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TAIL DEPENDENCE IN BIVARIATE DISTRIBUTIONS

Abstract

In the paper the problem of tail dependence for bivariate data is considered. The review of different approaches is given. The particular emphasis is put on the conditional correlation coefficients and tail dependence coefficients. It is shown how the latter can be analyzed through copula analysis.

Key words: tail dependence, copula analysis, bivariate distribution.

I. INTRODUCTION

The analysis of the dependence (relationship) between variables is one of the most important tasks in multivariate (particularly – bivariate) statistical analysis. One usually considers either the so called joint relationship or the so called conditional relationship. In the first type of relationship all variables are regarded as whole set, in the second type of relationship one (or more) variable is regarded as the dependent variable and the other variables are considered as the independent variables.

The analysis of the relationship is usually performed through two different quantitative approaches:

– modeling the relationship by a function – for example – regression function;

– measuring the relationship by a number – for example – correlation coefficient.

Very often, however, these two approaches are strictly connected, as it is in the case of regression function and correlation coefficient.

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When one assumes stochastic approach, the variables are treated as random variables, then all information about the relationship (dependence) is contained in the cumulative distribution function. For example, if we consider two variables, X and Y, then this information is given as:

– for joint relationship:

$$P(X < x, \ Y < y),$$

– for conditional relationship:

$$P(Y < y | X < x).$$

Of course, in the applications, the simplification is made. Instead of cumulative distribution function one takes into account only some parameters, usually moments of the distribution. Then we get for example:

– for joint relationship:

$$\mathrm{COV}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y),$$

– for conditional relationship:

$$\mathrm{E}(Y|X).$$

Such a simplification, however, may not capture the particular properties of the relationship, for example the tail relationship, that is the relationship existing between the very large (or very small) values of two variables. This is similar problem as in univariate analysis, where the classical "mean-based" analysis does not capture the extreme peculiarities.

In this paper we discuss the problem of tail dependence. For simplicity, we consider the case of bivariate distributions (two variables: X and Y).

II. MODELING TAIL DEPENDENCE – DIFFERENT APPROACHES

There are different approaches that can be used in the modeling of tail dependence. We divide them into three classes:

– separate modeling of center and tails of distribution;
– conditional dependence measures;
– tail dependence measures.

Separate modeling of center and tails of distribution

The first approach consists in the separation of data set into (usually) two classes. The first class contains the "center" (the "core") of the multivariate distribution, here the modeling of the relationships is done for the "typical" observations. The second class contains the tails (the "outliers") of the multivariate distribution, here the modeling of the relationships is done for the extreme values. It may also happen that the data set is separated into more than two classes (when more than one tail is considered). In this approach we can distinguish two groups of methods:

- clustering methods;
- mixture models.

Clustering methods aim at classifying the data set into classes, in such a way that the observations in the same class are as similar as possible, and the observations in different classes are as dissimilar as possible. In many methods, the clustering optimization criterion is defined. This criterion depends on the goal of classification and the understanding of similarity of the observations. For example, in the one of the most popular methods, k-means method, the similarity is measured through the Euclidean distance between the observations. This means that for the purpose of the modeling of the relationship one has to apply the suitable criterion.

The second group, mixture models, assumes stochastic approach. Here the multivariate distribution is treated as a mixture of distributions, where the respective components of the mixture correspond to the center and tails of the distribution. Mixture models are described for example by McLachlan and Peel (2000).

Conditional dependence measures

Here one considers the conditional distribution of two variables given that one of these variables takes the value from the tail. As the natural dependence measure the so called conditional correlation coefficient can be used. It is given by the following formula (without the loss of generality we consider the upper tail):

$$\rho_C = \frac{\mathrm{COV}(X, Y|X > s)}{\sqrt{V(X|X > s)V(Y|X > s)}}. \tag{1}$$

Here s denotes the large value of the variable. Therefore this conditional correlation coefficient is defined given that one variable takes large (extreme) value. It can be proved that for some bivariate distributions the conditional correlation coefficient is related to correlation coefficient.

For bivariate (standard) normal distribution we have:

$$\rho_C = \frac{\rho}{\sqrt{\rho^2 + \dfrac{1-\rho^2}{V(X \mid X > s)}}}. \tag{2}$$

In the limiting case – if s goes to infinity we get:

$$\rho_C \to \frac{\rho}{\sqrt{1-\rho^2}} \frac{1}{s}. \tag{3}$$

Therefore the conditional correlation coefficient converges to zero regardless of the value of (unconditional) correlation coefficient.

On the other hand, for bivariate t distribution with degrees of freedom we have in limiting case:

$$\rho_C \to \frac{\rho}{\sqrt{\rho^2 + (\nu - 1)}\sqrt{\dfrac{\nu-2}{\nu}(1-\rho^2)}}. \tag{4}$$

Table 1 presents the limit of conditional correlation coefficient for bivariate t distribution – different number degrees of freedom in columns and different unconditional correlation coefficient in rows.

Table 1. Conditional correlation coefficient – limit in the case of bivariate t distribution

	3	4	10	20	30
–0.9	–0.79	–0.68	–0.43	–0.31	–0.25
–0.5	–0.45	–0.35	–0.19	–0.13	–0.10
0	0	0	0	0	0
0.5	0.45	0.35	0.19	0.13	0.10
0.9	0.79	0.68	0.43	0.31	0.25

In all presented cases – except for 0 – the conditional correlation coefficient is lower than (unconditional) correlation coefficient. It also gets lower when the number of degrees of freedom increases – but it is different from 0.

Sometimes the other concept of conditional correlation coefficient is used, where instead of one variable, conditioning is on both variables. Then we get the following version of conditional correlation coefficient:

$$\rho_{CC} = \frac{\mathrm{COV}(X, Y|X > s, Y > s)}{\sqrt{V(X|X > s, Y > s)V(Y|X > s, Y > s)}}. \tag{5}$$

It can be proved that in the case of bivariate normal distribution we get in the limiting case:

$$\rho_C \to \rho \frac{1+\rho}{1-\rho} \frac{1}{s^2}. \tag{6}$$

So also here the conditional correlation coefficient converges to zero regardless of the value of (unconditional) correlation coefficient. Also the properties of this version of conditional correlation coefficient are similar to the one in the previous version. More detailed description of conditional correlation coefficients is given by Malevergne and Sornette (2002).

It should be noted, that in both version of conditional correlation coefficient we still are limited to linear relationship, which is the main drawback of this approach. This drawback (as well as some others) does not exist in the other approach, tail dependence measures, described below.

III. TAIL DEPENDENCE MEASURES

This is the other group of dependence measures, where one looks directly into tails of the bivariate distribution. There are two coefficients of tail dependence, namely:

– coefficient of lower tail dependence, given as:

$$\lambda_L = \lim_{u \to 0} \mathrm{P}(Y \leqslant G^{-1}(u)|X \leqslant F^{-1}(u)), \tag{7}$$

– coefficient of upper tail dependence, given as:

$$\lambda_U = \lim_{u \to 1} \mathrm{P}(Y > G^{-1}(u)|X > F^{-1}(u)), \tag{8}$$

Here F and G denote the cumulative marginal distribution function of X and Y, respectively, and u denotes the probability.

Both tail dependence coefficients have rather clear interpretation. They show the probability that one variable takes extremely large value (case of upper tail dependence) or extremely small value (case of lower tail dependence) given the other variables takes extremely large value (case of upper tail dependence) or extremely small value (case of lower tail dependence). This probability is taken as limiting probability and in fact one speaks about asymptotic tail dependence (or independence).

As one can see from (7) and (8) the extremely large or extremely small values are taken as high or low quantile and in the limit these quantiles converge to (plus or minus) infinity. In addition, it should be mentioned that:

– the tail dependence coefficient falls into interval [0;1];

– we speak of asymptotic tail independence if tail dependence coefficient is equal to 0;

– we speak of asymptotic tail dependence if tail dependence coefficient is higher than 0.

In practice, the calculation of tail dependence coefficient is not easy. There is one important case, when this coefficient is given through the analytical formula. This refers to bivariate elliptically symmetric distributions. The density of elliptically symmetric distribution is given as (see e.g. Jajuga, 1993):

$$f(x) = c|\Sigma|^{-0.5}h[(x-\mu)^T\Sigma^{-1}(x-\mu)]. \tag{9}$$

Among the members of this family are the following multivariate distributions:

– normal distribution, and more general – Kotz type distribution;

– Cauchy distribution, and more general t distribution and even more general Pearson type VII distribution;

– Pearson type II distribution;

– logistic distribution, etc.

The upper tail dependence coefficient for bivariate elliptically symmetric distributions depends on the correlation coefficient and is given as (Embrechts, McNeil, Straumann, 1999):

$$\lambda_U = \frac{\int_{(\pi/2-\arcsin\rho)/2}^{\pi/2} \cos^\alpha t dt}{\int_0^{\pi/2} \cos^\alpha t dt}. \tag{10}$$

Here α denotes the tail index of the distribution (for example, in the case of t distribution is equal to the number of degrees of freedom).

From the formula (10) it can be proved that for the normal distribution we have the asymptotic tail independence, if the correlation coefficient is different from +1 and –1. Therefore, bivariate normal distribution is not the suitable model to capture tail de; endence. On the other hand, for bivariate t distribution, if the correlation coefficient is different from minus 1, we have asymptotic tail dependence.

Table 2 (taken from Embrechts, McNeil, Straumann, 1999) presents the upper tail dependence coefficients for bivariate t distribution – different number degrees of freedom in columns and different unconditional correlation coefficient in rows.

Table 2. Upper tail dependence coefficient – case of bivariate t distribution

	-0.5	0	0.5	0.9
2	0.06	0.18	0.39	0.72
4	0.01	0.08	0.25	0.63
10	0	0.01	0.08	0.46

In general situation the analytical formulas for tail dependence coefficients are not given. However, in some cases on still can arrive at the solution by applying the so called copula analysis. The presentation of this idea is given below.

IV. COPULA ANALYSIS AND TAIL DEPENDENCE

The idea of copula analysis lies in the decomposition of the multivariate distribution into two components. The first component consists of the marginal distributions. The second component – the crucial one – is the function linking these marginal distributions in multivariate distribution. This function reflects the structure of the relationship between the components of the multivariate random vector. For simplicity, we consider the bivariate case.

This idea is reflected in Sklar theorem, given through the following formula:

$$H(x_1, x_2) = C(F_1(x_1), F_2(x_2)), \tag{11}$$

where:

H – the multivariate distribution function;

F_i – the distribution function of the i-th marginal distribution;

C – copula function.

Thus the bivariate distribution function is the function of the univariate (marginal) distribution functions. This function is called copula function and it reflects the structure of the relationships between the univariate components. In the case of bivariate continuous distribution the presentation given by (11) is unique.

The presentation given by (11) can be reverted. Here the copula function is given as bivariate distribution function defined for the quantiles of the marginal distributions. It is given as:

$$C(u_1, u_2) = H(F_1^{-1}(u_1), F_2^{-1}(u_2)). \tag{12}$$

Among the particular cases are (already discussed) bivariate normal distribution and bivariate t distribution. When this distribution is decomposed according to Sklar theorem, we get the so called normal copula and t copula. Their analytical form is given as:

– normal copula:

$$C(u_1, u_2) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right) dxdy, \tag{13}$$

– t copula:

$$C(u_1, u_2) = \int_{-\infty}^{t^{-1}(u_1)} \int_{-\infty}^{t^{-1}(u_2)} \frac{1}{2\pi\sqrt{1-\rho^2}} \left(1 + \frac{x^2 - 2\rho xy + y^2}{\rho(1-\rho^2)}\right)^{-(\rho+2)/2} dxdy. \tag{14}$$

So we see that in both cases, numerical procedures are needed to calculate the values of copula function.

Among the other interesting types of copulas, it is worth to mention the so called Archimedean copulas. They are defined for strictly decreasing and convex function in the following way:

$$C(u_1, u_2) = \psi^{-1}(\psi(u_1) + \psi(u_2)), \tag{15}$$

where:

$\psi : [0;\ 1] \rightarrow [0;\ \infty)$

$\psi(1) = 0.$

The most popular case of Archimedean copula is Gumbel copula, where:

$$\psi(t) = -(\ln(t))^{\beta}. \tag{16}$$

Here parameter β is interpreted as a measure of dependence, taking values from 1 to infinity. The value equal to 1 means independence, and the closer is this value to infinity, the closer is to the strict positive dependence.

Gumbel copula can be written in the following form:

$$C(u_1, u_2) = \exp(-((-\ln u_1)^{\beta} + (-\ln u_2)^{\beta})^{1/\beta}). \tag{17}$$

The crucial property of the copula function refers to the tail dependence coefficients. It turns out that both, upper tail and lower tail dependence coefficients can be expressed through the copula function, in the following way:

- the lower tail dependence coefficient:

$$\lambda_L = \lim_{u \to 0} [C(u, u))/u], \tag{18}$$

- the upper tail dependence coefficient:

$$\lambda_U = \lim_{u \to 1} [(1 - 2u + C(u, u))/(1 - u)]. \tag{19}$$

Using (19), it can be proved that for the Gumbel copula we get asymptotic tail dependence, if:

$$\beta > 1.$$

Then the upper tail dependence coefficient is equal to:

$$\lambda_U = 2 - 2^{1/\beta}. \tag{20}$$

In practice, the important issue is, of course, the identification of suitable copula function for given bivariate data sets – this helps to determine the asymptotic tail dependence coefficients.

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ZALEŻNOŚĆ W OGONIE DLA ROZKŁADÓW DWUWYMIAROWYCH

Streszczenie

W artykule rozpatrywany jest problem zależności w ogonie dla rozkładów dwuwymiarowych. Przedstawiono przegląd różnych podejść do analizowania tej zależności. Szczególna uwaga poświęcona została warunkowym współczynnikom korelacji oraz współczynnikom zależności w ogonie. Wskazano, jak te współczynniki mogą być analizowane za pomocą tzw. analizy połączeń.