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Jacek Białek[*]

SPECIAL CASES OF SOME GENERAL FORMULA FOR PRICE INDICES

Abstract. In the paper we present a general formula for aggregative price indices that satisfies most postulates coming from the axiomatic price index theory. We show that a lot of known and useful price indices are particular cases of the discussed formula.

Key words: price indices, Laspeyres index, Paasche index, Fisher index, Marhall-Edgeworth index, Geary-Khamis index, Walsh index.

I. INTRODUCTION

The history of price indices is quite long – Dutot[1] presented his index in 1738, M. W. Drobisch published his formulas in 1871, Laspeyres and Paasche indices have been known since the 19-th century. From a theoretical point of view, a proper index should satisfy a group of postulates (tests) coming from the axiomatic index theory (see Balk (1995)). A system of minimum requirements of an index comes from Marco Martini (1992). According to the mentioned system a price index should satisfy at least three conditions: *identity*, *commensurability* and *linear homogeneity*. German index theoreticians – Eichhorn and Voeller (1976) – introduced a more generally acceptable system (EV) of five, and later also of four axioms: *strict monotonicity, price dimensionality*, *commensurability, identity* and (optionally) *linear homogeneity* (see also Białek (2005), von der Lippe (2007)). In the literature we can also meet other systems – for example Bernhard Olt (1996) examined several systems that provide less restrictive requirements than EV-systems.

Let us consider a group of N components observed at times s, t and let us denote[2]:

[*] Ph. D., Chair of Statistical Methods, University of Łódź .

[1] „Reflexions politiques sur les finances et le commerce", The Hague 1738.

[2] The time moment s we consider as the *basis*, i.e. the reference situation, for the comparison.

$P^s = [p_1^s, p_2^s, ..., p_N^s]'$ – a vector of components' prices at time s;

$P^t = [p_1^t, p_2^t, ..., p_N^t]'$ – a vector of components' prices at time *t*;

$Q^s = [q_1^s, q_2^s, ..., q_N^s]'$ – a vector of components' quantities at time s;

$Q^t = [q_1^t, q_2^t, ..., q_N^t]'$ – a vector of components' quantities at time t.

Using the above denotations the Paasche price index can be defined as follows:

$$I_{Pa}^P(Q^t, P^s, P^t) = \frac{Q^t \circ P^t}{Q^t \circ P^s}, \tag{1}$$

and the Laspeyres price index:

$$I_L^P(Q^s, P^s, P^t) = \frac{Q^s \circ P^t}{Q^s \circ P^s}, \tag{2}$$

where "$\circ$" denotes an outer product of two vectors.

As it is known, the indices (1) and (2) are very important from a practical point of view and they do not satisfy some of the axioms. For example, *time reversibility* for the price index I^P, described by the formula (3)

$$I^P(Q^s, Q^t, P^s, P^t) = \frac{1}{I^P(Q^t, Q^s, P^t, P^s)}, \tag{3}$$

is not satisfied. But none of the *reversal tests* (time and factor reversal test) or the *circular test* is mentioned in the EV-systems. Fisher proposed another definition based on Paasche and Laspeyres formulas (see Fisher (1922)). His formula I_F^p is a geometric mean of Paasche and Laspeyres indices:

$$I_F^P(Q^s, Q^t, P^s, P^t) = \sqrt{I_L^p(Q^s, P^s, P^t) I_{Pa}^p(Q^t, P^s, P^t)}. \tag{4}$$

The Fisher's definition is called an "ideal formula", because it satisfies the *factor reversal test*. In the literature we can also meet a general version of the Fisher index (see (5)):

$$\widetilde{I}^{P}_{F,\alpha}(Q^s,Q^t,P^s,P^t) \equiv \widetilde{I}^{P}_{F}(\alpha) = \\ = [I^P_L(Q^s,P^s,P^t)]^{\alpha} \cdot [I^P_{Pa}(Q^t,P^s,P^t)]^{1-\alpha}, \ \alpha \in [0,1] \tag{5}$$

where

$$\widetilde{I}^{P}_{F}(\alpha = 0) = I^P_{Pa}(Q^t,P^s,P^t), \tag{6}$$

$$\widetilde{I}^{P}_{F}(\alpha = 0{,}5) = I^P_{F}(Q^s,Q^t,P^s,P^t), \tag{7}$$

$$\widetilde{I}^{P}_{F}(\alpha = 1) = I^P_{L}(Q^s,P^s,P^t). \tag{8}$$

In this paper we present a more general class of price indices satisfying most of the mentioned tests and some special cases of indices belonging to this class.

II. GENERAL FORMULA FOR PRICE INDICES

Let $f_j : R^N_+ \times R^N_+ \to R^N_+$, for $j = 1,2,...,m$ with some fixed m, be such functions that, for any $X = [x_1, x_2,..., x_N]'$ and $Y = [y_1, y_2,..., y_N]'$, it holds:

$f_j(\lambda X, \lambda Y) = \lambda f_j(X,Y)$, (9)

where λ is an $N \times N$ diagonal matrix with elements $\lambda_1, \lambda_2,..., \lambda_N$.

Certainly the set of functions satisfying (9) is not empty – for example we could assume $f_j(X,Y) = c_j(X+Y)$ for $j = 1,2,...,m$ and some $c_j \in R_+$. In the paper of Białek (2010a) we present the following, general formula for price indices:

$$I^P(Q^s,Q^t,P^s,P^t) = [\prod_{j=1}^{m} \frac{f_j(Q^s,Q^t) \circ P^t}{f_j(Q^s,Q^t) \circ P^s}]^{\frac{1}{m}}. \tag{10}$$

Let us notice that if we assume $I^P_j(Q^s,Q^t,P^s,P^t) = \frac{f_j(Q^s,Q^t) \circ P^t}{f_j(Q^s,Q^t) \circ P^s}$ for $j = 1,2,...,m$ then the formula I^P can be written as a geometric mean[3] of indices I^P_j. It is easy to prove (see Białek (2010b)) the theorem 1:

[3] Thus the formula (6) is much more general than the generalized Fisher index: $I^P_{GF}(Q^s,Q^t,P^s,P^t) = (I^P_L(Q^s,P^s,P^t))^{\alpha}(I^P_{Pa}(Q^t,P^s,P^t))^{1-\alpha}$.

Theorem 1

Let I^P be the aggregative price index with the structure described by (10) with an additional condition (9). Then, the I^P index satisfies tests coming from the EV-system (*strict monotonicity, price dimensionality*, *commensurability, identity* and *linear homogeneity)*.

Moreover, if we additionally assume that for any $X,Y,Z \in R_+^N$ it holds

$$\sum_{j=1}^{m} \ln \frac{f_j(X,Y) \circ Z}{f_j(Y,Z) \circ Z} = 0, \tag{11}$$

or equivalently

$$\prod_{j=1}^{m} \frac{f_j(X,Y) \circ Z}{f_j(Y,X) \circ Z} = 1, \tag{12}$$

then we have *time reversibility* (3) satisfied:

In fact, if the condition (12) holds then we have:

$$I^P(Q^s,Q^t,P^s,P^t) \cdot I^P(Q^t,Q^s,P^t,P^s) =$$

$$= [\prod_{j=1}^{m} \frac{f_j(Q^s,Q^t) \circ P^t}{f_j(Q^s,Q^t) \circ P^s}]^{\frac{1}{m}} [\prod_{j=1}^{m} \frac{f_j(Q^t,Q^s) \circ P^s}{f_j(Q^t,Q^s) \circ P^t}]^{\frac{1}{m}} =$$

$$= \{\prod_{j=1}^{m} \frac{f_j(Q^s,Q^t) \circ P^t}{f_j(Q^t,Q^s) \circ P^t} \prod_{j=1}^{m} \frac{f_j(Q^t,Q^s) \circ P^s}{f_j(Q^s,Q^t) \circ P^s}\}^{\frac{1}{m}} = 1^{\frac{1}{m}} = 1.$$

However, *time reversibility* seems to be too restrictive and relatively unimportant. Firstly, it rules out many reasonable and useful index functions like Laspeyres or Paasche. Secondly, the history takes one direction only. And finally, from an economical point of view, there is no need for "symmetry" described by (12) and *time reversibility.* Thus, in the next part of the paper, we assume only the condition (9).

III. SPECIAL CASES OF THE GENERAL FORMULA

Firstly, let us notice that each of unweighted indices (like the index of Dutot (1738)) is out of the considered class because the assumption (9) is too restrictive for this kind of indices. Let us consider the case of $m = 2$ and define

$$f_1(X,Y) = X, \tag{13}$$

$$f_2(X,Y) = Y, \tag{14}$$

where $X = (x_1, x_2, .., x_N)'$, $Y = (y_1, y_2, .., y_N)'$. Certainly both functions f_1 and f_2 satisfy the assumption (9) and moreover for any $Z \in R_+^N$ we have:

$$\prod_{j=1}^{2} \frac{f_j(X,Y) \circ Z}{f_j(Y,X) \circ Z} = \frac{f_1(X,Y) \circ Z}{f_1(Y,X) \circ Z} \cdot \frac{f_2(X,Y) \circ Z}{f_2(Y,X) \circ Z} = \frac{X \circ Z}{Y \circ Z} \cdot \frac{Y \circ Z}{X \circ Z} = 1. \tag{15}$$

Using the formula (10) for $m = 2$, f_1 and f_2 (where $f_1(Q^s, Q^t) = Q^s$ and $f_2(Q^s, Q^t) = Q^t$) we get the following structure of a price index:

$$I^P(Q^s, Q^t, P^s, P^t) = [\prod_{j=1}^{2} \frac{f_j(Q^s, Q^t) \circ P^t}{f_j(Q^s, Q^t) \circ P^s}]^{\frac{1}{2}} = \sqrt{\frac{Q^s \circ P^t}{Q^s \circ P^s} \cdot \frac{Q^t \circ P^t}{Q^t \circ P^s}} =$$

$$= \sqrt{I_L^p(Q^s, P^s, P^t) I_{Pa}^p(Q^t, P^s, P^t)} = I_F^p(Q^s, Q^t, P^s, P^t) . \tag{16}$$

Thus, the Fisher index is a special case of the general class defined in (10) and it satisfies tests from the EV-system and even *time reversibility*. Taking $m = 1$ and functions from (13)–(14) we get Laspeyres nad Paasche formulas. Let us also notice that if we assumed $m = 1$ and $f_1(X,Y) = \frac{1}{2}(X + Y)$, where $X = (x_1, x_2, .., x_N)'$ and $Y = (y_1, y_2, .., y_N)'$, we would have the condition (9) satisfied and from (10) we would get

$$I^P(Q^s, Q^t, P^s, P^t) = \frac{f_1(Q^s, Q^t) \circ P^t}{f_1(Q^s, Q^t) \circ P^s} = \frac{\frac{1}{2}(Q^s + Q^t) \circ P^t}{\frac{1}{2}(Q^s + Q^t) \circ P^s}. \tag{17}$$

The formula (17) is a definition of the Marhall-Edgeworth index (see von der Lippe (2007)) which is presented in the literature of the subject as

$$I_{ME}^{P}(Q^{s},Q^{t},P^{s},P^{t})=\frac{\sum_{i=1}^{N}\frac{q_i^s+q_i^t}{2}p_i^t}{\sum_{i=1}^{N}\frac{q_i^s+q_i^t}{2}p_i^s}. \tag{18}$$

Thus, from (18) we have the additional conclusion: the Marhall-Edgeworth index is a special case of the general formula defined in (10). Morover, if we defined the following functions ($m=1$):

$$\tilde{f}_1(Q_s,Q_t)=\tilde{Q},\ \hat{f}_1(Q_s,Q_t)=\hat{Q}, \tag{19}$$

where

$$\tilde{Q}_i=\sqrt{q_i^s\cdot q_i^t},\ \ \hat{Q}_i=\frac{q_i^s\cdot q_i^t}{q_i^s+q_i^t},\text{, for } i=1,2,...,N, \tag{20}$$

we would get

$$I^{P}(Q^{s},Q^{t},P^{s},P^{t})=\frac{\sum_{i=1}^{N}p_i^t\sqrt{q_i^s\cdot q_i^t}}{\sum_{i=1}^{N}p_i^s\sqrt{q_i^s\cdot q_i^t}}\ \text{(in case of } \tilde{f}_1\text{)}, \tag{21}$$

and

$$I^{P}(Q^{s},Q^{t},P^{s},P^{t})=\frac{\sum_{i=1}^{N}p_i^t\frac{q_i^s\cdot q_i^t}{q_i^s+q_i^t}}{\sum_{i=1}^{N}p_i^s\frac{q_i^s\cdot q_i^t}{q_i^s+q_i^t}}\ \text{(in case of } \hat{f}_1\text{)}. \tag{22}$$

Hence, we get the Walsh index (formula (21)) and the Geary-Khamis index (formula (22)).

IV. REMARKS AND CONCLUSIONS

The presented, general formula for aggregative price indices satisfies all the postulates coming from the EV-system. Thus, we have a practical conclusion: it is easier to prove that a given index belongs to the considered class than verify that the axioms in question are satisfied. It is shown that the Laspeyres, Paasche, Fisher, Marhall-Edgeworth and some other indices are particular cases of the proposed formula. Moreover, using the general formula we can easily define new indices, which also satisfy the given postulates. For example taking $m = 4$,

$$f_1(X,Y) = \frac{1}{2}(X+Y), \quad f_2(X,Y) = f_1(X,Y), \quad f_3(X,Y) = X \quad \text{and} \quad f_4(X,Y) = Y$$

we get the following structure of a price index (the assumption (9) is certainly satisfied):

$$I_{New}^{P}(Q^s, Q^t, P^s, P^t) = [\prod_{j=1}^{4} \frac{f_j(Q^s, Q^t) \circ P^t}{f_j(Q^s, Q^t) \circ P^s}]^{\frac{1}{4}} =$$

$$= \sqrt[4]{\frac{0.5(Q^s + Q^t) \circ P^t}{0.5(Q^s + Q^t) \circ P^s} \cdot \frac{0.5(Q^s + Q^t) \circ P^t}{0.5(Q^s + Q^t) \circ P^s} \cdot \frac{Q^s \circ P^t}{Q^s \circ P^s} \cdot \frac{Q^t \circ P^t}{Q^t \circ P^s}} =$$

$$\sqrt[4]{(I_{ME}^{p})^2 (I_{F}^{p})^2} = \sqrt{I_{ME}^{p} I_{F}^{p}} \,. \quad (23)$$

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Jacek Białek

SZCZEGÓLNE PRZYPADKI PEWNEJ OGÓLNEJ FORMUŁY INDEKSÓW CEN

W pracy prezentujemy ogólną formułą dla agregatowych indeksów cen, która spełnia większość postulatów wywodzących się z aksjomatycznej teorii indeksów. Pokazano, że wiele powszechnie znanych i użytecznych indeksów statystycznych stanowi szczególny przypadek omawianej formuły.