ACTA UNIVERSITATIS LODZIENSIS
FOLIA OECONOMICA 194, 2005

*Grażyna Trzpiot**

EFFECTIVENESS OF STOCHASTIC DOMINANCE IN FINANCIAL ANALYSIS

Abstract

Portfolio analysis can be regarded as a problem of choosing the best investment project from all possible investments. This choice depends on, the unique for each investor, utility function and the distribution of the return of the investment project. Unlike MV criterion, SD criterion is optimal for a class of utility function and additionally we elaborate with all value of the return of the investment project. We will present the results of analysis the properties of the optimal efficient set according SD criteria for asymmetric distribution.

Key words: asymmetric distribution, stochastic dominance criterion, efficient set.

I. INTRODUCTION

Portfolio analysis poses the problem of choosing of the best prospect from all possible alternative random prospects (portfolio). This selection depends on the investor's utility function and on probability distribution of the prospects. In general the analysis proceeds in two steps. First for a group of investor having the same class of utility function all possible alternative random prospects we divide in two sets: efficient set and inefficient set. The set are constructed so that for any prospect *G* in the inefficient set exist at least one prospect *F* in efficient set with the property that no investor prefers *G* to *F* and there is at least one investor who prefers *F* to *G*. Secondly, an individual investor chooses his most preferred portfolio from an efficient set according his individual utility function.

In this paper we deal with characterization of optimal efficient set. The efficient set is optimal when it is a subset (not necessarily proper) of every

* Professor, Department of Statistics, University of Economics in Katowice.

possible efficient set. Each no proper subset of an optimal efficient set is a subset (not necessarily proper) of every possible efficient set. Markowitz (1952) and Tobin (1958) introduce MV criterion (MV – mean variance) for characterize an efficient set. According to this criterion, prospect belong to efficient set if there no other prospect with the same or larger mean and a smaller variance or the same or smaller variance and a larger mean. The efficient set is optimal if either the class of utility function is quadratic or prospects are normally distributed. Limitations of quadratic utility function have been discussed by: Pratt (1964), Arrow (1965), Hanoch and Levy (1970). The recent empirical study (Mandelbrot, 1963; Fama, 1965) suggests that the distribution of stock price, the area in which portfolio analysis has been applied – are essentially non-normal.

Stochastic dominance as a criterion in portfolio analysis was introduced by Quirk and Saposnik (1962), Fishburn (1964), Hadar and Russelll (1969), Hanoch and Levy (1969, 1970) and many more. In contrast to efficient set for criterion MV and the efficient set for SD criterion is optimal for whole class of utility functions (not only for quadratic one). Additionally SD criterion ulilizes all information in the probability distributions. There are many empirical works describing relationship between the optimal efficient set for U_1 and U_2, the efficient set for criterion MV and the efficient set for SD criterion. In this paper we derive parametric criteria for optimal efficient set when we have a set of prospects with asymmetric distributions.

II. SOME THEOREMS ON STOCHASTIC DOMINANCE

Definition 1. For two random variables X and Y with distributions F and G, we say that X FSD Y if and only if

$$F(x) \leqslant G(x) \quad \text{for all} \quad x \in R.$$

Definition 2. For two random variables X and Y with distributions F and G, we say that X SSD Y if and only if

$$\int_{-\infty}^{t} F(x)dx \leqslant \int_{-\infty}^{t} G(x)dx \quad \text{for all} \quad t \in R,$$

if both integrals exists.

Definition 3. U_1 is the set of all non-decreasing utility function $U_1 = \{u: u' \geqslant 0\}$.

Definition 4. U_2 is the set of all non-decreasing and concave utility function $U_2 = \{u: u' \geqslant 0, u'' \leqslant 0\}$.

Definition 5. For two prospects X and Y with distributions F and G, an investor with utility function u prefers F if and only if

$$\mathrm{E}(u(X)) \geqslant \mathrm{E}(u(Y)).$$

Definition 6. For two prospects X and Y with distributions F and G, X dominates Y in U, a class of utility function if and only if

$$\mathrm{E}(u(X)) \geqslant \mathrm{E}(u(Y)),$$

with a strict inequality for some u.

Definition 7. An efficient set for a class of utility function U is defined as a set of prospects with the property that for any prospect G outside the set, there exists a prospect F in the set, which dominates G in U.

Definition 8. An efficient set is optimal if and only if no proper subset of it is efficient.

Definition 9. The MV efficient set is defined as a set of prospects with the property that for any prospect G outside the set, there exists a prospect F in the set such that

$$\mathrm{E}(X) \geqslant \mathrm{E}(Y) \quad \text{and} \quad V(X) \leqslant V(Y)$$

with at least one strict inequality (the existence of integrals is assumed).

Definition 10. The FSD efficient set is defined as a set of prospects with the property that for any prospect G outside the set, there exists a prospect F in the set such that F FSD G.

Definition 11. The SSD efficient set is defined as a set of prospects with the property that for any prospect G outside the set, there exists a prospect F in the set such that F SSD G.

The following theorems can be used to characterize the optimal efficient set for U_1 and U_2.

Theorem 1. For two random variables X and Y with distributions F and G. Random variable X dominates Y by first stochastic dominance (X FSD Y), if $\mathrm{E}(u(X)) \geqslant \mathrm{E}(u(Y))$ in U_1 (the existence of integrals is assumed).

Theorem 2. For two random variables X and Y with distributions F and G. Random variable X dominates Y by second stochastic dominance (X SSD Y), if $\mathrm{E}(u(X)) \geqslant \mathrm{E}(u(Y))$ in U_2 (the existence of integrals is assumed).

III. EFFECTIVENNES ANALYSIS FOR ASYMMETRIC DISTRIBUTIONS

We derive parametric criteria for optimal efficient set when we have a set of prospects with asymmetric distributions. We specify efficient sets for both criterion and then we compare it. The following theorem specifies the optimal efficient set for gamma distribution.

Theorem 3. Let $F_{\alpha,\beta}$ be a family of gamma distribution with positive parameters α and β the density functions

$$f_{\alpha,\beta}(x) = (\alpha^{\beta}/\Gamma\beta)e^{-\alpha x}x^{\beta-1}, \quad x > 0$$

Then:

(a) $F_{\alpha_1,\beta}$ FSD $F_{\alpha_2,\beta}$ if and only if $\alpha_1 < \alpha_2$,
(a′) $F_{\alpha_1,\beta}$ SSD $F_{\alpha_2,\beta}$ if and only if $\alpha_1 < \alpha_2$,
(b) F_{α,β_1} FSD F_{α,β_2} if and only if $\beta_1 > \beta_2$,
(b′) F_{α,β_1} SSD F_{α,β_2} if and only if $\beta_1 > \beta_2$,
(c) F_{α_1,β_1} FSD F_{α_1,β_2} if and only if $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_2$ with at least one strict inequality,
(d) F_{α_1,β_1} SSD F_{α_1,β_2} if and only if $\beta_1/\beta_2 \geqslant \max(1, \alpha_1/\alpha_2)$ with strict inequality at least when $\alpha_1/\alpha_2 = 1$.

Let the prospect have gamma distribution with parameters (α, β) then any prospect can be identified by corresponding values (α, β). According to a) and a′) for two prospects with gamma distributions, which differs only by parameter α the one with the smaller α is preferable. Similarly according to b) and b′) for two prospects with gamma distributions, which differs only by parameter β, and α is the same for both prospects, the larger β is preferable. Utility function, on which the preference is based, can be chosen arbitrarily from U_1 or U_2. As the mean and the variance of a prospects with gamma distributions with parameters (α, β) are respectively $EX = \beta/\alpha$ and $D^2X = \beta/\alpha^2$, it follows that in either case – prospects differing in α or β, the preferred prospects has the larger mean and the larger variance. For any risk averters interpreting variance as a measure of risk, increasing mean of the prospect compensates larger risk.

If we have two prospects differing in both parameters, then from c) we have that for all non-decreasing utility function, the preferred prospect should not have either the larger α or smaller β. If one of them has both the larger α and the larger β, then no preference can be established. These conditions can be relaxed if we consider a risk averse utility function (a class of DARA functions). In this case form d), prospect with the smaller β is never preferred or prospect with the larger α is preferred only when it is compensated by increased β.

The optimal and MV efficient set

If the prospects differs only in α, then from a) and a′) the FSD efficient set and the SSD efficient set consist only the prospects with the smallest α. Hence, the optimal efficient set for utility function from U_1 or U_2 consist only one prospect. As mean and variance of the prospect with parameters (α, β) are respectively $EX = \beta/\alpha$ and $D^2X = \beta/\alpha^2$, so MV efficient set consist all possible prospect.

Similarly, if the prospects differs only in β from b) and b′), the optimal efficient set for utility function from U_1 or U_2 contain only one prospect with the largest β, but all prospects belong to efficient set for MV criterion. In case, if prospects differ only one parameters, α or β, optimal efficient set for U_1 or U_2 are identical and they are a subset of MV efficient set.

Suppose that prospects differs both in α and β. Then according c) prospect belong to the FSD efficient set if and only if, there is no other prospect with the same or the smaller α and the larger β or the same or the larger β and the smaller α. Thus, the optimal efficient set for U_1 can contain more than one prospect. The optimal efficient set is a subset of the MV efficient set.

Let as consider the group of investors from U_2 and the prospects differs in both α and β. Then according part d) SSD efficient set can be characterised in the following way. For any two distributions F_{α_1,β_1} and F_{α_1,β_2}, F_{α_1,β_1} eliminates F_{α_1,β_2} from SSD efficient set if and only if

$$\beta_1/\beta_2 \geqslant \max(1,\ \alpha_1/\alpha_2)$$

with strict inequality if $\alpha_1/\alpha_2 = 1$.

As

$$E_{F_{\alpha,\beta}}(X) = \beta/\alpha$$

and

$$V_{F_{\alpha,\beta}}(X) = \beta/\alpha^2$$

F_{α_1,β_1} eliminates F_{α_1,β_2} from the MV efficient set if

$$\beta_1/\alpha_1 \geqslant \beta_2/\alpha_2$$

and

$$\beta_1/\alpha_1^2 \geqslant \beta_2/\alpha_2^2$$

with at least one strict inequality. However, if these condition hold, we must have:

$$\beta_1/\alpha_1 \geqslant \beta_2/\alpha_2 \quad \text{and} \quad \alpha_1 > \alpha_2$$

or

$$\beta_1/\beta_2 \geqslant \max(1, \ \alpha_1/\alpha_2) \quad \text{and} \quad \alpha_1 > \alpha_2.$$

From criterion MV we have SSD criterion, but we have not the inversion. We can observe this relation on the example when $\alpha_1 < \alpha_2$ and $\beta_1/\beta_2 \geqslant \max(1, \alpha_1/\alpha_2)$. The MV efficient set contains the optimal efficient set for U_2. As an example we choose a prospect with gamma distribution with parameters (α, β) defined as:

$$0 < \alpha_0 < \alpha < \alpha_1 < \infty,$$

and

$$0 < \beta_0 < \beta < \beta_1 < \infty.$$

Efficient set for SSD criterion contains only one prospect F_{α_0, β_1}, but the MV efficient set contains prospect such a that

$$\beta = \beta_1, \quad \alpha_0 < \alpha < \alpha_1,$$

and

$$\alpha = \alpha_1, \quad \beta_0 < \beta < \beta_1.$$

The MV efficient set includes not only the optimal efficient set for U_2, but can be larger.

Theorem 4. Let $F_{\alpha,\beta}$ be a family of beta distribution with positive parameters α and β the density functions

$$f_{\alpha,\beta}(x) = (\Gamma(\alpha+\beta)/\Gamma\alpha\Gamma\beta)x^{\beta-1}(1-x)^{\alpha-1}, \quad 0 < x < 1$$

Then:

(a) $F_{\alpha_1,\beta}$ FSD $F_{\alpha_2,\beta}$ if and only if $\alpha_1 < \alpha_2$,
(a′) $F_{\alpha_1,\beta}$ SSD $F_{\alpha_2,\beta}$ if and only if $\alpha_1 < \alpha_2$,
(b) F_{α,β_1} FSD F_{α,β_2} if and only if $\beta_1 > \beta_2$,
(b′) F_{α,β_1} SSD F_{α,β_2} if and only if $\beta_1 > \beta_2$,
(c) F_{α_1,β_1} FSD F_{α_1,β_2} if and only if $\alpha_1 \leqslant \alpha_2$ and $\beta_1 \geqslant \beta_2$ with at least one strict inequality,
(d) F_{α_1,β_1} SSD $F_{\alpha,_1\beta_2}$ if and only if $\beta_1/\beta_2 \geqslant \max(1, \alpha_1/\alpha_2)$ with strict inequality at least when $\alpha_1/\alpha_2 = 1$.

The method of analysis between the various optimal and the MV efficient set are the same when the prospect has gamma distribution. This theorem describes efficient set for FSD and SSD criterion. As an application of theorem 3 we have the following results:

Theorem 5. Let $F_{\alpha,\beta}$ be a family of χ^2 distribution in real positive parameters β the density functions

$$f_\beta(x) = (1/\Gamma(\beta/2))(1/2)e^{-(x/2)}(x/2)^{\frac{\beta}{2}-1}, \quad x > 0$$

Then:

(a) F_{β_1} FSD F_{β_2} if and only if $\beta_1 > \beta_2$,
(b) F_{β_1} SSD F_{β_2} if and only if $\beta_1 > \beta_2$.

The proof is based on following relation that $X/2$ is a gamma distribution with parameters $\alpha' = 1$, $\beta' = \beta/2$. According to this theorem, either U_1 or U_2, if the prospect has χ^2 distribution with parameter β, then the prospect with the largest β is the most preferable. Optimal efficient sets for U_1 and U_2 have only one prospect. The prospect with the largest parameter β has the largest mean and variance. According the MV criterion all prospects belong to this efficient set. The MV efficient set includes the optimal efficient set.

The method used before we can adopt for the following theorem, which characterizes the optimal efficient set for the prospect with F-distribution.

Theorem 6. Let $F_{\alpha,\beta}$ be a family of F-distribution in real positive parameters α and β the density functions

$$f_{\alpha,\beta} = \frac{\Gamma\{(\alpha+\beta)/2\}\alpha\{(\alpha/\beta)x\}^{(\alpha/2)-1}}{\Gamma(\alpha/2)\Gamma(\beta/2)\beta\{1+(\alpha/\beta)x\}^{(\alpha/2)/2}}$$

Assume that $E_{F\alpha,\beta}(X)$ exists $(\beta > 2)$ then:

(a) $F_{\alpha_1,\beta}$ SSD $F_{\alpha_2,\beta}$ if and only if $\alpha_1 > \alpha_2$,
(b) F_{α,β_1} SSD F_{α,β_2} if and only if $\beta_1 > \beta_2$.

If prospect have F distribution with parameters α and β, then we can't in simple waypoint out the conditions for the optimal sets either U_1 or U_2. However, if we consider only U_2, the class of risk averters, and to the prospects with the same β then the largest α is preferable (part a)) and considering prospects with the same α, the smallest β is preferable (part b)). It can be shown that the prospects with the largest α and β, when $\alpha = \beta$, have the same mean as another prospects, but prospects the smallest variance. So, if prospects have the same parameter β, the MV efficient set and optimal efficient set for U_2 are identical. In contrast, if prospects have the same parameter α, then the one with the smallest parameter β, which belong to optimal efficient set for U_2, has not only the largest mean, but also the largest variance. The MV efficient set contains all prospects and it is much largest than optimal efficient set. In the case then β are bigger than 4 the variance does not exist and this comparison are not valid.

REFERENCES

Arrow K.J. (1965), *Aspects of the Theory of Risk Bearing*, Yrjo Jahssonin Saatjo, Helsinki.

Bradley M.G., Lehman D.E. (1988), Instrument effects and stochastic dominance, *Insurance Mathematic and Economics*, **7**, 185–191.

Fama E.F. (1965), The behaviour of stock market prices, *Journal of Business*, **38**, 34–105.

Fishburn P.C., Lavalle I. (1995), Stochastic dominance and undimensional grids, *Management Science*, **20**, 3, 513–525.

Fishburn P.C. (1964), *Decision and Value Theory*, John Wiley and Sons, New York.

Fishburn P.C. (1990), Stochastic dominance and moments of distributions, *Mathematics of Operation Research*, **5**, 1, 94–100.

Hadar J., Russell W.R. (1969), Rules for ordering uncertain prospects, *American Economic Review*, **59**, 25–34.

Hanoch G., Levy H. (1969), The efficiency analysis of choices involving risk, *Review of Economic Studies*, **36**, 335–346.

Hanoch G., Levy H. (1970), Efficient portfolio selection with quadratic and cubic utility function, *Journal of Business*, **43**, 181–189.

Levy H. (1996), Investment Diversification and Investment Specialization and the Assumed Holding Period, *Applied Mathematical Finance*, **3**, 117–134.

Levy H., Kroll Y. (1970), Ordering dominance with riskless assets, *Journal of Financial and Quantitative Analysis*, **11**, 743–773.

Mandelbrot B. (1963), The variation of certain speculative prices, *Journal of Business* **36**, 394–419.

Markowitz H.M. (1952), Portfolio selection, *Journal of Finance*, **7**, 77–91.

Markowitz H.M. (1987), *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, Blackwell, Oxford.

Pratt J.W. (1964), Risk aversion in the small and in the large, *Econometrica*, **32**, 122–136.

Quirk J.P., Saposnik R. (1962), Admissibility and measurable utility functions, *Review of Economics Study*, (Feb.) **29**, 140–146.

Skórnik A., Trzaskalik T., Trzpiot G. (1999), Dominacje stochastyczne w teorii portfela akcji na przykładzie Giełdy Papierów Wartościowych w Warszawie, *Badania Operacyjne i Decyzje*, **3–4**, 5–19.

Tobin J. (1958), Liquidity preference as behaviour towards risk, *Review of Economic Studies*, **25**, 65–86.

Trzaskalik T., Trzpiot G., Zaras K. (1998), *Modelowanie preferencji z wykorzystaniem dominacji stochastycznych*, AE, Katowice.

Trzpiot G. (1999), Analiza szeregów czasowych z wykorzystaniem stochastycznych relacji, *Prace Naukowe AE Wrocław*, **817**, 189–196

Trzpiot G. (2002), Multicriterion analysis based on marginal conditional stochastic dominance in financial analysis, [in:] *Multiple Objective and Goal Programming*, ed. T. Trzaskalik, J. Michnik, Ser. Advances in Soft Computing, Springer – Verlag, New York, 401–412.

Grażyna Trzpiot

ANALIZA EFEKTYWNOŚCI DOMINACJI STOCHASTYCZNYCH W ZASTOSOWANIACH FINANSOWYCH

Streszczenie

Analiza portfelowa stawia problem wyboru najlepszego spośród możliwych losowych projektów inwestycyjnych. Wybór ten zależy od, jedynej dla każdego inwestora, funkcji użyteczności oraz od rozkładu prawdopodobieństwa rozważanej inwestycji. W niniejszym opracowaniu skoncentrowano się na scharakteryzowaniu zbioru optymalnych efektywnych inwestycji. W odróżnieniu od zbioru efektywnych inwestycji zgodnego z kryterium momentów MV, zbiór efektywnych inwestycji zgodny z kryterium SD jest optymalny dla całych ogólnych klas funkcji użyteczności (nie tylko dla funkcji kwadratowej). Dodatkowo kryterium SD wykorzystuje wszystkie wartości rozkładu prawdopodobieństwa projektu inwestycyjnego. Wiele prac empirycznych omawia zależności pomiędzy zbiorem efektywnych inwestycji z kryterium momentów MV a zbiorem efektywnych inwestycji zgodnym z kryterium SD. W tym artykule przedstawione zostały wyniki analiz wybranych typów rozkładów asymetrycznych.