&1_1&0&1_2&0&0&1_2&1_2&1_1&0&0 \\ 0&1_2&0&0&1_2&1_1&0&1_2&0&1_2&0&0&1_1&0&0&0&0&1_2&1_2&1_1&0 \\ 0&0&1_2&0&0&1_2&1_1&0&1_2&0&1_2&0&0&1_1&0&0&0&0&1_2&1_2&1_1 \end{bmatrix},$$

where 1_1 and 1_2 denote the object belonging to the first and second subblock, respectively. Then we form the design matrix $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ in (8) (cf. Theorem 6(ix)), where

$$\mathrm{X}_1' = \begin{bmatrix} 1&-1&-1&-1&1&-1&1 \\ 1&1&-1&-1&-1&1&-1 \\ -1&1&1&-1&-1&-1&1 \\ 1&-1&1&1&-1&-1&-1 \\ -1&1&-1&1&1&-1&-1 \\ -1&-1&1&-1&1&1&-1 \\ -1&-1&-1&1&-1&1&1 \end{bmatrix}$$

and

$$
\mathbf{X}_2' = \begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1
\end{bmatrix}.
$$

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(Streszczenie)

W artykule rozważa się zagadnienie estymacji nieznanych miar poszczególnych obiektów w chemicznym układzie wagowym. Zakłada się, że nie w każdej operacji pomiaru wszystkie przedmioty są uwzględniane oraz że błędy mają jednakowe wariancje i są równo skorelowane. Podane zostały warunki konieczne i dostateczne, przy spełnieniu których wariancja estymatorów osiąga dolne ograniczenie. Do konstrukcji macierzy układu przy podanych wyżej założeniach wykorzystuje się macierze incydencji układów zrównoważonych o blokach niekompletnych i dwudzielnych układów bloków.