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EFFICIENCY OF METHODS OF ESTIMATION OF TÖRNQVIST FUNCTION PARAMETERS

1. INTRODUCTION

Törnqvist functions belong to a class of simple non-linear models describing relationship between consumer's demand and income.

Analytic forms of these functions are as follows

$$y = \frac{\alpha_o x}{x + \alpha_1} \tag{1}$$

for necessaries,

$$y = \frac{\alpha_o (x - \alpha_2)}{x + \alpha_1} \tag{2}$$

for basic commodities,

$$y = \frac{\alpha_o x (x - \alpha_2)}{x + \alpha_1} \tag{3}$$

for luxury commodities, where y and x denote demand and income respectively.

Functions (1)-(3) can have an economic sense when values of

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the variables and parameters are positive i.e. $x > 0$, $y > 0$ (the last relationship implies $x > \alpha_2$) and $\alpha_o > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$.

In econometric investigations a random term is attached to Törnqvist functions what creates a stochastic structure and allows to use statistical method to estimate parameters α_o, α_1, α_2.

Methods of estimation of Törnqvist function parameters are the subject of many works published in econometric literature. The choice of one of the methods depends on the stochastic n structure of the model.

In this paper we present some results of an extended Monte-Carlo experiment aimed to compare the efficiency of basic estimation methods of Törnqvist function (1).

2. TÖRNQVIST FUNCTION PARAMETERS ESTIMATION METHODS

Similarly as in the case of many non-linear models, the demand functions can be estimated using iterative methods or a linearizing transformation.

Törnqvist functions can be transformed to forms which are linear in relation to the parameters and, in consequence, the classical estimatiom methods based on least squares can be applied. However, the transformations have an influence on the stochastic structure of the model. Thus, in transformed Törnqvist functions not all classical assumptions required for OLS method hold good, and this is why OLS Method does not lead to efficient estimators.

Let us consider some method of Törnqvist function estimation (1).

Assume that we have the demand model of the form

$$y_i = \frac{\alpha_o x_i}{x_i + \alpha_i} + \eta_i \tag{4}$$

where:

y_i - denotes the amount of consumers demand of a given good measured as expenditures spent for the good in a fixed time unit i;

x_i - consumer's income in the period i;

α_o, α_1 - positive structural parameters of the model ($\alpha_o > 0$, $\alpha_1 > 0$);

η_i - random term.

2.1. Iterative method of model (4) parameters estimation

Estimators a_o, a_1 of parameters α_o, α_1 of model (4), according to the idea of least squares, can be found as the values minimizing the quadratic form

$$Q(a_o, a_1) = \sum_{i=1}^{n} (y_i - \frac{a_o x_i}{x_i + a_1})^2 \tag{5}$$

Comparing with zero the partial derivations of (5) we obtain the two-equation system

$$a_o \sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^2} - \sum_{i=1}^{n} \frac{x_i y_i}{x_i + a_1} = 0 \tag{6}$$

$$a_o \sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^3} - \sum_{i=1}^{n} \frac{x_i y_i}{(x_i + a_1)^2} = 0 \tag{7}$$

Thus, to get estimates a_o, a_1 it is sufficient to solve that system and check the Hessian minimum condition:

$$\frac{\partial^2 Q(a_o, a_1)}{\partial a_o^2} > 0 \tag{8}$$

$$\det H(Q(a_o, a_1)) > 0.$$

Because

$$H(Q(a_o, a_1)) = \begin{bmatrix} \frac{\partial^2 Q(a_o, a_1)}{\partial a_o^2} & \frac{\partial^2 Q(a_o, a_1)}{\partial a_o \partial a_1} \\ \frac{\partial^2 Q(a_o, a_1)}{\partial a_1 \partial a_o} & \frac{\partial^2 Q(a_o, a_1)}{\partial^2 a_1^2} \end{bmatrix} =$$

$$= \begin{bmatrix} \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^2}; & 2\sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^2} - 4\sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3} \\ 2\sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^2} - 4a_o \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3}; & -4a_o \sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^2} + 6a_o \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3} \end{bmatrix}$$

the necessary and sufficient condition of (8) is holding the formula

$$a_o^2 \left[\sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^2} \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^4} - 4\left(\sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3} \right)^2 \right] +$$

$$+ a_o \left[4 \sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^2} \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3} - 2 \sum_{i=1}^{n} \frac{x_i^2}{(a_1 + x_i)^3} \sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^3} \right] +$$

$$- \left[\sum_{i=1}^{n} \frac{x_i y_i}{(a_1 + x_i)^2} \right]^2 > 0 \tag{9}$$

in a stationary point (a_o, a_1).

We have directly

$$a_o = \frac{\sum_{i=1}^{n} \frac{x_i y_i}{x_i + a_1}}{\sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^2}} \tag{10}$$

Taking it into account and multiplying (7) by $\sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^2}$ we obtain the equation for a_1:

$$\sum_{i=1}^{n} \frac{x_i y_i}{(x_i + a_1)^2} \sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^2} - \sum_{i=1}^{n} \frac{x_i y_i}{x_i + a_1} \cdot \sum_{i=1}^{n} \frac{x_i^2}{(x_i + a_1)^3} = 0 \tag{11}$$

To find the value of a_1 satisfying (11) a bisection procedure was applied.

2.2. Estimating linear transformation of Törnqvist function

Linear transformation of Törnqvist functions allows to use the Ordinary Least Square Method to estimate their parameters. However, as we have already mentioned, it may raise some problems connected with stochastic structure of the model. Thus, instead of transforming (4), model (1) is transformed and then a random term is added. There are three main possibilities

$$\frac{1}{y} = \beta_4 + \beta_3 \frac{1}{x} + \varepsilon^{(1)} \tag{12}$$

$$\frac{x}{y} = \beta_4 x + \beta_3 + \varepsilon^{(2)} \tag{13}$$

$$y = \beta_2 + \beta_1 xy + \varepsilon^{(3)} \tag{14}$$

where

$$\beta_4 = \frac{1}{\alpha_o}, \quad \beta_3 = \frac{\alpha_1}{\alpha_o}, \quad \beta_2 = \frac{\alpha_o}{\alpha_1}, \quad \beta_1 = -\frac{1}{\alpha_1}.$$

Parameters of models (12)-(14) can be estimated using be Ordinary Least Square Method. But the estimators of β_1 and β_2 do not satisfy consistency conditions because the exogenous variable xy is correlated with random term ε.

2.3. Instrumental variable methods

When an exogenous variable is strongly correlated with the random term the instrumental variable methods (IVM) are recommended. In case of model (14) instead of OLS-estimator

$$\hat{\beta} = (X^T X)^{-1} X^T y \tag{15}$$

the estimator

$$\hat{\beta}\ \text{IVM} = (Z^T X)^{-1} Z^T y \tag{16}$$

is used where X - the exogenous variable observation matrix, y - endogenous variable observation vector, and Z is obtained from the matrix X after inserting values of some variable z instead of xy variable observations.

In our research we have taken into account Wald's and Bartlett's instrumental variable methods. According to Wald $z_i = -1$, when the value $x_i y_i$ is below the median and $z_i = 1$ otherwise.

Similarly, in Bartlett method the ordered sequence of $x_i y_i$ is divided into three groups with equal (or almost equal) numbers of observations and $z_i = -1$, $z_i = 0$, $z_i = 1$ over the first, second and third group, respectively.

In our investigations we compare the efficiency of the following methods

- met n based on formulae (8)-(9),
- met 11 based on formula (12),
- met 12 based on formula (13),
- met 13 based on formula (14),
- met w - Wald IVM,
- met b - Bartlett IVM.

3. THE EXPERIMENTS

Model (4) has been assumed as the "true" model. For the given

- sample size n;
- values of income x_i, $i = 1, \ldots, n$;
- parameters α_o, α_1;
- random term variance σ_η^2

a set of p samples was generated. It consisted of n pairs

(y_i^s, x_i) $i = 1, 2, \ldots, n$; $s = 1, 2, \ldots, S$; where y_i^s was computed using the formula (4) with independently generated η_i.

For each sample the six listed above methods were applied to estimate the parameters α_o and α_1 and random term variance σ_η^2. The sets of estimates was a base of estimation methods efficiency analysis.

Our research covers 90 experiments defined by

- three sample sizes $n = 10, 30, 50$;
- two sequences $\{x_i\}$: uniform and based on lognormal quantiles;
- five values $\alpha_1 = 0.1, 0.25, 0.5, 0.75, 1$;
- three values of determination coefficient $R^2 = 0.90$, 0.95, 0.99 (basing on them the variances σ_η^2 were determined);
- the value $\alpha_o = 1$ was accepted.

In each experiment 100 samples were generated. In computation our own software in Turbo-Pascal 3.0 was applied.

4. THE RESULTS

As it was said above in each experiment 100 samples were drawn and estimates of parameters α_o, α_1 and σ^2_η, using six methods, were computed.

In some samples we did not obtain reasonable results applying all the methods. These samples were eliminated from considerations concerning the unsuccessful method. For the methods n, 13, b, w there were only a few such cases in all the experiment. For methods 11 and 12 the number of the rejected samples in some experiments was quite large. Extreme cases are shown in Table 1.

Table 1

The number of samples when the estimation failed (extremal cases), $\alpha_1 = 1.0$ $R^2 = 0.90$

n	met 11		met 12	
	x - uniform	x - lognormal	x - uniform	x - lognormal
10	50	31	8	9
50	30	44	5	3

S o u r c e: The author's calculations.

Table 2

Efficiency (MSE) of estimator a_1

	$\alpha_1 = 0.5$				$R^2 = 0.95$		
	n	met n	met 11	met 12	met 13	met b	met w
x-uniform	10	0.05	4.64	0.53	0.31	4.01	4.65
	30	0.09	0.22	0.10	0.25	3.78	4.88
	50	0.04	1.11	0.28	0.12	0.22	0.24
x-lognormal	10	0.09	2.28	0.10	0.15	0.65	0.72
	30	0.05	1.02	0.11	0.27	3.95	4.72
	50	0.04	3.40	0.06	0.13	0.17	0.20

S o u r c e: The author's calculations.

The basic measure of goodness of an estimator is the mean square error (MSE). The estimated values of MSE are included into the Tables 2-10.

Table 3

Efficiency (MSE) of estimator a_1

$\alpha_1 = 0.5$				$n = 30$			
	R^2	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.90	0.13	0.45	0.15	1.46	7.44	10.40
	0.95	0.09	0.22	0.10	0.25	3.78	4.88
	0.99	0.04	0.06	0.04	0.06	3.40	4.26
x-lognormal	0.90	0.08	7.28	1.24	0.74	4.94	5.71
	0.95	0.05	1.02	0.11	0.28	3.95	4.72
	0.99	0.02	0.20	0.03	0.06	3.49	4.14

Source: The author's calculations.

Table 4

Efficiency (MSE) of estimator α_1

$n = 30$				$R^2 = 095$			
	α_1	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.10	0.01	0.01	0.01	0.05	11.91	2.27
	0.25	0.03	0.05	0.03	0.10	2.51	3.26
	0.50	0.09	0.22	0.10	0.25	3.78	4.88
	0.75	0.14	0.77	0.23	0.29	5.05	6.54
	1.00	0.22	0.92	0.36	0.51	6.88	8.58
x-lognormal	0.10	0.01	0.01	0.01	0.12	2.03	2.33
	0.25	0.02	0.08	0.03	0.17	2.72	3.15
	0.50	0.05	1.02	0.11	0.28	3.95	4.72
	0.75	0.08	2.74	6.37	0.38	5.36	6.41
	1.00	0.15	1.44	1.21	0.55	7.02	8.37

Source: The author's calculations.

In most cases the larger is the sample size the smaller is MSE (Table 2). Besides, larger values of MSE can be observed for log-normal distribution of x_i than in the case of the uniform one.

MSE (a_1) decreases when R^2 increases (Table 3). The rate of decrease is different and relatively the biggest using the method n. And vice versa, MSE increases when the true value of α_1 increases (Table 4).

Similar tendencies can be observed studying results obtained for MSE (a_o). Namely

- decreasing of MSE when the sample size increases (Table 5);
- decreasing of MSE when the determination coefficient R^2 increases (Table 6);

Table 5

Efficiency (MSE) of estimator a_o

$\alpha_1 = 0.5$					$R^2 = 0.95$		
	n	met n	met 11	met 12	met 13	met b	met w
x-uniform	10	0.03	4.06	0.32	0.16	1.72	2.10
	30	0.06	0.16	0.07	0.14	1.63	2.29
	50	0.03	1.04	0.10	0.06	0.53	0.051
x-lognormal	10	0.06	1.91	0.07	0.09	0.31	0.27
	30	0.03	0.86	0.15	0.15	1.69	2.14
	50	0.03	3.40	0.03	0.05	0.58	0.58

Source: The author's calculations.

Table 6

Efficiency (MSE) of estimator a_o

$\alpha_1 = 0.5$					n = 30		
	R^2	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.90	0.09	0.32	0.10	0.78	3.74	5.38
	0.95	0.06	0.16	0.07	0.15	1.63	2.29
	0.99	0.03	0.04	0.03	0.04	1.42	1.95
x-lognormal	0.90	0.05	5.88	0.67	0.39	2.22	2.67
	0.95	0.03	0.85	0.15	0.14	1.69	2.14
	0.999	0.02	0.17	0.02	0.03	1.45	1.84

Source: The author's calculations.

Table 7

Efficiency (MSE) of estimator a_o

	α_1	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.10	0.01	0.02	0.02	0.04	0.74	1.06
	0.25	0.03	0.04	0.03	0.06	1.07	1.61
	0.50	0.06	0.16	0.07	0.15	1.63	2.29
	0.75	0.08	0.48	0.12	0.14	2.06	2.82
	1.00	0.11	0.54	0.17	0.23	2.66	3.43
x-lognormal	0.10	0.01	0.02	0.02	0.08	0.81	1.06
	0.25	0.02	0.09	0.02	0.11	1.18	1.49
	0.50	0.03	0.86	0.15	0.15	1.69	2.14
	0.75	0.04	1.77	2.87	0.17	2.17	2.69
	1.00	0.07	0.92	0.57	0.23	2.66	3.26

Source: The author's calculations.

Table 8

Efficiency (MSE) of estimator S^2_e

	$\alpha_1 = 0.5$				$R^2 = 0.95$		
	n	met n	met 11	met 12	met 13	met b	met w
x-uniform	10	0.00	0.09	0.02	0.01	0.12	0.11
	30	0.01	0.02	0.01	0.02	0.14	0.13
	50	0.01	0.12	0.02	0.01	0.28	0.27
x-lognormal	10	0.01	0.12	0.01	0.02	0.27	0.25
	30	0.01	0.06	0.01	0.01	0.12	0.12
	50	0.00	0.16	0.01	0.01	0.29	0.28

Source: The author's calculations.

- increasing of MSE when the value of α_1 increases (Table 7).

Analogous conclusions could be formed observing MSE when the sample size (Table 8) and the determination coefficient (Table 9) increase. However, when the value of α_1 increases, it is rather difficult to notice any clear tendency of changes of MSE of σ^2_η estimator.

Table 9

Efficiency (MSE) of estimator S_e^2

	$\alpha_1 = 0.5$				n = 30		
	R^2	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.90	0.01	0.02	0.01	0.03	0.13	0.12
	0.95	0.01	0.01	0.01	0.02	0.14	0.13
	0.99	0.00	0.01	0.00	0.00	0.16	0.13
x-lognormal	0.90	0.01	0.08	0.03	0.03	0.11	0.11
	0.95	0.01	0.06	0.01	0.01	0.12	0.12
	0.99	0.02	0.02	0.00	0.00	0.14	0.14

Source: The author's calculations.

Table 10

Efficiency (MSE) of estimator S_e^2

	n = 30				$R^2 = 0.95$		
	l	met n	met 11	met 12	met 13	met b	met w
x-uniform	0.10	0.01	0.01	0.01	0.02	0.38	0.35
	0.25	0.01	0.01	0.01	0.02	0.25	0.28
	0.50	0.01	0.02	0.01	0.02	0.14	0.13
	0.75	0.01	0.02	0.01	0.01	0.09	0.09
	1.00	0.01	0.02	0.01	0.01	0.06	0.06
x-lognormal	0.10	0.00	0.00	0.00	0.05	0.34	0.32
	0.25	0.01	0.01	0.01	0.02	0.22	0.21
	0.50	0.01	0.06	0.01	0.01	0.12	0.12
	0.75	0.01	0.06	0.02	0.01	0.08	0.08
	1.00	0.00	0.04	0.02	0.01	0.05	0.05

Source: The author's calculations.

Taking the MSE as the measure of efficiency of estimation method, the six methods could be ordered as follows: method n, 12, 13, 11, w, b.

The next Tables 11-13 present empirical mean values of bias for estimators of α_o, α_1 and σ_η^2, respectively. We limit our

Table 11

Average biases in non-linear method for x uniform and $R^2 = 0.95$ for estimator a_1

n \ α_1	0.10	0.25	0.50	0.75	1.00
10	0.001	0.003	0.004	0.005	0.008
30	0.000	-0.001	0.020	-0.003	0.038
50	0.000	0.000	0.005	-0.005	0.018

Source: The author's calculations.

Table 12

Average biases in non-linear method x uniform for estimator a_o

n \ α_1	0.10	0.25	0.50	0.75	1.00
10	0.001	0.002	0.000	0.004	0.004
30	0.000	0.000	0.012	-0.003	0.021
50	0.000	0.001	0.001	-0.007	0.006

Source: The author's calculations.

Table 13

Average biases in non-linear method x uniform for estimator S_e^2

n \ α_1	0.10	0.25	0.50	0.75	1.00
10	0.000	-0.000	0.000	0.000	-0.001
30	-0.001	-0.002	0.000	-0.001	-0.003
50	-0.000	-0.001	-0.001	-0.001	-0.001

Source: The author's calculations.

considerations only to the method n which turned out to be the most efficient. It can be seen, that the observed values of the bias are quite small and statistically insignificant.

5. FINAL REMARKS

Basing on our research concerning properties of the estimators of the first type of Törnqvist function it became possible to choose, among the recommended in the literature estimation methods, the most efficient one. The nonlinear iterative method seems to be the best one. Although it was not preferred in practical research because of some computational difficulties, the advantage of using it is rather clear. The further analysis of this method properties should be based on

- larger number of samples S,
- extended range of experiments.

Besides it seems interesting to take into account different kinds of stochastic structure of the model.

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EFEKTYWNOŚĆ ESTYMACJI PARAMETRÓW FUNKCJI TÖRNQVISTA

Funkcje Törnqvista są często stosowane do modelowania relacji popyt - dochód. W literaturze ekonometrycznej można znaleźć szereg propozycji metod estymacji tych funkcji, brak jest jednakże analizy własności otrzymanych estymatorów.

Autorzy podjęli próbę wypełnienia tej luki przy użyciu metody Monte-Carlo. Otrzymane wyniki wykazują, że efektywność niektórych zalecanych w literaturze metod jest dość mała.