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THE PRODUCTION ANALYSIS BY DUAL DYNAMIC PROGRAMMING

Abstract. The aim of this paper is to present a concept of a duality theory for dynamic production processes with constraints i.e. production processes described by nonconvex dynamical mathematical models (models depending on time).

1. INTRODUCTION

The production function plays the crucial role in economics. Enterprises aim to minimize their costs (for a given level of production), but the main motivation of their action is to maximize the profits. An analysis of the cost production is very often led by the production function. In static economy, if we are going to analyse costs through a production function and prices then a duality theorem is especially useful (see e.g. Leidler, Estrin, 1989). The essence of duality theory is that all elements of the production technology (available for enterprises) which are important for economists can be simply placed in the cost function. This statement has important consequences for the choice of the quantity of expenditure and the level of production. For example, optimal quantity of production factors we can gain directly from the cost function. Empirical investigations also acknowledge a great interest in the study of duality theories, for instance: very often it is difficult to collect credible information concerning the production factors (capital, labour). All available data concerning the labour expenditure take into account the number of the workers, but they do not take into account their qualifications and their intensity of work. Such problems are much more serious in the case of capital. A capital value is difficult to be measured and information concerning usefulness of

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the capital is almost unavailable. The enterprises have much more information about their costs. The duality theory allows us to infer properties of the production function from the cost function, where the information is available, more readable and reliable. About the relation of the costs with production we can think in two ways: 1) we find maximized level of production, when the cost production is constant, 2) we find minimized level of the cost, when the production is constant. In each case we have the same result. The analysed cost function has a dual form with relation to the production function. Of course, such a duality theorem exists, up to now, but only for production processes described by static (linear or convex) mathematical models i.e. models which do not depend on time.

The aim of this note is to present a concept of a duality theory for dynamic production processes i.e. production processes described by non-convex dynamical mathematical models (models depending on time). In that dual model the constraints appear, which cannot be taken into account in primal model. Such constraints appear every time in each production process. The companies which are interested in their own profits know, that they do not possess inexhaustible resources and that they do not have endless and unlimited funds. Their scale of production is constrained, like all their possibilities.

2. DUAL APPROACH

Let us consider the cost functional:

$$J(x, u) = \int_0^T L(t, x)(t), u(t))dt + l(x(T)) \tag{1}$$

depending on the state $x(t)$ and control $u(t)$ which measure the level of a cost of a production or a quantity of the production (depending on the way we think about the relations between the cost and the production). The state $x(t)$ denotes the expenditure of an enterprise depending on time i.e. they may change with time. We admit a possibility to control by $u(t)$ the types of transitions of $x(t)$ in time. It is natural to require, that state $x(t)$ varies dynamically, i.e. that $x(t)$ and $u(t)$ are subject to some differential equation:

$$x(t) = f(t, x(t), u(t)) \quad \text{a.e. in} \quad [0, T] \tag{2}$$

where $f:[0, T] \times R^n \times R^m \to R^n$ is to measure the speeds of changes of the expenditure $x(t)$ in time. We assume that f, $L:[0, T] \times R^n \times R^m \to R$ and $l:R^n \to R$ are continuous function and the controls $u:[0, T] \to U \subset R^m$ are measurable functions, $t \in [0, T]$. We shall also assume that the expenditure at time $t = 0$ has a given value c i.e.

$$x(0) = c, \quad c \in R^n \tag{3}$$

Moreover, we shall admit that the expenditure $x(t)$ is also subject to some constraints:

$$g(x(\cdot)) = 0 \tag{4}$$

where $g: R^n \to R^k$. A pair $x(t)$, $u(t)$ satisfying the constraints (2), (4) will be called *admissible* and corresponding $x(t)$ *an admissible state* or *an admissible trajectory*, see (Fleming, Rishel, 1975).

Our goal is to minimize the functional (1) in the space of absolutely continuous states $x(t)$ and measurable controls $u(t)$ subject to the constraints (2), (3), (4).

The classical method to study such problems is to define in some open set $G \subset R^{n+1}$ of the variables (t, x), the value function of our problem. The value function $S(t, x)$ in the classical approach is defined as follows:

$$S(t, x) = \inf\left\{\int_t^T L(\tau, x(\tau), u(\tau))d\tau + l(x(T))\right\},$$

where the inferior is taken over all pairs $x(\tau), u(\tau), \tau \in [t, T]$, whose states start at the point $(t, x) \in G$ and their graphs are contained in G. The next step is the following: if $S(t, x)$ is continuously differentiable then it must satisfy *the partial differential equation of Hamilton-Jacoby-Bellman type*:

$$S_t(t, x) + H(t, x, S_x(t, x)) = 0 \quad (t, x) \in G,$$

where $H(t, x, y) = yf(t, x, u(t, x)) - y^0 L(t, x, u(t, x))$, y, y^0 are multipliers and $u(t, x)$ is an optimal control. The value function satisfies also *the partial differential equation of dynamic programming*:

$$\inf\{S_t(t, x) + S_x(t, x)f(t, x, u) - y^0 L(t, x, u) : u \in U\} = 0.$$

This approach has many disadvantages. First of all, it is a very rare case that the value function is continuously differentiable in some open set G when the constraints (especially state constraints) are included in optimization problems. The second is that there is no suitable duality theory for production analysis with the above approach. In fact that approach cannot be in general applied to the problem (1), (2), (4) just because of (4).

A non-classical method to study the problem (1), (2), (4) by dynamical approach is to carry out all explorations concerning dynamic programming from the (t, x) – space to the space of multipliers $((t, y^0, y)$ – space). Let us explain it briefly. Let be given an open set $P \subset R^{n+2}$ of the dual space of the variables $(t, y^0, y) = (t, p)$, $y^0 \leqslant 0$ and a function $x(t, p)$, defined in P, $x(t, p)$: $P \subset R^{n+2} \rightarrow R^n$, such that $x(\cdot, p)$ satisfies (4) for each p, such that $(t, p) \in P$. Then in the set P we define *a dual value function*:

$$S_D(t, p) = \inf \left\{ - y^0 \int_t^T L(\tau, x(\tau), u(\tau)) d\tau - y^0 l(x(T)) \right\} \tag{5}$$

where the inferior is taken over all pairs $x(\tau), u(\tau), \tau \in [t, T]$, whose states start at $(t, x(t, p))$ and their graphs are contained in the set of values of the mapping $(t, p) \rightarrow (t, x(t, p))$, $(t, p) \in P$. Next we define a new function:

$$V(t, p) = - S_D(t, p) - x(t, p) y,$$

about which we assume that it is subject to satisfy the condition:

$$V(t, p) = V_{y^0}(t, p) y^0 + V_y(t, p) y = V_p(t, p) p \tag{6}$$

where: $- S_D(t, p) = V_{y^0}(t, p) y^0$, $- x(t, p) = V_y(t, p)$, $(t, p) \in P$.
We shall require that $V(t, p)$ satisfies *the dual partial differential equation of Hamilton-Jacoby-Bellman type*:

$$V_t(t, p) + H(t, - V_y(t, p), p) = 0, \quad (t, p) \in P \tag{7}$$

and the state constraint:

$$g(- V_y(\cdot, p)) = 0,$$

where $H(t, x, p) = y f(t, x, u(t, p)) + y^0 L(t, x, u(t, p))$, y, y^0 are multipliers and $u(t, p)$ is a dual optimal control. The function $V(t, p)$ must satisfy also *the dual partial differential equation of dynamic programming*:

$$\sup \{ V_t(t, p) + y f(t, - V_y(t, p), u) + y^0 L(t, - V_y(t, p), u) : u \in U \} = 0 \tag{8}$$

and the state constraint:

$$g(- V_y(\cdot, p)) = 0 \tag{9}$$

The non-classical approach has several advantages. Now we need not require that the set G has a nonempty interior. We do not require the value function $S(t, x)$ to be differentiable in G. The state constraints are in a natural way included in the dynamic programming equation. The most important advantage is that we have a duality theory which associates the value functions: primal and dual.

3. A VERIFICATION THEOREM

In this section we will give the main theorem about the dual sufficient conditions of optimality.

Let $G \subset R^{n+1}$ denote a set covered by the graphs of all admissible trajectories.

Let $P \subset R^{n+2}$ be a set of variables (t, p), $t \in [0, T]$, with $y^0 \leqslant 0$ and have a nonempty interior. Take a function $x(t, p)$ defined in P such that $(t, x(t, p)) \in G$, $(t, p) \in P$ and $g(x(\cdot, p)) = 0$.

Let the function $x(t, p)$ satisfy the following assumptions:

1) for each admissible trajectory $x(t)$ lying in G there exists an absolutely continuous function $p(t) = (y^0, y(t))$, lying in P such that: $x(t) = x(t, p(t))$,

2) if all trajectories $x(t)$ start at the same (t_0, x_0), then all the corresponding them trajectories $p(t)$ have the same first coordinate y^0.

Let $S_D(t, p)$ be as in (5). We see that:

$$S_D(t, p) = -y^0 S(t, x(t, p)), \quad (t, p) \in P.$$

Now we will give the proposition, which will be used in the proof of the main theorem of this section.

Theorem 1. Let $W(t, p) = -y^0 Z(t, x(t, p))$ be a real-valued function in P such that $W(T, p) = -y^0 l(x(T, p))$. Let $(t_0, x_0) \in G$ be given initial condition. Suppose that for each absolutely continuous function $p(t) = (y^0, y(t))$, $t \in [t_0, T]$, with graph lying in P, the function $x(t) = x(t, p(t))$, $t \in [t_0, T]$, $x(t_0) = x_0$, is an admissible trajectory lying in G and that:

$$W(t, p(t)) + y^0 \int_t^T L(\tau, x(\tau), u(\tau)) d\tau$$

is non-decreasing on $[t_0, T]$. If $\overline{p}(t) = (\overline{y}^0, \overline{y}(t)$, $t \in [t_0, T]$ is absolutely continuous function and if $\overline{x}(t) = x(t, \overline{p}(t))$, $t \in [t_0, T]$, $\overline{x}(t_0) = x_0$ is an admissible trajectory in G and is such that:

$$W(t, \overline{p}(t)) + y^0 \int_t^T L(\tau, \overline{x}(\tau), \overline{u}(\tau))d\tau$$

is constant in $[t_0, T]$, then $\overline{x}(t)$ is an optimal trajectory and $W(t_0, \overline{p}(t_0)) = S_D(t_0, \overline{p}(t_0))$, where $\overline{u}(t)$ is an optimal control corresponding to $\overline{x}(t)$.

Proof. For any function $p(t)$, $t \in [t_0, T]$ described above: $-y^0 Z(t_0, x_0) \leqslant -y^0 \int_{t_0}^T L(\tau, x(\tau), u(\tau))d\tau - y^0 l(x(T))$, where $u(t)$ is a control feasible for $x(t)$. For the function $\overline{p}(t)$:

$$-y^0 Z(t_0, x_0) = -\overline{y}^{\,0} \int_{t_0}^T L(\tau, \overline{x}(\tau), \overline{u}(\tau))d\tau - \overline{y}^{\,0} l(\overline{x}(T))$$

so $W(t_0, \overline{p}(t_0)) = S_D(t_0, \overline{p}(t_0))$ and $\overline{x}(t)$, $\overline{u}(t)$ is an optimal pair for the problem

$$\inf\{ -\overline{y}^{\,0} \int_t^T L(\tau, x(\tau) u(\tau))d\tau - \overline{y}^{\,0} l(x(T)) \colon x(t),\ u(t),\ t \in [t_0, T], \text{ are admissible pairs with } x(t_0) = x_0 \text{ and } x(t) \text{ lying in } G\}.$$

Now we will formulate the main theorem (sufficient optimality conditions) which is a counterpart for the dual version of the verification theorem from (Fleming, Rishel, 1975, Theorem 4.4, p. 87).

Theorem 2. Let $V(t, p)$, $(t, p) \in P$, $t \in [0, T]$, be a continuously differentiable solution of (8), (9) with the boundary condition: $y^0 V_{y^0}(T, p) = y^0 l(-V_y(T, p))$, $(T, p)P$, and satisfying the relation:

$$V(t, p) = V_p(t, p)p, \ (t, p) \in P \tag{10}$$

Let $x(t)$, $u(t)$ be an admissible pair whose graph of the trajectory $x(t)$ is contained in $\overline{G} = \{(t, x) : x = -V_y(t, p),\ (t, p) \in P\}$ and such that there exists an absolutely continuous function $p(t)$ lying in P and satisfying:

$$x(t) = -V_y(t, p(t)) \tag{11}$$

Then:

$$\overline{W}(t, p(t)) = -y^0 V_{y^0}(t, p(t)) + y^0 \int_t^T L(\tau, x(\tau), u(\tau))d\tau \tag{12}$$

is a non-decreasing function of t.

Let now $\overline{x}(t)$, $\overline{u}(t)$, $t\in[0,T]$, $\overline{x}(0)=c$ be an admissible pair with $\overline{x}(t)$ lying in $\overline{G}$ and let $\overline{p}(t)$, $t\in[0,T]$, be a nonzero absolutely continuous function lying in P such that: $\overline{x}(t)=-V_y(t,\overline{p}(t))$, $t\in[0,T]$. Let for all $t\in[0,T]$:

$$V_t(t,\overline{p}(t))+\overline{y}(t)f(t,-V_y(t,\overline{p}(t)),\ \overline{u}(t))+\overline{y}^{\,0}L(t,-V_y(t,\overline{p}(t)),\ \overline{u}(t))=0 \quad (13)$$

Then $\overline{x}(t)$, $\overline{u}(t)$, $t\in[0,T]$ is an optimal pair for the problem (1), (2), (4) relative to all admissible pairs $x(t), u(t), t\in[0,T], x(0)=c$ whose graphs of trajectories $x(t)$ are contained in $\overline{G}$.

Moreover: $S_D(t,\overline{p}(t))=-\overline{y}^{\,0}S(t,x(t,\overline{p}(t)))=-\overline{y}^{\,0}V_{y^0}(t,\overline{p}(t))$ with $x(t,p)=-V_y(t,p)$ is the dual value function along $\overline{p}(t)$.

Proof. Let us differentiate both sides of (10) with respect to t along $p(t)$:

$$V_t(t,p(t))=y^0(d/dt)V_{y^0}(t,p(t))+y(t)(d/dt)V_y(t,p(t)).$$

From (2) and (11) we receive:

$$(d/dt)V_y(t,p(t))=-f(t,-V_y(t,p(t)),\ u(t)),$$

and from (12) we have:

$$(d/dt)y^0V_{y^0}(t,p(t))=-(d/dt)\overline{W}(t,p(t))-y^0L(t,-V_y(t,p(t)),\ u(t)).$$

Hence and from (8) we obtain that $(d/dt)\overline{W}(t,p(t))\geqslant 0$ for almost all $t\in[0,T]$. The above relations written for $\overline{p}(t)$, together with equation (13), imply that for all $t\in[0,T]$:

$$-\overline{y}^{\,0}V_{y^0}(t,\overline{p}(t))=-\overline{y}^{\,0}\int_t^T L(\tau,\overline{x}(\tau),\overline{u}(\tau))d\tau-\overline{y}^{\,0}l(-V_y(T,\overline{p}(T))).$$

Hence we get that $\overline{W}(t,\overline{p}(t))=-\overline{y}^{\,0}l(\overline{x}(T))$ for all $t\in[0,T]$, i.e. $\overline{W}(t,\overline{p}(t))$ is a constant function. This together with Theorem 1 implies the assertions of the theorem.

Remark 1. Solving (8), (9) we obtain much more information about our problem than in the classical dynamic programming. The function: $-V_y(t,p)$ defines the whole space of admissible states where our problem mathematically makes sense. The condition (10) extremely important in physics and mathematics, in economy was not included into consideration up to now. It shows the real production costs, dynamically changing in time, not only those which are placed into the cost functional. This condition tells us that the multipliers $(y^0,-y)$ are orthogonal to the epigraph of the minimized cost functional $S(t,x)$ at the point $(x(t,p),S(t,x(,p)))$. It may be interpreted economically as follows: multiplier y, which is equal to: $-S_x(t,x(t,p))$

(when $S(t, x)$ is differentiable with respect to x) equals the marginal cost in time t (or marginal product) (compare (Leidler, Estrin, 1989) in the static case).

3.1. Conclusions

In order to understand what the new function $V(t, p)$ means let us come back to the static problem of production analysis. Then the cost functional (1) reduces to the function $l(x)$, we have not dynamical equations (2) but we have constraint (4). Usually to make an analysis of production through the costs and the level of production the Lagrange function is formed:

$$L(x, y) = l(x) + yg(x) \tag{14}$$

and then suitable calculations on this function are made. Our new function for this simple case has the form:

$$V(p) = y^0 l(x(p)) - yx(p) \tag{15}$$

where $p = (y^0, y)$ and $x(p)$ is a parametric description of the constraint (4), but the parameter is just the multiplier p. In fact (15) is a dual functional exactly in the same sense as it is in linear programming problems (see Aubin, (1979, 1997), Schiller (1989)).

Usually in duality theories multiplier y means the prices of some quantity x. Because $y = (y^1, ..., y^n)$, so the dual variable $y^i (i = 1, ..., n)$ denotes (according to neoclassical theory of economy) *the marginal productivity* of the i-th resource of production. In (15) y can be interpreted as a price of the quantity $x(p)$, like for example in Leidler, Estrin (1989). That is why: $-V(p)$ is just a full cost of the whole production process. We observe that studying (14) we cannot derive this type of duality results (see Leidler, Estrin (1989)).

4. EXAMPLE

Let us consider the problem of minimizing the cost functional:

$$J(x, u) = (1/2)\int_0^{\pi}(-x^2(t) + u^2(t))dt + l(x(\pi)) \tag{16}$$

where:

$$l(x)\pi)) = \begin{cases} 0, & \text{if} \quad x(\pi) = 0, \\ +\infty & \text{on the contrary,} \end{cases} \tag{17}$$

but we assume, that expenditure of an enterprise changes in time and we admit a control of them. Expenditure is described by the following dynamic:

$$x(t) = B(t)u(t) \quad \text{a.e. in} \quad [0, \pi] \tag{18}$$

where:

$$B(t) = \begin{cases} 1, & \text{for} \quad t \in [0, \pi], \\ 0, & \text{for} \quad t = 0, \end{cases}$$

$$u(t) \in [0, 1], \quad t \in [0, \pi] \tag{19}$$

$$x(0) = c \tag{20}$$

$$g(x(\cdot)) = 0 \tag{21}$$

The constraints (21) are defined as follows:

Let g be an indicator function of the set D, i.e. it equals zero on the set D and equals one out of D (on the plane R^2), where the set

$$D = \{(t, x) : t \in [0, \pi], \ -t^2 \leqslant x \leqslant t^2\}.$$

It means that if the graph of $x(t)$, $t \in [0, \pi]$ lies in D, then $g(x(\cdot)) = 0$.

The condition (17) means that all admissible trajectories (for our problem) must be in the point π equal zero.

To find an optimal control we can use Pontryagin's Maximum Principle (necessary optimality conditions) for a problem (16)–(20) – compare (Fleming, Rishel, 1975) – we can also simply guess a certain family of the trajectories, which we "suspect" of the extreme, which is dependent on changing initial conditions.

So, we receive the following functions: $x(t)$, $u(t)$, $p(t) = (y^0, y(t))$

1) $x(t, c_1) = c_1 \sin t$, $u(t, c_1) = c_1 \cos t$, $y^0 = -e$, $y(t, ec_1) = ec_1 \cos t$, where $t \in [0, \pi]$, $c_1 \in (-1, 1)$, $e \in \left(\frac{1}{2}, \frac{3}{2}\right)$,

2) $x(t, e) = 0$, $u(t, e) = 0$, $y^0 = -e$, $y(t, e) = 0$, where $t \in [0, \pi]$, $e \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

Because our trajectories must satisfy constraints (21) so, the above functions $x(t)$, $u(t)$, $p(t)$ reduce to:

1) $x(t, c_1) = c_1 \sin t$, $u(t, c_1) = c_1 \cos t$, $y^0 = -e$, $y(t, ec_1) = ec_1 \cos t$,

where $t \in [t(c_1), \pi]$, $c_1 \in (-1, 1)$, $e \in \left(\frac{1}{2}, \frac{3}{2}\right)$,

where $t(c_1)$ is a solution of equation $c_1 \sin t = t^2$ with respect to t in $[0, \pi]$ depending on c_1;

2) $x(t, e) = 0$, $u(t, e) = 0$, $y^0 = -e$, $y(t, e) = 0$,

where $t \in [0, \pi]$, $e \in \left(\frac{1}{2}, \frac{3}{2}\right)$.

We can easily check that the trajectories:

1) $x(t, c_1) = c_1 \sin t$, where $t \in [t(c_1), \pi]$, $c_1 \in (-1, 1)$,

2) $x(t, e) = 0$, where $t \in [0, \pi]$, $e \in \left(\frac{1}{2}, \frac{3}{2}\right)$

satisfy constraints (21).

Let us define a control (taking into account above functions):

$$u(t, y^0, y) = \begin{cases} -y/y^0), & \text{if } \; t \in [0, \pi], \; y^0 \in \left(-\frac{3}{2}, -\frac{1}{2}\right), \; |y| < \frac{3}{2}|\cos t|, \\ 0, & \text{if } \; t \in [0, \pi], \; y^0 \in \left(-\frac{3}{2}, -\frac{1}{2}\right), \; y = 0. \end{cases} \tag{22}$$

Now we will define $x(t, y^0, y)$ and $\overline{V}(t, y^0, y)$ in the same set of variables t and (y^0, y) respectively as:

$$x(t, y^0, y) = \begin{cases} (-y/y^0)\operatorname{tg} t, \\ 0, \end{cases} \tag{23}$$

$$\overline{V}(t, y^0, y) = \begin{cases} (y^2/2y^0)\operatorname{tg} t, \\ 0. \end{cases} \tag{24}$$

Substituting $x(t, y^0, y)$ and $\overline{V}(t, y^0, y)$ to the assertions of the Theorem 2 we see that $\overline{V}(t, y^0, y)$ defined by (24) and $V_y(t, y^0, y) = -x(t, y^0, y)$ defined by (23) satisfy these assertions, and also these assertions are satisfied by the pair $x(t) = 0$, $u(t) = 0$. So, from the Theorem 2, this pair is optimal.

The above statement denotes that, if expenditure starts from the value zero and after time must be also equal zero in the problem (16)–(21), so they must be all the time equal zero, without action of a control i.e. a control must be equal zero. Intuitively this fact is obvious, but this example proves that mathematically there is no other possibility.

5. REFERENCES

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