A C T A U N I V E R S I T A T I S L O D Z I E N S I S
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SOME CONSTRUCTION OF REGULAR A–OPTIMAL SPRING BALANCE WEIGHING DESIGNS FOR EVEN NUMBER OF OBJECTS

ABSTRACT. In the paper, the problem of construction of the spring balance weighing designs satisfying the criterion of A-optimality is discussed. The incidence matrices of the partially incomplete block designs are used for constructing the regular A-optimal spring balance weighing design.

Key words: A–optimal design, partially balanced incomplete block design, spring balance weighing design

I. INTRODUCTION

We study the experiment in which using n measurement operations we determine unknown measurements of p objects. The results of experiment can be written as $\mathbf{y} = \mathbf{Xw} + \mathbf{e}$, where $\mathbf{y}$ is an $n \times 1$ random vector of the observations, $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$, where $\mathbf{\Phi}_{n \times p}(0, 1)$ denotes the class of $n \times p$ matrices $\mathbf{X} = (x_{ij})$, $i = 1,2,...,n$, $j = 1,2,..., p$, having entries $x_{ij} = 1$ or 0 depending upon whether the jth object is included or excluded on the ith weighing operation, $\mathbf{w} = (w_1, w_2,..., w_p)'$ is a vector representing unknown measurements of objects and $\mathbf{e}$ is the $n \times 1$ random vector of errors. We assume that there are not systematic errors and the errors are uncorrelated and have constant variance σ^2, i.e. $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2 \mathbf{I}_n$, where $\mathbf{0}_n$ is $n \times 1$ vector of zeros, $\mathbf{I}_n$ is the $n \times n$ identity matrix. If the design matrix $\mathbf{X}$ is of full column rank, then all w_j are estimable and the variance matrix of their best linear unbiased estimator is $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$. The matrix $(\mathbf{X}'\mathbf{X})^{-1}$ is called the information matrix of $\mathbf{X}$.

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In many problems concerning weighing experiments the A–optimal designs are considered. There are designs for that the trace of $\left(\mathbf{X}'\mathbf{X}\right)^{-1}$ is minimal. The concept of A–optimality was considered, for instance, in the books of Raghavarao (1971), Banerjee (1975) and in the paper of Jacroux and Notz (1983). Moreover, the design for which the sum of variances of estimated measurements attains the lower bound is called the regular A–optimal design.

The main purpose of this paper is to obtain a new construction method, which gives the regular A–optimal spring balance weighing designs.

II. A–OPTIMAL DESIGN

For even p, let us consider the design matrix $\mathbf{X}_1 \in \mathbf{\Phi}_{h\times p}(0,1)$ satisfying the condition given by Jacroux and Notz (1983)

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{hp}{4(p-1)}\mathbf{I}_p + \frac{h(p-2)}{4(p-1)}\mathbf{1}_p\mathbf{1}_p' \tag{1}$$

where $\frac{hp}{4(p-1)}$, $\frac{h(p-2)}{4(p-1)}$ are some integers, $\mathbf{1}_p$ is the $p\times 1$ vector of ones. Moreover, for $h=n$ let

$$\mathbf{X} = \mathbf{X}_1, \tag{2}$$

for $h = n-1$ let

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \end{bmatrix}, \tag{3}$$

where $\mathbf{x}$ is the $p\times 1$ vector of elements equal to 0 or 1.

The theorem below is from Graczyk (2010).

Theorem 1. For even p, any nonsingular spring balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ of the form (3) is regular A–optimal if condition (1) is fulfilled and $\mathbf{x}'\mathbf{1}_p = \frac{p}{2}$.

Some method of construction of the regular A–optimal spring balance weighing design for matrix $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) based on the incidence matrices of the balanced incomplete block designs is given in by Jacroux and Notz (1983). Graczyk (2010) gave the conditions determining the regular A–optimal spring balance weighing design $\mathbf{X}$ in the form (3).

In this paper, for even p, we propose new construction method of the regular A–optimal spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in the form (2) which widest the class of the design matrices given by Jacroux and Notz (1983) and in the form (3). This method is based on the incidence matrices of two group divisible designs with the same association scheme.

III. CONSTRUCTION OF THE DESIGN MATRIX

Now, we recall the definition of the partially balanced incomplete block design with two associate classes given, for instance in Raghavarao and Padgett (2005).

A partially balanced incomplete block design with two associate classes is an arrangement of v treatments in b blocks, each of size k such that every treatment occurs at most once in a block and occurs in r blocks. Each treatment has exactly n_h treatments which are its h^{th} associates, $h = 1,2$. Two treatments which are h^{th} associate occur together in exactly λ_h blocks. The numbers $v,\ b,\ r,\ k,\ \lambda_1,\ \lambda_2$ are the parameters of the partially balanced incomplete block design. This design is usually identified by the associate scheme of treatments.

A group divisible design is a partially balanced incomplete block design with two associate classes for which the $v = ms$ treatments may be divided into m groups of s distinct treatments each, such that treatments belonging to the same group are first associates and two treatments belonging to different groups are second associates, $n_1 = s-1,\ n_2 = s(m-1),\ (s-1)\lambda_1 + s(m-1)\lambda_2 = r(k-1)$.

Now, based on the incidence matrices of two group divisible designs with the same association scheme, we construct the regular A–optimal spring balance weighing design.

For $p = v,$ let us consider the design matrix with $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$, where $\mathbf{N}_t$ is the incidence matrix of the group divisible design with the same association scheme with the parameters $v,\ b_t,\ r_t,\ k_t,\ \lambda_{1t},\ \lambda_{2t},\ t = 1,2$ and let

$$\lambda_{11} + \lambda_{12} = \lambda_{21} + \lambda_{22} = \lambda\,. \tag{4}$$

Theorem 2. Let p be even. If there exist the incidence matrices $\mathbf{N}_1$ and $\mathbf{N}_2$ of the group divisible design with the same association scheme with the parameters $v,\ b_t,\ r_t,\ k_t,\ \lambda_{1t},\ \lambda_{2t},\ t = 1,2,$ and if the conditions

(i) $b_1 + b_2 = 2(r_1 + r_2)$

(ii) $4\lambda(v-1)=(v-2)(b_1+b_2)$

are fulfilled simultaneously and

(a) if $n=b_1+b_2$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ in the form (2),

(b) if $n=b_1+b_2+1$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ in the form (3) for $\mathbf{x}'\mathbf{1}_p=\frac{p}{2}$,

with $\mathbf{X}_1=\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\end{bmatrix}'$ is the regular A–optimal spring balance weighing design.

Proof. Let $p=v$ and $h=b_1+b_2$. From the condition (1) we have

$$\mathbf{X}_1'\mathbf{X}_1=\mathbf{N}_1\mathbf{N}_1'+\mathbf{N}_2\mathbf{N}_2'=\frac{(b_1+b_2)v}{4(v-1)}\mathbf{I}_v+\frac{(b_1+b_2)(v-2)}{4(v-1)}\mathbf{1}_v\mathbf{1}_v'. \quad (5)$$

On the other hand, $\mathbf{N}_1\mathbf{N}_1'+\mathbf{N}_2\mathbf{N}_2'=(r_1+r_2-\lambda)\mathbf{I}_v+\lambda\mathbf{1}_v\mathbf{1}_v'$. Thus (5) is satisfied if and only if $\lambda=\frac{(b_1+b_2)(v-2)}{4(v-1)}$, thus we are given in the condition (ii). Considering Theorem 1 and the equality $\frac{(b_1+b_2)v}{4(v-1)}=r_1+r_2-\lambda$ we obtain the condition (i). Hence the result.

Based on the book of Clatworthy (1973) we formulate next theorems giving the parameters of group divisible design having appropriate design numbers.

Theorem 3. Let $v=4$. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible design with the same association with the parameters

(i) $b_1=8,\ r_1=4,\ k_1=2,\ \lambda_{11}=2,\ \lambda_{21}=1$ (R1) and
$b_2=10, r_2=5, k_2=2, \lambda_{12}=1,\ \lambda_{22}=2$ (R3),

(ii) $b_1=8,\ r_1=4,\ k_1=2,\ \lambda_{11}=2,\ \lambda_{21}=1$ (R1) and
$b_2=16,\ r_2=8,\ k_2=2,\ \lambda_{12}=2,\ \lambda_{22}=3$ (R10),

(iii) $b_1=8,\ r_1=4,\ k_1=2,\ \lambda_{11}=2,\ \lambda_{21}=1$ (R1) and
$b_2=4,\ r_2=2,\ k_2=2,\ \lambda_{12}=0,\ \lambda_{22}=1$ (SR1),

(iv) $b_1=10,\ r_1=5,\ k_1=2,\ \lambda_{11}=3,\ \lambda_{21}=1$ (R2) and
$b_2=14,\ r_2=7,\ k_2=2,\ \lambda_{12}=1,\ \lambda_{22}=3$ (R7),

(v) $b_1=10,\ r_1=5,\ k_1=2,\ \lambda_{11}=3,\ \lambda_{21}=1$ (R2) and
$b_2=20,\ r_2=10,\ k_2=2,\ \lambda_{12}=2,\ \lambda_{22}=4$ (R17),

(vi) $b_1=10,\ r_1=5,\ k_1=2,\ \lambda_{11}=3,\ \lambda_{21}=1$ (R2) and
$b_2=8,\ r_2=4,\ k_2=2,\ \lambda_{12}=0,\ \lambda_{22}=2$ (SR2),

(vii) $b_1=10,\ r_1=5,\ k_1=2,\ \lambda_{11}=1,\ \lambda_{21}=2$ (R3) and

$b_2 = 14,\ r_2 = 7,\ k_2 = 2,\ \lambda_{12} = 3,\ \lambda_{22} = 2$ (R6),

(viii) $b_1 = 10,\ r_1 = 5,\ k_1 = 2,\ \lambda_{11} = 1,\ \lambda_{21} = 2$ (R3) and

$b_2 = 20,\ r_2 = 10,\ k_2 = 2,\ \lambda_{12} = 4,\ \lambda_{22} = 3$ (R16),

(ix) $b_1 = 12,\ r_1 = 6,\ k_1 = 2,\ \lambda_{11} = 4,\ \lambda_{21} = 1$ (R4) and

$b_2 = 12,\ r_2 = 6,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 3$ (SR3),

(x) $b_1 = 14,\ r_1 = 7,\ k_1 = 2,\ \lambda_{11} = 5,\ \lambda_{21} = 1$ (R5) and

$b_2 = 16,\ r_2 = 8,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 4$ (SR4),

(xi) $b_1 = 14,\ r_1 = 7,\ k_1 = 2,\ \lambda_{11} = 3,\ \lambda_{21} = 2$ (R6) and

$b_2 = 16,\ r_2 = 8,\ k_2 = 2,\ \lambda_{12} = 2,\ \lambda_{22} = 3$ (R10),

(xii) $b_1 = 14,\ r_1 = 7,\ k_1 = 2,\ \lambda_{11} = 3,\ \lambda_{21} = 2$ and (R6)

$b_2 = 4,\ r_2 = 2,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 1$ (SR1),

(xiii) $b_1 = 14,\ r_1 = 7,\ k_1 = 2,\ \lambda_{11} = 1,\ \lambda_{21} = 3$ (R7) and

$b_2 = 16,\ r_2 = 8,\ k_2 = 2,\ \lambda_{12} = 4,\ \lambda_{22} = 2$ (R9),

(xiv) $b_1 = 16,\ r_1 = 8,\ k_1 = 2,\ \lambda_{11} = 6,\ \lambda_{21} = 1$ (R8) and

$b_2 = 20,\ r_2 = 10,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 5$ (SR5),

(xv) $b_1 = 16,\ r_1 = 8,\ k_1 = 2,\ \lambda_{11} = 4,\ \lambda_{21} = 2$ (R9) and

$b_2 = 20,\ r_2 = 10,\ k_2 = 2,\ \lambda_{12} = 2,\ \lambda_{22} = 4$ (R17),

(xvi) $b_1 = 16,\ r_1 = 8,\ k_1 = 2,\ \lambda_{11} = 4,\ \lambda_{21} = 2$ (R9) and

$b_2 = 8,\ r_2 = 4,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 2$ (SR2),

(xvii) $b_1 = 16,\ r_1 = 8,\ k_1 = 2,\ \lambda_{11} = 2,\ \lambda_{21} = 3$ (R10) and

$b_2 = 20,\ r_2 = 10,\ k_2 = 2,\ \lambda_{12} = 4,\ \lambda_{22} = 3$ (R16),

(xviii) $b_1 = 18,\ r_1 = 9,\ k_1 = 2,\ \lambda_{11} = 5,\ \lambda_{21} = 2$ (R12) and

$b_2 = 18,\ r_2 = 9,\ k_2 = 2,\ \lambda_{12} = 1,\ \lambda_{22} = 4$ (R13),

(xix) $b_1 = 18,\ r_1 = 9,\ k_1 = 2,\ \lambda_{11} = 5,\ \lambda_{21} = 2$ (R12) and

$b_2 = 12,\ r_2 = 6,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 3$ (SR3),

(xx) $b_1 = 20,\ r_1 = 10,\ k_1 = 2,\ \lambda_{11} = 6,\ \lambda_{21} = 2$ (R15) and

$b_2 = 16,\ r_2 = 8,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 4$ (SR4),

(xxi) $b_1 = 20,\ r_1 = 10,\ k_1 = 2,\ \lambda_{11} = 4,\ \lambda_{21} = 3$ (R16) and

$b_2 = 4,\ r_2 = 2,\ k_2 = 2,\ \lambda_{12} = 0,$

$b_2 = 4,\ r_2 = 2,\ k_2 = 2,\ \lambda_{12} = 0,\ \lambda_{22} = 1$ (SR1)

and let $\mathbf{X}_1 = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \end{bmatrix}'$.

(a) If $n = b_1 + b_2$ then $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0,1)$ in the form (2),

(b) if $n = b_1 + b_2 + 1$ then $\mathbf{X} \in \mathbf{\Phi}_{n \times p}(0, 1)$ in the form (3) for $\mathbf{x}'\mathbf{1}_p = \frac{v}{2}$ is the regular A–optimal spring balance weighing design.

Proof. This is proved by checking that the parameters given in (i) – (xxi) satisfy conditions (i) and (ii) of Theorem 2.

Theorem 4. Let $v = 6$. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible design with the same association with the parameters

(i) $b_1 = 12,\ r_1 = 6,\ k_1 = 3,\ \lambda_{11} = 3,\ \lambda_{21} = 2$ (R43) and
$b_2 = 18,\ r_2 = 9,\ k_2 = 3,\ \lambda_{12} = 3,\ \lambda_{22} = 4$ (R52),

(ii) $b_1 = 4,\ r_1 = 2,\ k_1 = 3,\ \lambda_{11} = 0,\ \lambda_{21} = 1$ (SR18) and
$b_2 = 6,\ r_2 = 3,\ k_2 = 3,\ \lambda_{12} = 2,\ b_2 = 6,\ r_2 = 3,\ k_2 = 3,\ \lambda_{12} = 2,\ \lambda_{22} = 1$ (R42),

(iii) $b_1 = 4,\ r_1 = 2,\ k_1 = 3,\ \lambda_{11} = 0,\ \lambda_{21} = 1$ (SR18) and
$b_2 = 16,\ r_2 = 8,\ k_2 = 3,\ \lambda_{12} = 4,\ \lambda_{22} = 3$ (R48),

(iv) $b_1 = 8,\ r_1 = 4,\ k_1 = 3,\ \lambda_{11} = 0,\ \lambda_{21} = 2$ (SR19) and
$b_2 = 12,\ r_2 = 6,\ k_2 = 3,\ \lambda_{12} = 4,\ b_2 = 12,\ r_2 = 6,\ k_2 = 3,\ \lambda_{12} = 4,\ \lambda_{22} = 2$ (R44),

(v) $b_1 = 12,\ r_1 = 6,\ k_1 = 3,\ \lambda_{11} = 0,\ \lambda_{21} = 3$ (SR20) and
$b_2 = 18,\ r_2 = 9,\ k_2 = 3,\ \lambda_{12} = 6,\ \lambda_{22} = 3$ (R50),

(vi) $b_1 = 6,\ r_1 = 3,\ k_1 = 3,\ \lambda_{11} = 2,\ \lambda_{21} = 1$ (R42) and
$b_2 = 14,\ r_2 = 7,\ k_2 = 3,\ \lambda_{12} = 2,\ \lambda_{22} = 3$ (R46),

(vii) $b_1 = 12,\ r_1 = 6,\ k_1 = 3,\ \lambda_{11} = 4,\ \lambda_{21} = 2$ (R44) and
$b_2 = 18,\ r_2 = 9,\ k_2 = 3,\ \lambda_{12} = 2,\ \lambda_{22} = 4$ (R51),

(viii) $b_1 = 14,\ r_1 = 7,\ k_1 = 3,\ \lambda_{11} = 2,\ \lambda_{21} = 3$ (R46) and
$b_2 = 16,\ r_2 = 8,\ k_2 = 3,\ \lambda_{12} = 4,\ b_2 = 16,\ r_2 = 8,\ k_2 = 3,\ \lambda_{12} = 4,\ \lambda_{22} = 3$ (R48),

and let $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$.

(a) If $n = b_1 + b_2$ then $\mathbf{X} \in \mathbf{\Phi}_{n \times v}(0, 1)$ in the form (2),

(b) if $n = b_1 + b_2 + 1$ then $\mathbf{X} \in \mathbf{\Phi}_{n \times v}(0, 1)$ in the form (3) for $\mathbf{x}'\mathbf{1}_v = \frac{v}{2}$ is the regular A–optimal spring balance weighing design.

Proof. It is easily seen that the parameters given in (i) – (viii) satisfy conditions (i) and (ii) of Theorem 2.

Theorem 5. Let $v = 8$. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible design with the same association with the parameters

(i) $b_1 = 12,\ r_1 = 6,\ k_1 = 4,\ \lambda_{11} = 2,\ \lambda_{21} = 3$ (SR38) and
$b_2 = 16,\ r_2 = 8,\ k_2 = 4,\ \lambda_{12} = 4$, $b_2 = 16,\ r_2 = 8,\ k_2 = 4,\ \lambda_{12} = 4,\ \lambda_{22} = 3$ (R98),

(ii) $b_1 = 6,\ r_1 = 3,\ k_1 = 4,\ \lambda_{11} = 3,\ \lambda_{21} = 1$ (S6) and
$b_2 = 8,\ r_2 = 4,\ k_2 = 4,\ \lambda_{12} = 0,\ \lambda_{22} = 2$ (SR36),

(iii) $b_1 = 12,\ r_1 = 6,\ k_1 = 4,\ \lambda_{11} = 6,\ \lambda_{21} = 2$ (S7) and
$b_2 = 16,\ r_2 = 8,\ k_2 = 4,\ \lambda_{12} = 0$, $\lambda_{22} = 4$ (SR39),

(iv) $b_1 = 8,\ r_1 = 4,\ k_1 = 4,\ \lambda_{11} = 0,\ \lambda_{21} = 2$ (SR36) and
$b_2 = 20,\ r_2 = 10,\ k_2 = 4,\ \lambda_{12} = 6$, $\lambda_{22} = 4$ (R103),

(v) $b_1 = 12,\ r_1 = 6,\ k_1 = 4,\ \lambda_{11} = 0,\ \lambda_{21} = 3$ (SR37) and
$b_2 = 16,\ r_2 = 8,\ k_2 = 4,\ \lambda_{12} = 6$, $\lambda_{22} = 3$ (R99),

(vi) $b_1 = 10,\ r_1 = 5,\ k_1 = 4,\ \lambda_{11} = 3,\ \lambda_{21} = 2$ (R97) and
$b_2 = 18,\ r_2 = 9,\ k_2 = 4,\ \lambda_{12} = 3$, $\lambda_{22} = 4$ (R101),

and let $\mathbf{X}_1 = [\mathbf{N}_1 \quad \mathbf{N}_2]'$.

(a) If $n = b_1 + b_2$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times v}(0,1)$ in the form (2),

(b)if $n = b_1 + b_2 + 1$ then $\mathbf{X} \in \mathbf{\Phi}_{n\times v}(0,1)$ in the form (3) for $\mathbf{x}'\mathbf{1}_v = \frac{v}{2}$ is the regular A–optimal spring balance weighing design.

Proof. An easy computation shows that the parameters given in (i) – (vi) satisfy conditions (i) and (ii) of Theorem 2.

Theorem 6. Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be the incidence matrices of the group divisible design with the same association with the parameters

(i) $v = 10,\ b_1 = 8,\ r_1 = 4,\ k_1 = 5,\ \lambda_{11} = 0,\ \lambda_{21} = 2$ (SR52) and
$v = 10,\ b_2 = 10,\ r_2 = 5$, $k_2 = 5$, $\lambda_{12} = 4,\ \lambda_{22} = 2$ (R139),

(ii) $v = 10,\ b_1 = 16,\ r_1 = 8,\ k_1 = 5,\ \lambda_{11} = 0,\ \lambda_{21} = 4$ (SR54) and
$v = 10,\ b_2 = 20,\ r_2 = 10$, $k_2 = 5\ \ \lambda_{12} = 8,\ \lambda_{22} = 4$ (R142),

(iii) $v = 12,\ b_1 = 10,\ r_1 = 5,\ k_1 = 6,\ \lambda_{11} = 5,\ \lambda_{21} = 2$ (S28) and
$v = 12,\ b_2 = 12,\ r_2 = 6$, $k_2 = 6$, $\lambda_{12} = 0,\ \lambda_{22} = 3$ (SR67),

(iv) $v = 14,\ b_1 = 12,\ r_1 = 6,\ k_1 = 7,\ \lambda_{11} = 0,\ \lambda_{21} = 3$ (SR81) and
$v = 14,\ b_2 = 14,\ r_2 = 7$, $k_2 = 7$, $\lambda_{12} = 6,\ \lambda_{22} = 3$ (R177),

(v) $v=16,\ b_1=14,\ r_1=7,\ k_1=8,\ \lambda_{11}=7,\ \lambda_{21}=3$ (S63) and
$v=16,\ b_2=16,\ r_2=8,\ k_2=8,\ \lambda_{12}=0,\ \lambda_{22}=4$ (SR92),
(vi) $v=18,\ b_1=16,\ r_1=8,\ k_1=9,\ \lambda_{11}=0,\ \lambda_{21}=4$ (SR100) and
$v=18,\ b_2=18,\ r_2=9,\ k_2=9,\ \lambda_{12}=8,\ \lambda_{22}=4$ (R197),
(vii) $v=20,\ b_1=18,\ r_1=9,\ k_1=10,\ \lambda_{11}=9,\ \lambda_{21}=4$ (S109) and
$v=20,\ b_2=20,\ r_2=10,\ k_2=10,\ \lambda_{12}=0,\ \lambda_{22}=5$ (SR108),
and let $\mathbf{X}_1=\left[\mathbf{N}_1\quad \mathbf{N}_2\right]'$.

(a) If $n=b_1+b_2$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times v}(0,1)$ in the form (2),

(b)if $n=b_1+b_2+1$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times v}(0,1)$ in the form (3) for $\mathbf{x}'\mathbf{1}_v=\frac{v}{2}$ is the regular A–optimal spring balance weighing design.

Proof. The main idea of proof is to show that the parameters given in (i) – (vii) satisfy conditions (i) and (ii) of Theorem 2.

IV. EXAMPLE

Let us consider the class $\mathbf{\Phi}_{11\times 6}(0,1)$. Let note that $n=11$ and $p=6$. For $h=11$, according to (1) $\frac{hp}{4(p-1)}$ is not integer. Hence $\mathbf{X}\in\mathbf{\Phi}_{11\times 6}(0,1)$ in the form (2) doesn't exist. On the other hand, for $h=10$, according to (1) $\frac{hp}{4(p-1)}$ is integer. Thus we consider $\mathbf{X}\in\mathbf{\Phi}_{11\times 6}(0,1)$ in the form (3) for $b_1+b_2=10=n-1$. Based on Theorem 4 (ii) there exist the group divisible block design with the association scheme with the parameters $v=6,\ b_1=4,\ r_1=2,\ k_1=3,\ \lambda_{11}=0,\ \lambda_{21}=1$ (SR18) and $v=6,\ b_2=6,\ r_2=3,\ k_2=3, \lambda_{12}=2$, $v=6,\ b_2=6,\ r_2=3,\ k_2=3,\ \lambda_{12}=2,\ \lambda_{22}=1$ (R42) given by the incidence matrices

$$\mathbf{N}_1=\begin{bmatrix}1&1&0&0\\1&0&1&0\\1&0&0&1\\0&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}\ \text{and}\ \mathbf{N}_2=\begin{bmatrix}1&0&0&1&0&1\\1&1&0&0&1&0\\0&1&1&0&0&1\\1&0&1&1&0&0\\0&1&0&1&1&0\\0&0&1&0&1&1\end{bmatrix}$$ with the same association

scheme $\begin{matrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{matrix}$. Hence $\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \in \mathbf{\Phi}_{11\times 6}(0,1)$ for $\mathbf{x}' = [1\ 1\ 0\ 0\ 1\ 0]$ is the regular A–optimal spring balance weighing design.

V. CONCLUSIONS

The problem of estimation of unknown measurements of objects in the model of spring balance weighing design is presented. Of particular interest is new construction method of design matrix $\mathbf{X}$ which allows to determine optimal design in the class of matrices $\mathbf{\Phi}_{n\times p}(0,1)$ in the cases not considered in literature.

REFERENCES

Banerjee K.S. (1950), How balanced incomplete block designs may be made to furnish orthogonal estimates in weighing designs, *Biometrica,* 37, 50 – 58.

Clatworthy W.H. (1973), *Tables of Two-Associate-Class Partially Balanced Designs,* NBS Applied Mathematics Series 63.

Graczyk M. (2010), Some notes about spring balance weighing design *(to appear*.)

Jacroux M., Notz W. (1983), On the optimality of spring balance weighing designs, *The Annals of Statistics*, 11, 970-978.

Raghavarao D. (1971), *Constructions and Combinatorial Problems in Designs of Experiments*, John Wiley Inc., New York.

Raghavarao D., Padgett L.V. (2005), *Block Designs, Analysis, Combinatorics and Applications,* Series of Applied Mathematics, 17, Word Scientific Publishing Co. Pte. Ltd.

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O PEWNEJ KONSTRUKCJI REGULARNEGO A–OPTYMALNEGO SPRĘŻYNOWEGO UKŁADU WAGOWEGO

W pracy przedstawiono zagadnienie konstrukcji sprężynowego układu wagowego spełniającego kryterium A–optymalności. Do konstrukcji macierzy układu wykorzystano macierze incydencji częściowo zrównoważonych układów bloków.