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DYNAMIC BAYESIAN INFERENCE IN GARCH PROCESSES WITH SKEWED-T AND STABLE CONDITIONAL DISTRIBUTIONS**

Abstract. In AR(1)-GARCH(1, 1) framework for daily returns, proposed and adopted by Bauwens and Lubrano (1997), Bauwens et al. (1999), Osiewalski and Pipień (2003), we considered two types of conditional distribution. In the first model (M_1) we assumed conditionally skewed-*t* distribution (defined by Fernández and Steel 1998) while the second GARCH specification (M_2) is based on the conditional stable distribution. We present Bayesian updating technique in order to check sensitivity of the posterior probabilities of considered specifications with respect to new observations included into dataset. We also study differences between Bayesian inference about tails and asymmetry of the conditional distribution of daily returns and between one-day predictive densities of growth rates obtained from both models. The results of dynamic Bayesian estimation, prediction and comparison of explanatory power of models M_1 and M_2 are based on very volatile daily growth rates of the WIBOR one-month interest rates and daily returns on the PLN/USD exchange rate.

Keywords: stable distributions, skewed-*t* distributions, Bayesian updating, univariate GARCH.
JEL Classification: C11, C32, C52.

1. INTRODUCTION

Commonly used tool in forecasting the volatility of the financial time series, namely GARCH processes, were initially defined as a white noise stochastic processes with conditionally heteroscedastic normal distribution. After Bollerslev's (1986) definition of GARCH scheme, more leptokurtotic conditional distributions (than those of normal) have been also proposed

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and applied. For example, Bollerslev (1987) presented estimation of the conditionally Student-*t* GARCH (with unknown degrees of freedom parameter). Nelson (1991) considered GARCH-type process with generalised error distribution (GED). Rachev and Mittnik (2002) present results of modeling the volatility of the daily returns using GARCH processes with conditional Weibull, Double Weibull, mixture of normals and Laplace distributions.

GARCH processes with conditional stable distributions have been also considered (e.g. McCulloch 1985, Liu and Brorsen 1995, Panorska et al. 1995, Mittnik et al. 2002 and Rachev and Mittnik 2002). The main advantage of stable GARCH processes is the fact that conditional normality can be tested in this framework. Additionally, stable distributions are able in general to capture heavy tailedness and possible skewness of the conditional distribution of returns.

Fernández and Steel (1998) proposed a generalization of Student-*t* distribution, namely the skewed Student-*t* distribution, which allowed in a very simple way for heavy tails as well as for possible distributional asymmetry. Osiewalski and Pipień (2003) presented Bayesian estimation and forecasting in GARCH models with conditional Skewed-*t* distribution. The main purpose of Pipień (2004) was Bayesian comparison of AR(1)-GARCH(1, 1) models with skewed-*t* and stable conditional distributions. Pipień (2004) presented posterior probabilities of models, posterior distributions of common and model specific parameters as well as discussed differences between predictive distributions generated from both specifications. These empirical results were based on three time series, namely daily returns of the PLN/USD exchange rate, daily returns on the Warsaw Stock Exchange index (WIG) and daily growth rates of the WIBOR one month zloty interest rate.

The main goal of this paper is to apply Bayesian updating technique in order to check sensitivity of the posterior probabilities of skewed-*t* and stable GARCH models, with respect to new observations included into dataset. The results of dynamic Bayesian comparison of explanatory power of conditionally skewed-*t* GARCH(1, 1) model (M_1) and conditionally stable GARCH(1, 1) model (M_2) are based on daily growth rates of the WIBOR one-month interest rates (dataset A) and daily returns on the PLN/USD exchange rate (dataset B). In both cases (A and B) starting from time series consisting of 100 observations, every time when we updated daily observation into dataset, we recalculated posterior distribution of parameters and posterior probabilities of models M_1 and M_2. We also study differences between Bayesian inference about tails and asymmetry of the conditional distribution of daily returns obtained from both models. As a result of application of dynamic Bayesian inference, we present highest posterior density intervals of tail and asymmetry parameters for model M_1 and M_2 and one-step predictive densities of daily growth rates.

2. SKEWED STUDENT-T AND STABLE DISTRIBUTION

Following the definition in Fernández and Steel (1998) let denote by z a random variable with skewed-t distribution with $\nu > 0$ degrees of freedom, modal parameter μ, inverse precision h0 and asymmetry parameter $h > 0$ $\gamma > 0$ ($z \sim Skt((\nu, \mu, h, \gamma)$. The density function of the distribution of z is given by the formula:

$$f_{Sks}(z|\nu,\mu,h,\gamma) = \frac{2}{(\gamma+\gamma^{-1})}\Big\{f_s((z-\mu)/\gamma|\nu,\mu,h)\gamma^2 I_{(-\infty,0)}(z-\mu) + \\ + f_s((z-\mu)\gamma|\nu,\mu,h)\gamma^{-2} I_{[0,+\infty)}(z-\mu)\Big\}, \tag{1}$$

where $f_s(x|\nu,\mu,h)$ denotes the value of the density function of the Student-t distribution with $\nu > 0$ degrees of freedom, modal parameter μ and inverse precision $h > 0$, calculated at point x:

$$f_s(x|\nu,\mu,h) = \frac{\Gamma((\nu+1)/2))}{\Gamma(\nu/2)\sqrt{\pi\nu h}}[1+(h\nu)^{-1}.(x-\mu)^2]^{-(\nu+1)/2}.$$

The shape parameter $\nu > 0$ controls tail behavior, mode μ and inverse precision h are the location and dispersion measures. Parameter γ captures possible asymmetry. In general γ^2 is the ratio of the probability masses on the right and on the left side of the mode of the distribution of z. Hence, if $\gamma = 1$, then z follows symmetric Student-t distribution. Under symmetry ($\gamma = 1$) it is also clear, that, for $\nu > 1$, $E(z)$ exists and is equal to μ.

The class of stable distributions is defined as a parametric family of continuous random variables closed with respect to the operation of summing. Hence, for any finite subset $\{w_1, ..., w_n\}$ of stable random variables, the linear combination $w = \alpha_1 w_1 + ... + \alpha_n w_n$ has also stable distribution ($\alpha_1, ..., \alpha_n$ are real numbers). Analytic expression for the characteristic function of stable random variable is given as follows:

$$\varphi(t) = \exp\left\{i\mu t - |ht|^2\left[1 - i\beta\frac{t}{|t|}\omega(|t|,\alpha)\right]\right\}, \tag{2}$$

$$\omega(|t|,\alpha) = \begin{cases} tan(\pi\alpha/2) & \text{if} \quad \alpha \neq 1 \\ -\frac{2}{\pi}\log|t| & \text{if} \quad \alpha = 1 \end{cases}$$

(e.g., Zolotarev 1961). In (2) the shape parameter $\alpha \in (0,2]$ called index of stability or characteristic exponent) defines the "fatness of tails" of density function (large α implies thin tails), μ and h are the location and scale parameters, $\beta \in [-1, 1]$ is the skewnes parameter (the symmetric stable distribution corresponds to $\beta = 0$). For $\alpha \in [1, 2]$ and $\beta = 0$ the location μ is also equal to the expected value of random variable w. We denote by $w \sim Sta(\alpha, \mu, h, \beta)$, that w is stably distributed with index of stability α, location parameter μ, scale h and skewness β. There are three cases, where the closed form expressions for the density of the stable random variable is known. A normal distribution is the case with $\alpha = 2$, a Cauchy distribution is the case with $\alpha = 1$ and $\beta = 0$, a Lévy distribution corresponds to the case with $\alpha = 0.5$ and $\alpha = 1$.

Practical application of stable random variables in econometric modeling requires deriving density function of random variable w. It can obtained as the integral of (2):

$$f_{Sta}(w|\alpha, \mu, h, \beta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{iwt} \varphi(t) dt, \tag{3}$$

and have to be approximated by numerical integration (e.g. Mittnik et al. 1999, Rachev and Mittnik 2002).

3. COMPETING GARCH SPECIFICATIONS

Let denote by x_j the value of a currency (stock market index, interest rate, exchange rate) at time j. Following Bauwens and Lubrano (1997), Bauwens et al. (1999), Osiewalski and Pipień (2003) let assume an AR(2) process for $\ln x_j$ with asymmetric GARCH(1, 1) error. In terms of logarithmic growth rates $y_j = 100 \ln (x_j / x_{j-1})$ our basic model framework is defined by the following equation:

$$y_j - \delta = \rho \cdot (y_{j-1} - \delta + \delta_1 \ln x_{j-1} + \varepsilon_j, \qquad j = 1, 2, \ldots \tag{4}$$

In the first model, M_1, we assume for the error term ε_j in (4), that $\varepsilon_j = z_j (h_j)^{0.5}$, where z_j are independent, skewed Student-t random variables, with $v > 0$ degrees of freedom parameter, mode $\zeta_1 \in (-\infty, +\infty)$, unit precision and asymmetry parameter $\gamma > 0$; i.e. $z_j \sim iiSkt(v, \zeta_1, 1, \gamma)$. Defining h_j we follow Glosten et al. (1993) asymmetric GARCH(1, 1) specification:

$$h_j = a_0 + a_1 \varepsilon_{j-1}^2 I(\varepsilon_{j-1} < 0) + a_1^+ \varepsilon_{j-t}^2 I(\varepsilon_{j-1} \geqslant 0) + b_1 h_{j-1}, \quad j = 1, 2, \ldots \tag{5}$$

which allows to model asymmetric reaction of conditional dispersion measure h_j to positive and negative sign of shock ε_{j-1}. The original GARCH(1, 1) formulation proposed by Bollerslev (1986) can be obtained from (5) by imposing restriction $a_1/a_1^+ = 1$. In (7) we also treat h_0 as an additional parameter. Model M_1 assumes, that the conditional distribution (given the past of the process, ψ_{j-1}, and the parameters) of the error term ε_j is the skewed-t distribution with $\nu > 0$ degrees of freedom parameter, mode $\zeta_1 \in (-\infty, +\infty)$, inverse precision h_j and asymmetry parameter $\gamma > 0$:

$$\varepsilon_j | \psi_{j-1}, M_1 \sim Skt(\nu, \zeta_1, h_j, \gamma). \qquad j = 1,\ 2,\ \ldots$$

In model M_2, $\varepsilon_j = w_j(h_j)^{0.5}$, where w_j are independent stable random variables with $\alpha \in (0,2]$, location parameter $\zeta_2 \in -\infty, +\infty)$, unit scale and skewness parameter $\beta \in [-1, 1]$; i.e. $w_j \sim iiSta(\alpha, \zeta_2, 1, \beta)$. Just like in model M_1 we assume for h_j asymmetric GARCH(1, 1) process, (5). In specification M_2 ε_j has conditional (with respect to ψ_{j-1} and the parameters) Stable distribution with $\alpha \in (0, 2]$, location $\zeta_2 \in (-\infty, +\infty)$, scale parameter $h_j^{0.5}$ and skewness $\beta \in [-1, 1]$:

$$\varepsilon_j | \psi_{j-1},\ M_2 \sim Sta(\alpha, \zeta_2, h_j^{0.5}, \beta), \qquad j=1,\ 2,\ \ldots$$

Let denote by $\theta = (\delta, \rho, \delta_1, a_0, a_1, a_1^+, b_1, h_0)$ the vector of all common parameters for both, M_1 and M_2, models. We denote by $\eta_1 = (\zeta_1, \nu, \gamma)$ the vector of model specific parameters in M_1; $\eta_2 = (\zeta_2, \alpha, \beta)$ groups additional parameters for M_2. In model M_1 the conditional distribution of y_j is the skewed-t distribution with $\nu > 0$ degrees of freedom parameter, mode $\mu_j^{(1)} = \delta + \rho(y_{j-1} - \delta) + \delta_1 \ln x_{j-1} + \zeta_1 h_j^{0.5}$, inverse precision h_j and asymmetry parameter $\gamma > 0$:

$$p(y_j | \psi_{j-1}, M_1, \theta, \eta_1) = f_{Sks}(y_j | \mu_j^{(1)}, h_j, \gamma), \qquad j = 1,\ 2,\ \ldots \tag{6}$$

In specification M_2 y_j has conditional stable distribution with $\alpha \in (0, 2]$, location $\mu_j^{(2)} = \delta + \rho(y_{j-1} - \delta) + \delta_1 \ln x_{j-1} + \zeta_2 h_j^{0.5}$, scale parameter $(h_j)^{0.5}$ and skewness $\beta \in [-1,1]$:

$$\mathrm{p}(y_j | \psi_{j-1}, M_2, \theta, \eta_2) = f_{Sta}(y_j | \alpha, \mu_j^{(2)}, h_j^{0.5}, \beta), \qquad j = 1,\ 2,\ \ldots \tag{7}$$

In both models the conditional distribution of y_t is heteroscedastic, where time varying dispersion measure h_j follows asymmetric GARCH(1, 1) equation (5). The degrees of freedom parameter, $\nu > 0$ and the characteristic exponent $\alpha \in (0, 2]$ enable also fat tails of $p(y_j | \psi_{j-1}, M_i, \theta, \eta_i)$ $(i = 1, 2)$. The possible asymmetry of conditional distribution of y_j can be modelled in M_1 by parameter $\gamma > 0$ or – in model M_2 – by $\beta \in [-1, 1]$. Hence, both sampling models are able to capture two generally appeared features of financial time series, i.e. heavy tails and asymmetry of the conditional distribution. For a discussion of potential differences in explanatory power of models M_1 and M_2 caused by definitions of stable and skewed-t families see Fernández and Steel (1998) and Pipień (2004).

4. COMPETING BAYESIAN MODELS AND DYNAMIC UPDATING

We denote by $y^{(t)} = (y_1, ..., y_t)$ the vector of observed up to day t (used in estimation in day t) daily growth rates and by $y_f^{(t)} = (y_{t+1}, ..., y_{t+k})$ the vector of forecasted observables at time t. The following density represents the i-th sampling model ($i = 1, 2$) at time t:

$$(8) \qquad p(y^{(t)}, y_f^{(t)} | M_i, \theta, \eta_i) = \prod_{j=1}^{t+k} p(y_j | \psi_{j-1}, M_i, \theta, \eta_i),$$

$$i = 1, 2, \quad t = T, \quad T+1, ..., T+T'$$

In specification M_1 the sampling model is based on the product of the appropriate skewed-t densities calculated at data point, namely on (6), while in model M_2 the density (8) is based on the product of stable densities (7). Constructed at time t Bayesian model M_i, i.e. the joint distribution of the observables $(y^{(t)}, y_f^{(t)})$ and the vector of parameters (θ, η_i):

$$(9) \qquad p(y^{(t)}, y_f^{(t)}, \theta, \eta_i | M_i) = p(y^{(t)}, y_f^{(t)} | \theta, \eta_i, M_i)\, p(\theta, \eta_i | M_i),$$

$$i = 1, 2, \quad t = T, \quad T+1, ..., T+T'$$

requires formulation of the prior distribution $p(\theta, \eta_i | M_i)$, which is invariant with respect to t. In both models we assume prior independence between vectors of common and model specific parameters. In each model we also assume the same proper prior structure for θ:

$$p(\theta, \eta_i | M_i) = p(\theta) \cdot p(\eta_i | M_i) \qquad i = 1, 2.$$

Our prior information about the common parameters is reflected by the following density $p(\theta)$:

(10) $$p(\theta) = p(\delta)p(\rho)p(\delta_1)p(a_0)p(a_1)p(a_1^+)p(b_1)p(h_0),$$

discussed in details in Osiewalski and Pipień (2003). In model M_1 we assume:

$$p(\eta_1|M_1) = p(\zeta_1, \nu, \gamma) = p(\zeta_1)p(\nu)p(\gamma),$$

where $p(\zeta_1)$ is standard normal, $p(\nu)$ is exponential with mean 10 and $p(\gamma)$ is log standard normal. The prior distribution of the model specific parameters in GARCH(1, 1) model, with stable conditional density, (M_2) is defined as follows:

$$p(\eta_2|M_2) = p(\zeta_2, \alpha, \beta) = p(\zeta_2)p(\alpha)p(\beta),$$

where $p(\zeta_2)$ is standard normal, $p(a)$ is uniform over interval $(0, 2]$ and $p(\beta)$ is uniform over $[-1,1]$.

The prior structure for common parameters as well as model specific prior assumptions for M_1 was presented in Osiewalski and Pipień (2003). Here we omit restrictions $\nu > 2$ and $\gamma \in (\exp(-2), \exp(2))$, imposed previously to guarantee existence of the second moment of $p(y_j|\psi_{j-1}, M_1, \theta, \eta_1)$. The prior distribution in model M_2 was discussed in details in Pipień (2004).

5. EMPIRICAL RESULTS

In this part we present an empirical example of dynamic Bayesian comparison of M_1 and M_2. We considered $T + T' + 1 = 1398$ observations of daily growth rates, y_j, of the WIBOR one month zloty interest rate from 20.03.1997 till 5.09.2002 (dataset A) and $T + T' + 1 = 1657$ observations of daily returns on the PLN/USD exchange rates from 5.02.1996 till 4.09.2002 (dataset B). Starting at $t = T = 100$ (which relates to 7.08.1997 for dataset A and to 25.06.1996 for dataset B) we calculated posterior probabilities of models M_1 and M_2, and posterior distribution of parameters based on dataset $y^{(t)}$, for each $t = 100$ up to $t = T + T' + 1$. As a result of daily updating observations into $y^{(t)}$ we obtained 1299 (for dataset A) and 1558 (for dataset B) posterior probabilities of models and posterior distributions of unknown parameters. The main purpose of the following presentation is to check sensitivity of the posterior probabilities (as well as of Bayesian inference about skewness and tails of conditional distribution of returns) with respect to new observations dynamically included into dataset $y^{(t)}$. We also study differences in the predictive distributions of future growth rates obtained from both models.

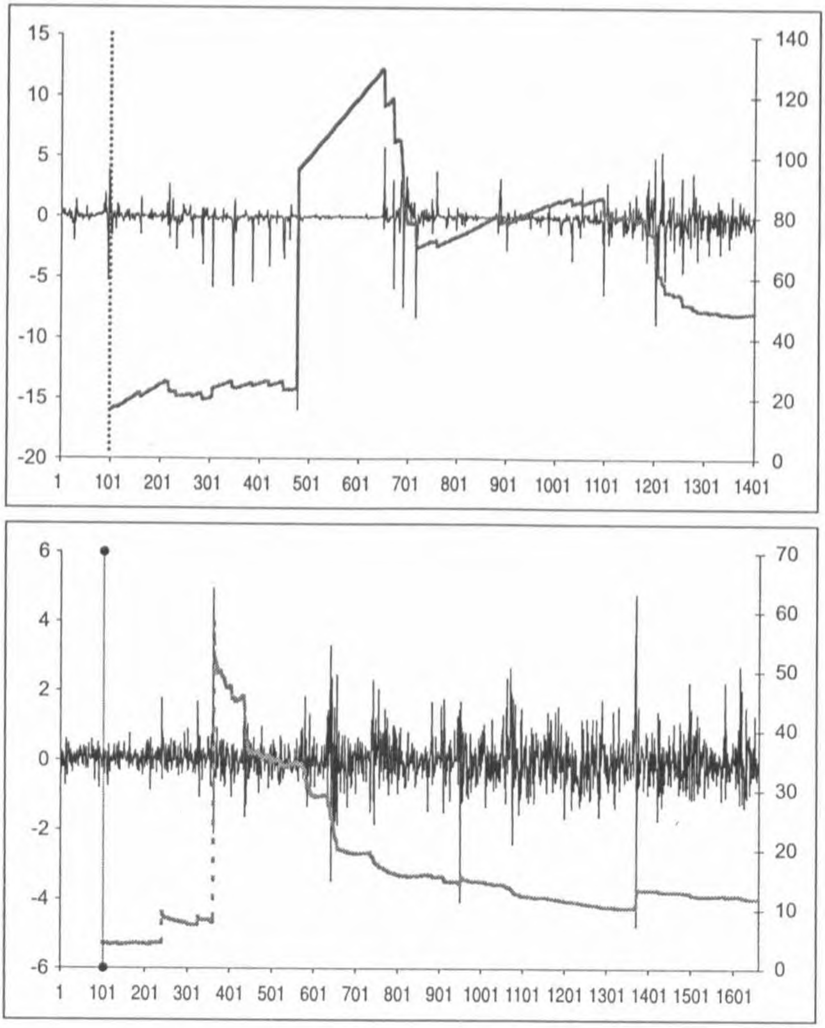

Fig. 1. Modeled time series with descriptive statistics

Figure 1 presents our both time series A and B. In Figures 1A and 1B on the left axis we plotted the vales of daily growth rates of the WIBOR one-month zloty interest rates and daily returns on the PLN/USD exchange rate (black line). In case of dataset A (Figure 1A) huge outliers in the plot of y_j, caused by changes in the monetary policy, together with the regions of almost no variability, depicts very anomalous behavior of daily changes of the Polish zloty middle term interest rate. Time series of daily growth rates of PLN/USD exchange rate is characterized by the presence of sparsely occured outliers with short-lived outbreaks of volatility. On the right axis in Figure 1A and 1B we plotted values of the sample kurtosis of $y^{(t)}$, $t = 100, ..., T + T'$ (grey line). In case of dataset A the fatness of tails of the empirical distribution of $y^{(t)}$ dramatically change with respect to t. In both cases we observe considerable variability of sample kurtosis, which – for dataset A – reaches values even greater than 130 and not less than 18. The vertical dotted lines in Figures 1A and 1B locates $t = 100$. It constitutes the shortest dataset used here in Bayesian inference in M_1 and M_2. Starting at this point, we recalculated posterior characteristics of models M_1 and M_2 every time the single observation of daily growth rates was included into $y^{(t)}$.

Figure 2 presents posterior probabilities $P(M_1|y^{(t)})$ (black line) and $P(M_2|y^{(t)})$ (grey line) obtained by assigning equal prior model probabilities ($P(M_1) = P(M_2) = 0.5$). In the first column of the first row of Figure 2 we present the results for dataset A, while the plot in the second column of the first row relates to the dataset B. The bottom plots of daily growth rates y_j ($j = 100$ to 1398) may help in visual assessment of the influence of new data included into $y^{(t)}$ on changes of the posterior probabilities. In case of datset A, the first 500 observations yield decisive support for GARCH model with skewed-t conditional distribution. Almost zero posterior probability $P(M_2|y^{(t)})$ makes stable GARCH completely improbable in the view of the data $y^{(t)}$, for $t = 100$ till about 560. For dataset A we also observe dramatic fall of the posterior probability $P(M_1|y^{(t)})$ for t greater than 600. It seems to be caused by the region of almost no variability of WIBOR one-month interest rate, which lies roughly between $t = 500$ and 650. Inclusion those observations into dataset makes $y^{(t)}$ (for $t = 650, ..., 700$) look like an almost non volatile series with huge negative outliers. Ever since, the data clearly support GARCH model with stable conditional distribution. We observe that, for $t > 1100$, the posterior probability of model M_1 again starts to lift, making this specification more likely *a posteriori*. Regular fluctuations of y_j for $j = 1100, ..., 1398$ supported GARCH model with skewed-t conditional distribution.

For dataset B we observe successive growth of the strength of the data support in favor of model M_1. Starting from $t = 100$ observations, for $t = 100, ..., 250$, skewed-t GARCH model quickly receives the majority of the posterior probability. For $t > 250$ some occasional outliers – and especially structural break at $t = 385$ – temporarily reduce posterior probability $P(M_1|y^{(t)})$, making specification M_2 more probable in view of the data. After including $t > 1100$ observations the posterior probabilities of both specifications become insensitive with respect to new observations included into dataset. For $t > 1100$ the dataset B decisively reject stable GARCH model.

In Figure 3 we present plots reflecting dynamic changes in location and dispersion of the marginal posterior distributions of tail and asymmetry parameters of the conditional distribution of y_j in models M_1 and M_2.

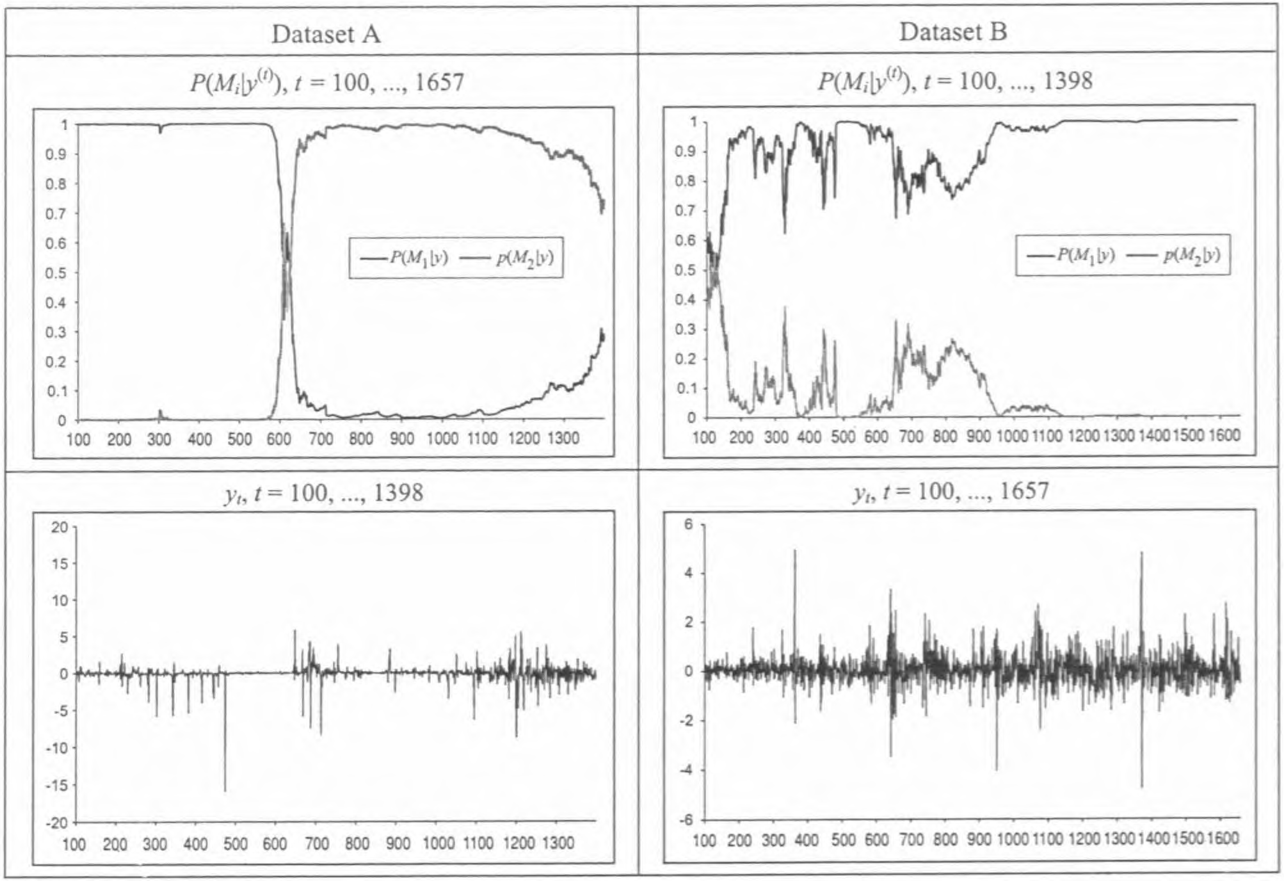

Fig. 2. Dynamic posterior probabilities of models M_1 and M_2 for datasets A and B

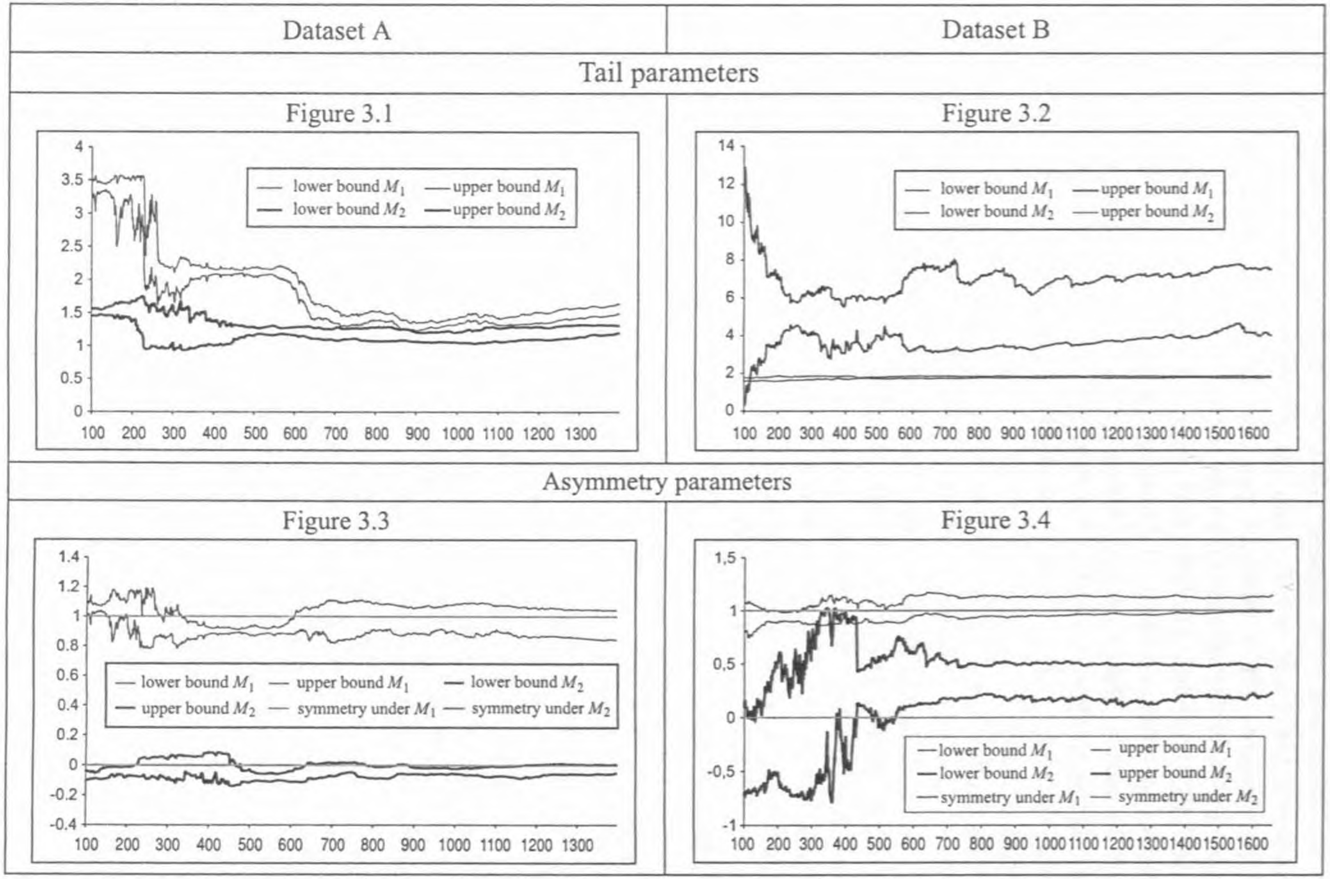

Fig. 3. HPD Intervals (for probability $1-\alpha=0.95$) of tail and asymmetry parameters obtained from models M_1 and M_2

For each dataset $y^{(t)}$ (for A $t = 100, ..., 1398$, and for B $t = 100, ..., 1657$) we calculated 95% highest posterior density (HPD) intervals for tail parameters α (M_2) and ν (M_1) and for asymmetry parameters β (M_2) and γ (M_1). Presented HPD intervals can be interpreted as the Bayesian 95% credible intervals for estimated parameters.

The HPD intervals, plotted on Figures 3.1 and 3.3 indicate fundamental differences in inference about the tails of the conditional distribution of y_j in case of dataset A. Based on time series $y^{(t)}$ both models, for $t = 100, ..., 650$, support different type of conditional distribution of return rates. Given model M_1, for $t = 100, ..., 650$, there is no doubt, that the second moment of the conditional distribution of y_j exists. At the same time, given model M_2, the data locate index α in the regions that would preclude conditional normality of y_j (cf. Figure 3.1). From the definition of stable random variables it is equivalent with non-existence of the second conditional moment. Similarly as for posterior probabilities of both models, inference about tails of the conditional distribution of y_j changes for t greater that 600 and additionally become quite unanimous. After updating about $t = 700$ observations the hypothesis of existence of the second conditional moment is strongly rejected in both models. For t greater than 700 HPD intervals for α (in M_2) and ν (in M_2) are both tightly located around the value 1.5 precluding existence of the variance of the conditional distribution of y_j.

Figure 3.2 in Table 3 presents the HPD intervals of tail parameters in M_1 and M_2 obtained in dataset B. Both models yield different information about existence of conditional moments of y_j. From the definition of stable family, stable GARCH specification precludes existence of second moment of $p(y_j|\psi_{j-1}, M_2, \theta, \eta_2)$. As seen from Figure 3.2 the HPD intervals for parameter α are very tight and located very close to value $\alpha = 2$. Additionally, location as well as spread of the HPD intervals of parameter α remains insensitive to new observations updated in dataset B. In spite of significant changes in dispersion of the HPD intervals of parameter ν in model M_1, there is no doubt that $p(y_j|\psi_{j-1}, M_2, \theta, \eta_2)$ posses variance (cf. Figure 3.2). The plot of lower bound of the HPD intervals of ν shows, that for $t > 120$ more than 95% of the posterior probability of $p(\nu|y^{(t)}, M_1)$ is concentrated on the left side of the value $\nu = 2$ (see Figure 3.2).

HPD 95% intervals of asymmetry parameters are presented in Figure 3.3 (dataset A) and Figure 3.4 (dataset B). By grey horizontal lines we located symmetric cases of the conditional distributions (for M_1 it is the case with $\gamma = 1$ and for M_2 it corresponds to $\beta = 0$). In case of dataset A, just like for tail parameters, both models yield different conclusions about asymmetry of the conditional distribution of y_j for $t = 100, ..., 600$. Under model M_1, dataset $y^{(t)}$ (for $t = 100, ..., 450$) build posterior distribution of γ with very volatile location and dispersion. It makes uncertainty about possible skewness of the conditional distribution of y_j (given M_1) very sensitive to new observations updated in dataset. Huge negative outliers, together with the region of no variability ($t = 500, ..., 650$) leads to very tight posterior distribution $p(\gamma|y^{(t)})$ for $t = 400, ..., 600$, where 95% of the posterior probability is located at the very small region of parameter space. Dataset $y^{(t)}$ (for $t = 400, ..., 600$) leaves no doubt that conditional distribution of daily growth rates (given model M_1) is skewed to the left. For t greater than 600 the HPD intervals for parameter γ quickly start to widen. Consequently, given model M_1, for t greater than 600, the data $y^{(t)}$ do not preclude symmetry of the conditional distribution, because the value $\gamma = 1$ lies among lower an upper bound of the 95% HPD interval. In model M_2 the HPD interval of the asymmetry parameter β seems to be more dispersed and less sensitive to new observations than the HPD interval for parameter γ in model M_1. Except for $t = 100, .., 150$ and $t = 480, ..., 650$, the dataset $y^{(t)}$ do not preclude symmetry of the (stable) conditional distribution of y_j. In most cases of t the value $\beta = 0$ lies either in the interior of 95% HPD interval or is very close to its upper bound. In model M_2, the data always support the hypothesis of left asymmetry of the conditional distribution of y_j, rather than right asymmetry. Except for a very few cases of t, the majority of the probability of the posterior distribution of β lies below the value $\beta = 0$ (see Figure 3.3).

Quite regular fluctuations of daily returns of PLN/ USD exchange rate (dataset B) makes, in model M_1, inference about possible skewness of $p(y_j|\psi_{j-1}, M_1, \theta, \eta_1)$ consistent with model M_2. From Figure 3.4 we see, that, for $t = 100, ..., 500$, both models support symmetric case, leaving great uncertainty about possible right or left asymmetry of $p(y_j|\psi_{j-1}, M_i, \theta, \eta_i)$. For $t = 100, ..., 500$ the HPD intervals of β and γ are very dispersed and its location and spread is very sensitive with respect to the new observations. But, for $t > 500$, both specifications support hypothesis of right asymmetry of conditional distribution of y_j. As in model M_2, for $t > 500$, the HPD intervals of asymmetry parameter β lies on the right side of the value $\beta = 0$, stable GARCH supports right asymmetry stronger than model M_1. In case of M_1 the HPD interval for asymmetry parameter γ includes symmetric case ($\gamma = 0$), but the majority of posterior probability mass of $p(\gamma|y^{(t)}, M_1)$ is concentrated on the right side of the value $\gamma = 0$.

Figure 4 presents quantiles of order 0.95 and 0.05 of the one-step predictive densities at time t (predictive distributions of y_{t+1} given $y^{(t)}$) obtained from both models in datasets A and B. Figures 4.1 and 4.3 plots the quantiles of $p(y_{t+1}|M_i, y^{(t)})$ in case of dataset A, while Figures 4.2 and 4.4 relates to dataset B. As usual, in the third row we put our time series (A and B) in order to asses sensitivity of spread of considered predictive distributions with respect to new observations y_j. Time varying inverse

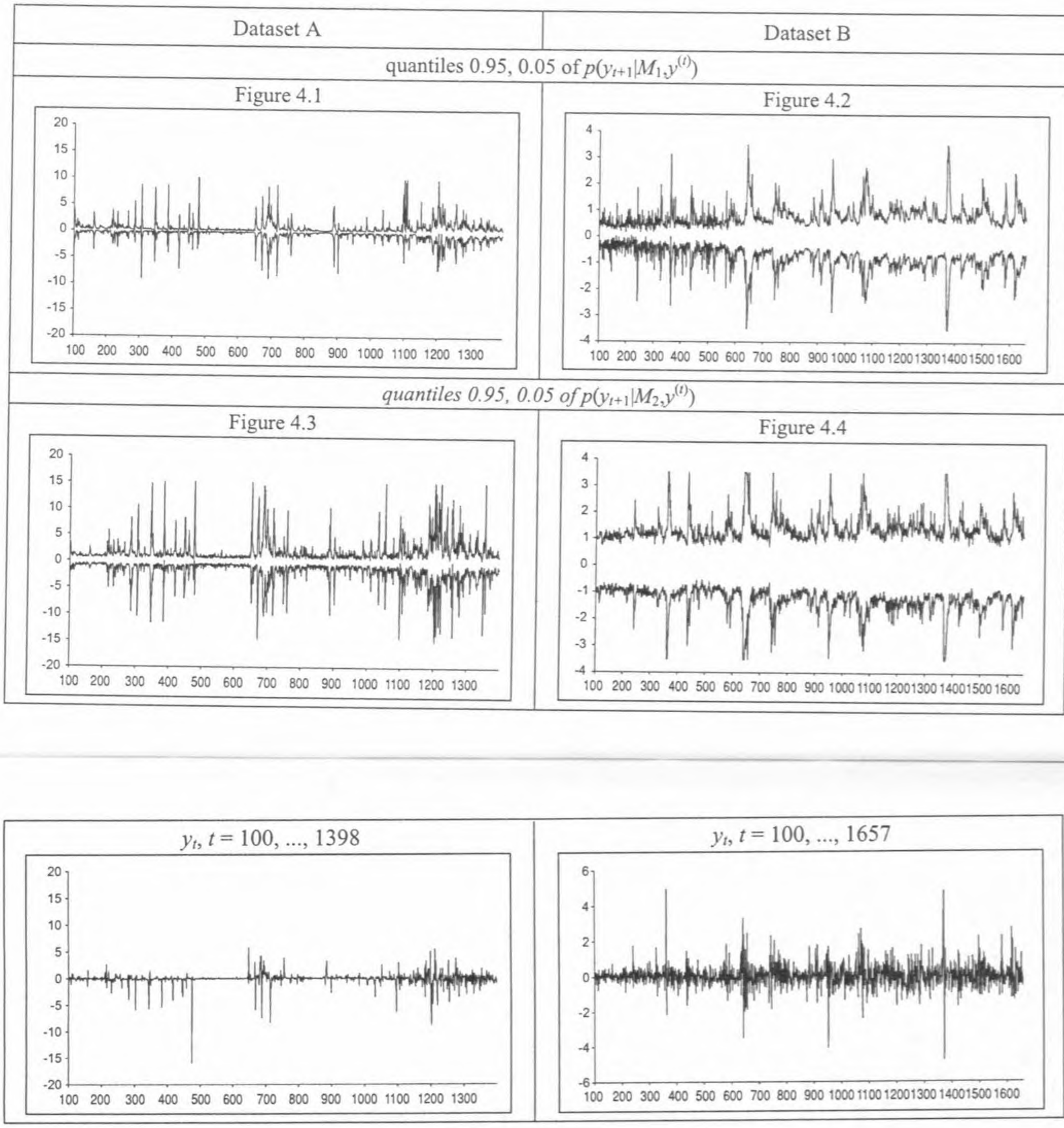

Fig. 4. Quantiles of orders 0.95 and 0.05 of the one-step predictive densities obtained from models M_1 (t = 100, ..., 1398) and M_2 (t = 100, ..., 1657)

precision in M_1 and scale parameter in M_2, which are both modeled by asymmetric-GARCH(1, 1) equation, make one day ahead predictive densities very sensitive to new observations included in observed time series. For both datasets, spread of $p(y_{t+1}M_i, y^{(t)})$ (as measured by quantiles of order 0.05 and 0.95) instantly responds to changes in the volatility (dataset B) or occasional huge outliers (dataset A). Additionally, either dataset A or B indicate, that stable GARCH model generate one day predictive densities more dispersed than those obtained from model M_1.

Visible difference in distance between quantiles of order 0.05 and 0.95 of the predictive distributions $p(y_{t+1}|M_i, y^{(t)})$ $(i = 1, 2)$ may be the crucial point in analyzing discrepancies of data support of skewed-*t* and stable GARCH models. In the constant location and scale framework Fernández and Steel (1998) compared sampling distributions obtained from skewed-*t* or stable assumption about the error term. The benchmark of comparison was empirical distribution of modeled time series. As a one of the results, which was also obtained for many time series by Rachev and Mittnik (2002), Fernández and Steel (1998) report almost imperceptible differences in data fit of skewed-*t* and stable regression models. The plots of sampling densities obtained from location and scale skewed-*t* and stable models were very similar, and fitted well to empirical density. As seen from Figure 4, taking into consideration the posterior uncertainty about parameters, makes the predictive densities (obtained from M_1 and M_2) very different. It seems that both models reflect different posterior information about common and model specific parameters. Consequently, M_1 and M_2 yield different *ex-ante* uncertainty about future growth rates.

6. CONCLUSIONS

In AR(1)-GARCH(1,1) framework for daily returns, proposed and adopted by Bauwens and Lubrano (1997), Bauwens et al. (1999) Osiewalski and Pipień (2003), there are considered in the paper two types of conditional distribution. In the first model (M_1) we assumed conditionally skewed-*t* distribution (defined by Fernández and Steel 1998) while the second GARCH specification (M_2) is based on the conditional stable distribution. We presented Bayesian updating technique in order to check sensitivity of the posterior probabilities of considered specifications, with respect to new observations included into dataset. We also studied differences between Bayesian inference about tails and asymmetry of the conditional distribution of daily returns and the one-step predictive distributions obtained from both models.

Based on very volatile daily growth rates of the WIBOR one-month interest rates (dataset A, 1398 observations) as well as on daily returns on the PLN/USD exchange rate (dataset B, 1657 observations), we calculated the posterior probabilities of models M_1 and M_2, and the posterior distribution of parameters using dataset $y^{(t)}$, for each $t = 100$ up to $t = T + T' + 1$ (which is equal to 1398 for dataset A and 1657 for B). The main empirical result of this paper is great sensitivity of the posterior model probabilities with respect to new observations of y_j included into dataset. Daily returns of dataset A characterized by very weak variability with unexpected huge negative outliers decisively supported GARCH model with conditional stable distribution. After including more volatile observations into dataset A, we observed that the posterior probability of model M_1 started to increase. For dataset B we observe successive growth of the strength of the data support in favor of model M_1. For $t > 1100$ observations of daily returns of the PLN/USD exchange rate, skewed-t GARCH model receives the whole posterior probability, making stable GARCH completely rejected by the dataset B.

We also checked conformity of inference about tails and asymmetry of the conditional distribution of daily returns. In case A, for short time series both models yielded different information about existence of moments as well as possible skewnes of $p(y_j|\psi_{j-1}, M_i, \theta, \eta_i)$. However, for datasets, which consisted more than 700 observations of daily growth rates of WIBOR1m, both models pointed to qualitatively similar results of the properties of the conditional distribution of y_j. For dataset B stable GARCH models was not able to model properly tails of the conditional distribution of returns. HPD intervals of the degrees of freedom parameter in model M_1 (skewed-t GARCH) decisively supported hypothesis, that the second and third conditional moment of $p(y_j|\psi_{j-1}, M_1, \theta, \eta_1)$ exist. However, tightly concentrated around value $v = 3$ posterior distribution $p(v|y^{(t)})$, precludes conditional normality. Dataset B supported more flexible skewed-t GARCH models, making conditional stability improbable *a posteriori*.

Both models built one day predictive distributions very sensitive to new observations included. We observed instant reaction of the spread of $p(y_{t+1}|M_i, y^{(t)})$ $(i = 1, 2)$ on occasionally appeared outliers or unexpected intensifications of volatility. For both datasets predictive distributions obtained from model M_2 has greater dispersion than those obtained from skewed-t GARCH model. It seems that, in building predictive distributions, posterior uncertainty about common and model specific parameters of specifications M_1 and M_2 lead up to different ex ante uncertainty about future growth rates.

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DYNAMICZNE WNIOSKOWANIE BAYESOWSKIE W PROCESACH GARCH ZE SKOŚNYMI T-STUDENTA I STABILNYM ROZKŁADEM WARUNKOWYM

(Streszczenie)

W artykule przedstawiono modele AR(1)-GARCH(1,1) dla dziennych stóp zmian (por. Bauwens i Lubrano 1997, Bauwens i in. 1999, Osiewalski i Pipień 2003) z różnymi typami rozkładu warunkowego. W pierwszym przypadku (model M_1) rozważono warunkowy rozkład skośny *t*-studenta (zdefiniowany przez Fernández i Steela 1998), podczas gdy model M_2 to proces GARCH o warunkowym rozkładzie α-stabilnym. Prezentujemy bayesowską aktualizację rozkładów *a posteriori* i predyktywnych (wraz z napływem nowych danych) w celu zbadania, czy typ rozkładu warunkowego zadany w procesie GARCH wpływa na wnioskowanie o naturze procesów opisujących zmienność finansowych szeregów czasowych o dużej częstotliwości. Rezultaty dynamicznej estymacji wykorzystującej podejście bayesowskie zilustrowano na przykładzie dwóch szeregów czasowych, tzn. dziennych stóp zmian kursu walutowego PLN/USD oraz oprocentowań jednomiesięcznych lokat międzybankowych (WIBOR1m).