A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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NOTES ON THE OPTIMUM CHEMICAL BALANCE WEIGHING DESIGN

Abstract. In the paper, the model of the chemical balance weighing design, i.e. model in that the result of experiment we can describe as linear function of unknown measurements of objects with known factors, is presented. Additionally, we assume that the measurement errors are uncorrelated and they have different variances. The problem is to estimate unknown measurements of objects. The existence conditions setting the optimum design and new construction method of the matrix determining the conditions of the experiment, are presented.

Key words: balanced bipartite weighing design, chemical balanced weighing design, ternary balanced block design.

I. INTRODUCTION

We consider the linear model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, that describe how to determine unknown measurements (weighings) of p objects using n measurement operations according to the design matrix $\mathbf{X} = \left(x_{ij}\right)$, where $x_{ij} = -1, 0, 1$, $i = 1,2,...,n$, $j = 1,2,..., p$. We will denote by $\mathbf{y}$ an $n \times 1$ random vector of the observed measurements, $\mathbf{w}$ is a $p \times 1$ vector representing unknown measurements of objects. It is assumed that there are not systematic errors, moreover the errors have different variances and they are uncorrelated, i.e. for the $n \times 1$ random vector of errors $\mathbf{e}$, $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{E}(\mathbf{e}\mathbf{e}') = \sigma^2\mathbf{G}$, where

$$\mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} \\ \mathbf{0}_{n_2}\mathbf{0}'_{n_1} & \mathbf{G}_2 \end{bmatrix}, \text{ where } \mathbf{G}_1 = \frac{1}{a}\mathbf{I}_{n_1}, \quad \mathbf{G}_2 = \mathbf{I}_{n_2}, \tag{1}$$

$a > 0$ is known scalar, $n = n_1 + n_2$, $\mathbf{0}_n$ is $n \times 1$ vector of zeros, $\mathbf{G}$ is an $n \times n$ positive definite diagonal matrix of known elements, $\mathrm{E}(\cdot)$ stands for the expectation

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of $(\cdot)$ and $(\cdot)'$ is used for the transpose of $(\cdot)$. It is obvious that we have many interesting possibilities of the patterns of the dispersion matrix. For each form of the matrix $\mathbf{G}$, the conditions determining the optimum chemical balance weighing design and the construction of the design matrix must be investigated separately. In the literature, for instance in the paper of Banerjee (1975), the matrix $\mathbf{X}$ of elements equal to –1, 0, 1 is called weighing matrix and can be interpreted as a weighing design for the two–pan scale or chemical scale, the relevant design is called a chemical balance weighing design. The structure of the matrix $\mathbf{G}$ given in (1) may be useful when the measurements are taken in two factories with different precisions. For the estimation of unknown measurement of objects, using the weighed least squares method we obtain $\hat{\mathbf{w}} = \left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\left(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\right)^{-1}$ provided $\mathbf{X}$ is full column rank, i.e. $\mathrm{r}(\mathbf{X}) = p$.

For $\mathbf{G} = \mathbf{I}_n$, some problems concerning the optimality criterions and the construction methods of the chemical balance weighing designs are considered in the literature. For details see Raghavarao (1971), Banerjee (1975), Wong and Masaro (1984), Shah and Sinha (1989), Pukelsheim (1993). For diagonal matrix $\mathbf{G}$, the conditions determining optimal design and the series of the optimal chemical balance weighing designs are given in Ceranka and Graczyk (2003).

Let us consider the design matrix of the chemical balance weighing design in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}, \tag{2}$$

where $\mathbf{X}_h$ is the $n_h \times p$ matrix of elements equal to – 1, 0, 1, $h = 1,2$.

For $\mathbf{X}$ in (2) and $\mathbf{G}$ in (1), we get from Ceranka and Graczyk (2003)

Definition 1. Any nonsingular chemical balance weighing design matrix $\mathbf{X}$ in (2) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), is called optimal if the variance of each estimator of unknown measurements of objects attains the lower bound, i.e. $\mathrm{Var}(\hat{w}_j) = \dfrac{\sigma^2}{am_1 + m_2}$, where $m_h = \max\{m_{1h}, m_{2h}, ..., m_{ph}\}$, m_{jh} denotes the number of nonzero elements in the j^{th} column of the matrix $\mathbf{X}_h$, $h = 1,2$, $j = 1,2,..., p$.

Also from Ceranka and Graczyk (2003) we have the following Theorem.

Theorem 1. Any nonsingular chemical balance weighing design matrix $\mathbf{X}$ in (2) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), is optimal if and only if

$$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = (am_1 + m_2)\mathbf{I}_p. \tag{3}$$

In practice, we are not able to construct the design matrix of the optimum chemical balance weighing design for any number of objects and any number of measurements. Because of this, the aim of the present paper is to wide the number of possible combination of p and n by constructing the design matrix of the optimum chemical balance weighing design for $p+1$ objects from the design matrix of the optimum design for p objects. Based on the matrix $\mathbf{X}$ in (2) of the optimum chemical balance weighing design we form the design matrix for $p+1$ objects as

$$\mathbf{X}^* = [\mathbf{X} \quad \mathbf{d}], \text{ where } \mathbf{d} = \left[\mathbf{1}_{n_1}' \quad \mathbf{0}_{n_2}'\right]', \tag{4}$$

where $\mathbf{1}_s$ denotes the $s \times 1$ vector of ones.

Theorem 2. Any nonsingular chemical balance weighing design $\mathbf{X}^*$ in (4) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), is optimal if

(i) the condition (3) is satisfied,

(ii) $\mathbf{X}_1'\mathbf{1}_{n_1} = \mathbf{0}_p$ and

(iii) $am_1 + m_2 = an_1$.

Proof. For the design matrix $\mathbf{X}^*$ in (4) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), we obtain

$$\mathbf{X}^{*'}\mathbf{G}^{-1}\mathbf{X}^* = \begin{bmatrix} \mathbf{X}'\mathbf{G}^{-1}\mathbf{X} & a\mathbf{X}_1'\mathbf{1}_{n_1} \\ a\mathbf{1}_{n_1}'\mathbf{X}_1 & n_1 \end{bmatrix}. \tag{5}$$

Considering (5) and the optimality condition (3) we have $\mathbf{X}_1'\mathbf{1}_{n_1} = \mathbf{0}_p$ and $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = an_1\mathbf{I}_p$, that complete the proof.

In the next sections, we will consider the methods of construction of the optimum chemical balance weighing design $\mathbf{X}^*$ based on the incidence matrices of the balanced bipartite weighing designs and the ternary balanced block designs.

II. BALANCED DESIGNS

Now, we remind the definition of the balanced bipartite weighing design given in Huang (1976) and of the ternary balanced block design given in Billington (1984).

The balanced bipartite weighing design is an arrangement of v treatments in b blocks such that each block containing k distinct treatments is divided into 2 subblocks containing k_1 and k_2 treatments, respectively, where $k = k_1 + k_2$. Each treatment appears in r blocks. For every pair of treatments, both treatments appear in the different subblocks in λ_1 blocks and in the same subblock in λ_2 blocks. The integers $v, b, r, k_1, k_2, \lambda_1, \lambda_2$ are called the parameters of the balanced bipartite weighing design. The parameters are not independent and they are related by the following identities $vr = bk$, $b = \frac{\lambda_1 v(v-1)}{2k_1k_2}$, $\lambda_2 = \frac{\lambda_1(k_1(k_1-1)+k_2(k_2-1))}{2k_1k_2}$, $r = \frac{\lambda_1 k(v-1)}{2k_1k_2}$. Let $\mathbf{N}^*$ be the incidence matrix of such a design with elements equal to 0 or 1 and $\mathbf{N}^*\mathbf{N}^{*'} = (r - \lambda_1 - \lambda_2)\mathbf{I}_v + (\lambda_1 + \lambda_2)\mathbf{1}_v\mathbf{1}_v'$. If in the balanced bipartite weighing design $k_1 \neq k_2$, then each object occurs in $r_{(1)}$ blocks in the first subblock and in $r_{(2)}$ blocks in the second subblock, $r_{(1)} + r_{(2)} = r$ and $r_{(1)} = \frac{\lambda_1(v-1)}{2k_2}$, $r_{(2)} = \frac{\lambda_1(v-1)}{2k_1}$.

Any ternary balanced block design is a design that describe how to replace v treatments in b blocks, each of size k in such a way that each treatment appears 0, 1 or 2 times in r blocks. Each of the distinct pairs of treatments appears λ times. Any ternary balanced block design is regular, that is, each treatment occurs alone in ρ_1 blocks and is repeated two times in ρ_2 blocks, where ρ_1 and ρ_2 are constant for the design. It is straightforward to verify that $vr = bk$, $r = \rho_1 + 2\rho_2$, $\lambda(v-1) = \rho_1(k-1) + 2\rho_2(k-2)$. $\mathbf{N}$ is the incidence matrix of such a design with elements equal to 0, 1 or 2 and moreover $\mathbf{N}\mathbf{N}' = (\rho_1 + 4\rho_2 - \lambda)\mathbf{I}_v + \lambda\mathbf{1}_v\mathbf{1}_v'$.

III. OPTIMAL DESIGNS

Let $\mathbf{N}_1^*$ be the incidence matrix of the balanced bipartite weighing design with the parameters $v, b_1, r_1, k_{11}, k_{21}, \lambda_{11}, \lambda_{21}$. From the matrix $\mathbf{N}_1^*$ we form the matrix $\mathbf{N}_1$ by replacing k_{11} elements equal to +1 of each column that correspond to the elements belonging to the first subblock by – 1. Thus each column of the matrix $\mathbf{N}_1$ will contain k_{11} elements equal to – 1, k_{21} elements equal to +1 and $v-k_{11}-k_{21}$ elements equal to 0. Let $\mathbf{N}_2$ be the incidence matrix of the ternary balanced block design with the parameters $v, b_2, r_2, k_2, \lambda_2, \rho_{12}, \rho_{22}$. Next, from the matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ we construct the chemical balance weighing design $\mathbf{X}^*$ in the form (4) for $\mathbf{X}_1 = \begin{bmatrix}\mathbf{N}_1 & -\mathbf{N}_1\end{bmatrix}'$, $\mathbf{X}_2 = \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v'$, $n_1 = 2b_1$, $n_2 = b_2$, $p = v+1$, as

$$\mathbf{X}^* = \begin{bmatrix} \mathbf{N}_1' & \mathbf{1}_{b_1} \\ -\mathbf{N}_1' & \mathbf{1}_{b_1} \\ \mathbf{N}_2' - \mathbf{1}_{b_2}\mathbf{1}_v' & \mathbf{0}_{b_2} \end{bmatrix}. \tag{6}$$

Lemma 1. Any chemical balance weighing design $\mathbf{X}^*$ in the form (6) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), is nonsingular if and only if $k_{11} \neq k_{21}$ or $v \neq k_2$.

Proof. For the design matrix $\mathbf{X}^*$ in (6) and $\mathbf{G}$ in (1) we have

$$\mathbf{X}^{*'}\mathbf{G}^{-1}\mathbf{X}^* = \begin{bmatrix} \mathbf{T} & \mathbf{0}_v \\ \mathbf{0}_v' & 2ab_1 \end{bmatrix}, \tag{7}$$

where

$\mathbf{T} = \eta\mathbf{I}_v + \left(2a(\lambda_{21}-\lambda_{11}) + b_2 - 2r_2 + \lambda_2\right)\mathbf{1}_v\mathbf{1}_v'$,

$\eta = 2a(r_1 - \lambda_{21} + \lambda_{11}) + r_2 + 2\rho_{22} - \lambda_2$.

Thus $\det\left(\mathbf{X}^{*'}\mathbf{G}^{-1}\mathbf{X}^*\right) = 2ab_1\eta^{v-1}\left(\frac{(v-1)\lambda_{11}(k_{11}-k_{21})^2}{2k_{11}k_{21}} + \frac{r_2(v-k_2)^2}{k_2}\right)$. Evidently $\eta > 0$. Hence $\det\left(\mathbf{X}^{*'}\mathbf{G}^{-1}\mathbf{X}^*\right) = 0$ if and only if $k_1 = k_2$ and $v = k_2$.

Theorem 3. Any nonsingular chemical balance weighing design $\mathbf{X}^*$ in the form (6) with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), is optimal if and only if the conditions

(i) $2a(b_1 - r_1) - b_2 + \rho_{12} = 0$ and

(ii) $2a(\lambda_{21} - \lambda_{11}) + b_2 + \lambda_2 - 2r_2 = 0$

are simultaneously fulfilled.

Proof. For $\mathbf{X}^*$ in (6) and $\mathbf{G}$ in (1), we obtain (7). The conditions (i)–(iii) of Theorem 2 imply the above result for $m_1 = 2r_1$, $m_2 = b_2 - \rho_{12}$.

For special forms of the matrix $\mathbf{G}$, based on Huang (1976), Billington and Robinson (1983), Ceranka and Graczyk (2004a,b, 2005) we formulate theorems given parameters of the optimum chemical balance weighing design. Based on the parameters of the balanced bipartite weighing design and the ternary balanced block design, we form appropriate incidence matrices and then the design matrix of the optimum chemical balance weighing design. The existence conditions of the balanced bipartite weighing designs are determined by the relation between subblock sizes. For any $a = \frac{b_2 - \rho_{12}}{2(b_1 - r_1)} > 0$ given by (i) of the Theorem 3, the condition (ii) will be always true if $\lambda_{21} - \lambda_{11} = 0$ and $b_2 + \lambda_2 - 2r_2 = 0$. If $\lambda_{21} - \lambda_{11} = 0$ then $k_{11} = \frac{c(c-1)}{2}$ and $k_{21} = \frac{c(c+1)}{2}$, $c = 2,3,\ldots$. Hence we obtain

Theorem 4. If the parameters of the balanced bipartite weighing design are equal to $v = t$, $b_1 = \frac{2st(t-1)}{c^2(c^2-1)}$, $r_1 = \frac{2s(t-1)}{c^2-1}$, $k_{11} = \frac{c(c-1)}{2}$, $b_1 = \frac{2st(t-1)}{c^2(c^2-1)}$, $r_1 = \frac{2s(t-1)}{c^2-1}$, $k_{11} = \frac{c(c-1)}{2}$, $k_{21} = \frac{c(c+1)}{2}$, $\lambda_{11} = \lambda_{21} = s$ and the parameters of the ternary balanced block design are of the form

(i) $v = t,\ b_2 = ut, r_2 = u(t-2),\ k_2 = t-2,\ \lambda_2 = \rho_{12} = u(t-4)$, $\rho_{22} = u,\ t = 5,6,\ldots$,
except the case $u = 1$ and $t = 5$, for $\varsigma = 4u$,

(ii) $v = t,\ b_2 = ut, r_2 = u(t-3), k_2 = t-3, \lambda_2 = u(t-6), \rho_{12} = u(t-9),\ \rho_{22} = 3u$, $t = 10,11,\ldots$, and for $\varsigma = 9u$,

(iii) $v = t, b_2 = ut, r_2 = u(t-4), k_2 = t-4, \lambda_2 = u(t-8), \rho_{12} = u(t-16), \rho_{22} = 6u$, $t = 17,18,\ldots$, and for $\varsigma = 16u$,

(iv) $v = t,\ b_2 = 4t,\ r_2 = 4(t-2),\ k_2 = t-2, \lambda_2 = \rho_{12} = 4(t-4),\ \rho_{22} = 4$, $t = 5,6,\ldots$, and for $\varsigma = 16$,

(v) $v = t, b_2 = 2t, r_2 = 2(t+1),\ k_2 = t+1,\ \lambda_2 = 2(t+2),\ \rho_{12} = 2(t-1), \rho_{22} = 2$, $t = 3,4,\ldots$, and for $\varsigma = 2$,

$s,u = 1,2,...$, $c = 2,3,...$, $t > c^2$, then $\mathbf{X}^*$ in the form (6) is the optimum chemical balance weighing design with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), where $\mathbf{G}_1 = \frac{4s(t-1)(t-c^2)}{\varsigma c^2(c^2-1)}\mathbf{I}_{2b_1}$.

Proof. It is easy to check that the parameters of the balanced bipartite weighing design and the ternary balanced block design satisfy the conditions of the theorem 3.

Theorem 5. If the parameters of the balanced bipartite weighing design are equal to $v = 2t^2$, $b_1 = \frac{4st^2(2t^2-1)}{c^2(c^2-1)}$, $r_1 = \frac{2s(2t^2-1)}{c^2-1}$, $k_{11} = \frac{c(c-1)}{2}$, $k_{21} = \frac{c(c+1)}{2}$, $\lambda_{11} = \lambda_{21} = s$ and the para-meters of the ternary balanced block design are of the form

(i) $v = 2t^2$, $b_2 = 4t^2$, $r_2 = 2t(2t+1)$, $k_2 = t(2t+1)$, $\lambda_2 = 4t(t+1)$, $\rho_{12} = 2t^2$, $\rho_{22} = t(t+1)$ and for $\tau = t^2$,

(ii) $v = 2t^2$, $b_2 = 4tu$, $r_2 = 2u(2t-1)$, $k_2 = t(2t-1)$, $\lambda_2 = 4u(t-1)$, $\rho_{12} = 2tu$, $\rho_{22} = u(t-1)$ and for $\tau = tu$,

$t,c = 2,3,...$, $2t^2 > c^2$, $s = 1,2,...$, then $\mathbf{X}^*$ in the form (6) is the optimum chemical balance weighing design with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), where $\mathbf{G}_1 = \frac{2s(2t^2-1)(2t^2-c^2)}{\tau c^2(c^2-1)}\mathbf{I}_{2b_1}$.

Proof. It is easy to see that the parameters of the balanced bipartite weighing design and the ternary balanced block design satisfy (i) and (ii) of the theorem 3.

Now, we consider the case $\lambda_{11} - \lambda_{21} \neq 0$ and $b_2 + \lambda_2 - 2r_2 \neq 0$ having the following theorems.

Theorem 6. If the parameters of the balanced bipartite weighing design are of the form

(i) $v = 9$, $b_1 = 18$, $r_1 = 10$, $k_{11} = 1$, $k_{21} = 4$, $\lambda_{11} = 2$, $\lambda_{21} = 3$,

(ii) $v = 9$, $b_1 = 36$, $r_1 = 28$, $k_{11} = 2$, $k_{21} = 5$, $\lambda_{11} = 10$, $\lambda_{21} = 11$

and the parameters of the ternary balanced block design are equal to $v = k_2 = 9$, $b_2 = r_2 = u + 8$, $\lambda_2 = u + 7$, $\rho_{12} = u$, $\rho_{22} = 4$, $u = 1,2,...$, then $\mathbf{X}^*$ in the form (6) is the optimum chemical balance weighing design with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), where $\mathbf{G}_1 = 2\mathbf{I}_{2b_1}$.

Proof. The proof is straightforward by checking that the parameters of the balanced bipartite weighing design and the ternary balanced block design satisfy the conditions of Theorem 3.

Theorem 7. For a given a, if the parameters of the balanced bipartite weighing design and of the ternary balanced block design are of the form

(i) $a = \frac{3}{2}$ and $v = 9, b_1 = 18, r_1 = 10, k_{11} = 2, k_{21} = 3, \lambda_{11} = 3, \lambda_{21} = 2$ and
$v = 9,\ b_2 = 27,\ r_2 = 15,\ k_2 = 5,\ \lambda_2 = 6,\ \rho_{12} = 3,\ \rho_{22} = 6$,

(ii) $a = \frac{1}{4}$ and $v = 9,\ b_1 = 36,\ r_1 = 16,\ k_{11} = 2, k_{21} = 2,\ \lambda_{11} = 4, \lambda_{21} = 2$ and
$v = 9,\ b_2 = 18,\ r_2 = 12,\ k_2 = 6,\ \lambda_2 = 7,\ \rho_{12} = 8,\ \rho_{22} = 2$,

(iii) $a = \frac{1}{3}$ and $v = 25,\ b_1 = 100, r_1 = 28, k_{11} = 1, k_{21} = 6, \lambda_{11} = 2, \lambda_{21} = 5$ and
$v = 25,\ b_2 = u + 49,\ r_2 = u + 49,\ k_2 = 25, \lambda_2 = u + 47,\ \rho_{12} = u + 1, \rho_{22} = 24$,

(iv) $a = \frac{1}{5}$ and $v = 49,\ b_1 = 294, r_1 = 54, k_{11} = 1, k_{21} = 8, \lambda_{11} = 2, \lambda_{21} = 7$ and
$v = 49,\ b_2 = u + 97,\ r_2 = u + 97,\ k_2 = 49,\ \lambda_2 = u + 95, \rho_{12} = u + 1, \rho_{22} = 48$,

$u = 1,2,...$, then $\mathbf{X}^*$ in the form (6) is the optimum chemical balance weighing design with $\sigma^2\mathbf{G}$ for $\mathbf{G}$ in (1), where $\mathbf{G}_1 = \frac{1}{a}\mathbf{I}_{2b_1}$.

Proof. Any computation show that the parameters of the balanced bipartite weighing design and the ternary balanced block design satisfy (i) and (ii) of the theorem 3.

IV. EXAMPLE

Let $n_1 = 20, n_2 = 10$ and $p = 6$. Then exists the balanced bipartite weighing design given in Theorem 4 (i) with $t = 5,\ s = 3,\ c = 2$ and with the parameters $v = 5,\ \ b_1 = 10,\ r_1 = 8,\ k_{11} = 1\ \ k_{21} = \lambda_{11} = \lambda_{21} = 3$ given by the incidence matrix

$$\mathbf{N}_1^* = \begin{bmatrix} 0 & 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 \\ 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 & 1_2 \\ 1_2 & 1_1 & 0 & 1_2 & 1_2 & 1_2 & 1_2 & 0 & 1_2 & 1_1 \\ 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 & 1_2 \\ 1_2 & 1_2 & 1_2 & 1_1 & 0 & 1_2 & 1_1 & 1_2 & 1_2 & 0 \end{bmatrix},$$

where 1_h denotes the element belonging to the h^{th} subblock, respectively, $h = 1,2$. From above matrix we form the matrix $\mathbf{N}_1$ as

$$\mathbf{N}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & -1 \\ 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 1 & 0 \end{bmatrix}.$$

From Theorem 4 (i) with $t = 5,\ u = 2$, there exists the ternary balanced block design with the parameters $v = 5,\ b_2 = 10, r_2 = 6, k_2 = 3, \lambda_2 = \rho_{12} = \rho_{22} = 2$ given by the incidence matrix

$$\mathbf{N}_2 = \begin{bmatrix} 2 & 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 1 & 1 \end{bmatrix}.$$

Form $\mathbf{N}_1$ and $\mathbf{N}_2$ we form the matrix $\mathbf{X}^*$ in (6). $\mathbf{X}^* = \begin{bmatrix} \mathbf{X}_1^* \\ \mathbf{X}_2^* \end{bmatrix}$ is the optimum chemical balance weighing design for $\sigma^2\mathbf{G}$, where $\mathbf{G}_1 = 0{,}5\mathbf{I}_{20}$, where

$$\mathbf{X}_1^* = \begin{bmatrix} 0 & -1 & 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 & 1 \\ -1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 1 & 1 & -1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 & 1 \\ 1 & 1 & -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & -1 & -1 & 1 \\ -1 & 0 & 1 & -1 & -1 & 1 \\ -1 & -1 & 0 & 1 & -1 & 1 \\ -1 & -1 & -1 & 0 & 1 & 1 \\ 1 & -1 & -1 & -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{X}_2^* = \begin{bmatrix} 0 & -1 & -1 & 1 & -1 & 1 \\ -1 & 0 & -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & -1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -1 & 1 \\ -1 & -1 & 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -1 & -1 & 0 \\ 1 & -1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & -1 & 1 & 0 \\ -1 & -1 & 1 & 0 & -1 & 0 \\ -1 & -1 & 1 & -1 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 \end{bmatrix}.$$

V. CONCLUSIONS

The theory and practice of the chemical balance weighing design is presented. There are given new construction methods of optimal design based on the incidence matrices of a balanced bipartite weighing design and a ternary balanced block design. The above example shows how to construct the design matrix and how estimate unknown measurement of objects in considered model.

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UWAGI O OPTYMALNYM CHEMICZNYM UKŁADZIE WAGOWYM

W pracy rozważa się model chemicznego układu wagowego, tzn. model w którym pomiar może być przedstawiony jako liniowa funkcja nieznanych miar obiektów o znanych współczynnikach. Dodatkowo zakłada się, że błędy wykonywanych pomiarów są nieskorelowane i mają różne wariancje. Naszym celem jest wyznaczenie nieznanych miar obiektów. W pracy podano warunki wyznaczające układ optymalny oraz konstrukcję macierzy, która opisuje sposób przeprowadzenia eksperymentu.